

Uniform Semi-Unification *

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Abstract

The notion of uniform semi-unification is extended by unification. We revisited symbolic semi-unification whose solvability coincides with that of uniform semi-unification (Aoto & Iwami, 2013). In this paper, we give the some proofs omitted in our previous work [1] due to the space limitation.

1 Introduction

The notion of semi-unification is extended by unification. If a semi-unifier exists, there exists a most general semi-unifier [3, 5, 9]. However, semi-unification is undecidable in general [5]. Hence, many decidable classes of semi-unification have been studied. For example, uniform semi-unification is decidable [2, 4, 8, 9, 10, 11]. We revisited symbolic semi-unification whose solvability coincides with that of uniform semi-unification [1].

In this paper, we give the some proofs omitted in [1] due to the space limitation. First, we consider symbolic semi-unification in section 2. In section 3, we introduce a rule-based symbolic semi-unification and show its partial correctness. In section 4, we discuss termination of symbolic semi-unification procedure on some derivation strategy. We refer to [1] omitted definitions in this paper.

2 Symbolic Semi-Unification

In this section, we consider a notion of symbolic semi-unification. We defined ∇ -term, ∇ -equation and ∇ -substitution in [1]. We refer to [1] omitted definitions.

Definition 2.1 ([1]) *For a set E of ∇ -equations, a semi-unifier of E is a ∇ -substitution σ such that $s\sigma^* = t\sigma^*$ for all $s \approx t \in E$; if E has a semi-unifier, E is said to be semi-unifiable. A symbolic semi-unification problem asks whether there exists a semi-unifier for a given set of ∇ -equations.*

Lemma 2.2 ([1]) *Let σ be a ∇ -substitution and s, t be ∇ -terms. If $s\sigma^* = t\sigma^*$ then $\nabla(s)\sigma^* = \nabla(t)\sigma^*$.*

Definition 2.3 ([1]) *For a set E of ∇ -equations, the ∇ -equality generated by E , denoted by $\approx_E$, is the smallest equivalence relation such that (i) $s \approx_E t$ for any $s \approx t \in E$, (ii) $s \approx_E t$ implies $\nabla(s) \approx_E \nabla(t)$, and (iii) for any $f \in \mathcal{F}$, $f(s_1, \dots, s_n) \approx_E f(t_1, \dots, t_n)$ iff, for any $i = 1, \dots, n$, $s_i \approx_E t_i$ holds.*

*This paper is revised version of [1].

Definition 2.4 ([1]) A set E of ∇ -equations is inconsistent if either (i) $x^i \approx_E s$ with $x^i \leq s \notin \mathcal{V}^*$, or (ii) $f(s_1, \dots, s_m) \approx_E g(t_1, \dots, t_n)$ with $f \neq g$ for some $f, g \in \mathcal{F}$. Furthermore, E is consistent if it is not inconsistent.

Since we gave only the proof sketch of the next lemma [1], we give the proof of it in detail.

Lemma 2.5 ([1]) Let E be a set of ∇ -equations. Suppose E is semi-unifiable and let σ be a semi-unifier of E . Then for any ∇ -terms u, v , $u \approx_E v$ implies $u\sigma^* = v\sigma^*$.

Proof. The proof proceeds by induction on the derivation of $u \approx_E v$. If $u \approx v \in E$ then the claim follows by assumption. If $u \approx_E v$ follows from $u' \approx_E v'$ where $u = \nabla(u')$ and $v = \nabla(v')$, then by induction hypothesis $u'\sigma^* = v'\sigma^*$, and hence by Lemma 2.2, $\nabla(u')\sigma^* = \nabla(v')\sigma^*$. Other cases follow easily. $\square$

Theorem 2.6 ([1]) For any terms $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, the following are equivalent: (i) $\{\nabla(s) \approx t\}$ is semi-unifiable, (ii) $\{s \leq t\}$ is semi-unifiable, and (iii) $\{\nabla(s) \approx t\}$ is consistent.

3 Partial Correctness of Symbolic Semi-Unification

In this section, we discuss a rule-based symbolic semi-unification procedure and prove its partial correctness. We refer to [1] omitted definitions.

Decompose	$\frac{\{f(s_1, \dots, s_n) \approx f(t_1, \dots, t_n)\} \uplus E}{\{s_1 \approx t_1, \dots, s_n \approx t_n\} \cup E} \quad f \in \mathcal{F}$
Reduce	$\frac{\{x^i \approx t, C[x^i] \approx u\} \uplus E}{\{x^i \approx t, C[t] \approx u\} \cup E} \quad x^i > t$
Delete	$\frac{\{x^i \approx x^i\} \uplus E}{E}$
Clash	$\frac{\{f(s_1, \dots, s_m) \approx g(t_1, \dots, t_n)\} \uplus E}{\perp} \quad f \neq g, f, g \in \mathcal{F}$
Check	$\frac{\{x^i \approx t\} \uplus E}{\perp} \quad t \notin \mathcal{V}^*, x^i \leq t$

Figure 1: Inference rules for symbolic semi-unification ([1])

Definition 3.1 ([1]) One step derivation using any of inference rules listed in Figure 1 is denoted by $\rightsquigarrow$. Here, the inference rules act on a finite set of ∇ -equations and $\uplus$ denotes the disjoint union. For an input of a finite set E_0 of ∇ -equations and the relation $>$, a symbolic semi-unification procedure non-deterministically constructs a derivation $E_0 \rightsquigarrow E_1 \rightsquigarrow \dots$ (possibly following some fixed derivation strategy). The derivation may be finite or infinite, and it is maximal if it does not end with E_k for which a further application of an inference rule is possible. A symbolic semi-unification procedure (following a fixed derivation strategy) terminates if any derivation (following that derivation strategy) is finite.

Remark 3.2 ([1]) We adopt a variant of Reduce using substitution (instead of the replacement):

$$\text{Reduce''} \frac{\{x^i \approx t\} \uplus E}{\{x^i \approx t\} \cup \{x^i := t\}(E)} x^i > t$$

Rule-based semi-unification calculi in [4, 8] use the replacement, and those in [6, 7, 11] use the substitution. We note that any substitution can be simulated by repeated applications of replacement.

Since we omitted the proof of the next lemma [1], we give the proof of it here.

Lemma 3.3 ([1]) Suppose $E \rightsquigarrow^* E'$ with $E' \neq \perp$. Then $\approx_E = \approx_{E'}$.

Proof. We show that $E \rightsquigarrow E'$ with $E' \neq \perp$ implies $\approx_E = \approx_{E'}$. Then the claim follows by induction on the length of $E \rightsquigarrow^* E'$. We distinguish the cases by the inference rule applied to $E \rightsquigarrow E'$. By our assumption that $E' \neq \perp$, inference rules Clash and Check are not used. Suppose that Delete is used. Let $E' \uplus \{x^i \approx x^i\} = E$. Then, since $s \approx_{E'} s$ for any s , the claim follows from the assumption immediately. Suppose Decompose is used. Let $E = F \uplus \{f(s_1, \dots, s_n) \approx f(t_1, \dots, t_n)\}$ and $E' = F \cup \{s_1 \approx t_1, \dots, s_n \approx t_n\}$. ($\approx_E \supseteq \approx_{E'}$) Since $f(s_1, \dots, s_n) \approx f(t_1, \dots, t_n) \in E$, $f(s_1, \dots, s_n) \approx_E f(t_1, \dots, t_n)$. Hence by the definition of $\approx_E$ ((iii) of Definition 2.3), $s_i \approx_E t_i$ for all $i = 1, \dots, n$. ($\approx_E \subseteq \approx_{E'}$) Since $s_i \approx t_i \in E'$, $s_i \approx_{E'} t_i$ for all $i = 1, \dots, n$. Hence by the definition of $\approx_{E'}$ ((iii) of Definition 2.3), $f(s_1, \dots, s_n) \approx_{E'} f(t_1, \dots, t_n)$. Suppose Reduce is used. Let $E = F \uplus \{x^i \approx t, C[x^i] \approx u\}$ and $E' = F \cup \{x^i \approx t, C[t] \approx u\}$. ($\approx_E \subseteq \approx_{E'}$) Since $x^i \approx t \in E'$ and $C[t] \approx u \in E'$, $x^i \approx_{E'} t$ and $C[t] \approx_{E'} u$. Then by the definition of $\approx_{E'}$ ((ii) and (iii) of Definition 2.3) $C[x^i] \approx_{E'} C[t]$ and hence by transitivity of $\approx_{E'}$, $C[x^i] \approx_{E'} u$. ($\approx_E \supseteq \approx_{E'}$) Since $x^i \approx t \in E$ and $C[x^i] \approx u \in E$, $x^i \approx_E t$ and $C[x^i] \approx_E u$. Then by the definition of $\approx_E$ ((ii) and (iii) of Definition 2.3) $C[x^i] \approx_E C[t]$ and hence by symmetricity and transitivity of $\approx_E$, $C[t] \approx_E u$. $\square$

Since we omitted the proof of the next corollary [1], we give the proof of it here.

Corollary 3.4 If $E \rightsquigarrow^* E' \neq \perp$, then E is semi-unifiable iff E' is semi-unifiable.

Proof. ($\Rightarrow$) Suppose E is semi-unifiable and $E \rightsquigarrow^* E' \neq \perp$. Let σ be a ∇ -substitution such that $s\sigma^* = t\sigma^*$ for all $s \approx t \in E$. For any $s \approx t \in E'$, $s \approx_{E'} t$ by the definition of $\approx_{E'}$, and hence by Lemma 3.3, $s \approx_E t$. Hence, by Lemma 2.5, $s\sigma^* = t\sigma^*$ for any $s \approx t \in E'$. Thus E' is semi-unifiable. ($\Leftarrow$) Suppose E' is semi-unifiable and $E \rightsquigarrow^* E' \neq \perp$. Let σ be a ∇ -substitution such that $s\sigma^* = t\sigma^*$ for all $s \approx t \in E'$. For any $s \approx t \in E$, $s \approx_E t$ by the definition of $\approx_E$ and hence by Lemma 3.3, $s \approx_{E'} t$. Hence, by Lemma 2.5, $s\sigma^* = t\sigma^*$ for any $s \approx t \in E$. Thus E is semi-unifiable. $\square$

Since we omitted the proof of the next theorem [1], we give the proof of it here.

Theorem 3.5 ([1]) Let E be a finite set of ∇ -equations. (1) If $E \rightsquigarrow^* \perp$ then E is not semi-unifiable. (2) If $E \rightsquigarrow^* E' \neq \perp$ and no inference rules are applicable to E' , then E is semi-unifiable.

Proof. (1) By our assumption, $E \rightsquigarrow^* E' \rightsquigarrow \perp$ for some E' . Then either $f(s_1, \dots, s_m) \approx g(t_1, \dots, t_n) \in E'$ with $f \neq g$ or $x^i \approx t \in E'$ with $t \notin \mathcal{V}^*$ and $x^i \trianglelefteq t$. In the former case, $f(s_1, \dots, s_m) \approx_E g(t_1, \dots, t_n)$ and in the latter case, $x^i \approx_E f(\dots, C[x^i], \dots)$, by Lemma 3.3.

Suppose E is semi-unifiable. Then, by Lemma 2.5, we have $f(s_1, \dots, s_m)\sigma^* = g(t_1, \dots, t_n)\sigma^*$ or $x^i\sigma^* = f(\dots, C[x^i], \dots)\sigma^*$, for a semi-unifier σ of E . But this is impossible. (2) By our assumption that no inference rules are applicable to E' , we have the following observations on E' : (a) One side of the equation is of the form x^i . (Otherwise Decompose rule should be applicable.) (b) If $x^i \approx t \in E'$ with $x^i \succ t$ then x^i does not occur in t or in other equations in E' ; this is because by (a) and the assumption that Check and Reduce can not be applied. Hence $\sigma = \{s := t \mid s \approx t \in E', s \succ t\}$ is a ∇ -substitution, and for any $s \approx t \in E'$, $s\sigma^* = (s\sigma)\sigma^* = t\sigma^*$. Thus E' is semi-unifiable. Hence by Corollary 3.4, E is semi-unifiable. $\square$

4 Termination of Symbolic Semi-Unification Procedure

In this section, we consider termination of symbolic semi-unification procedure on our derivation strategy [1]. We refer to [1] omitted definitions.

Theorem 4.1 ([1]) *Every derivation starting from a consistent finite set of ∇ -equations is finite.*

Definition 4.2 ([1]) *A derivation strategy is said to be refutationally complete if any maximal derivation starting from an inconsistent set of ∇ -equations and following that strategy is finite and ends with $\perp$.*

Lemma 4.3 ([1]) *A derivation strategy subject to using Reduce'' in place of Reduce and applying Check whenever possible is refutationally complete.*

Since we omitted the proof of the next theorem [1], we give the proof of it in detail.

Theorem 4.4 ([1]) *The symbolic semi-unification procedure terminates if it follows a refutationally complete derivation strategy; either the input E is semi-unifiable and any maximal derivation ends with a set of ∇ -equations or E is not semi-unifiable and any maximal derivation ends with $\perp$.*

Proof. If E is an inconsistent set of ∇ -equations, then by the refutational completeness of the derivation strategy then any derivation ends with $\perp$. Otherwise, E is a consistent set of ∇ -equations, and hence by Theorem 4.1, it stops. If the derivation ends with $\perp$, by Theorem 3.5, E is not semi-unifiable. Otherwise the derivation ends with a set of ∇ -equations and hence by Theorem 3.5, E is semi-unifiable. $\square$

Since we gave only the proof sketch of the next corollary [1], we give the proof of it in detail.

Corollary 4.5 ([1]) *Let E be a finite set of ∇ -equations. Then E is consistent iff E is semi-unifiable.*

Proof. $(\Rightarrow)$ Suppose E is not semi-unifiable. Take a refutationally complete strategy for the derivation. Then by Theorem 4.4, the derivation ends with $\perp$. Then $E \xrightarrow{*} E' \rightsquigarrow \perp$ for some E' , and thus either $f(s_1, \dots, s_m) \approx g(t_1, \dots, t_n) \in E'$ with $f \neq g$ or $x^i \approx f(\dots, C[x^i], \dots) \in E'$. Hence either $f(s_1, \dots, s_m) \approx_E g(t_1, \dots, t_n)$ or $x^i \approx_E f(\dots, C[x^i], \dots)$ by Lemma 3.3. Thus E is an inconsistent set of ∇ -equations. $(\Leftarrow)$ Let σ be a semi-unifier of E and suppose E is inconsistent. Then $f(s_1, \dots, s_m) \approx_E g(t_1, \dots, t_n)$ with $f \neq g$ or $x^i \approx_E f(\dots, C[x^i], \dots)$ by Definition 2.4. Then $f(s_1, \dots, s_m)\sigma^* = g(t_1, \dots, t_n)\sigma^*$ or $x^i\sigma^* = f(\dots, C[x^i], \dots)\sigma^*$ by Lemma 2.5. But this is a contradiction. $\square$

5 Conclusion

We revisited rule-based calculi for uniform semi-unification [1], on which efficient uniform semi-unification procedures [4, 8] are based. In this paper, we have given the some proofs omitted in [1] due to the space limitation.

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