

学位申請論文

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**Cancellation of Lattices and
Approximation Properties of Division Algebras.**

By Aiichi YAMASAKI

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0. Introduction

Let R be a Dedekind domain with the quotient field K . Let Λ be an R -order. In this general setting, it is proved in [3] that Roiter-Jacobinski type Divisibility Theorem holds for Λ -lattices. As a consequence, for a Λ -lattice L , the following two cancellation properties are equivalent.

(c) If L' is a local direct summand of $nL = L \oplus \cdots \oplus L$ for some $n \geq 0$, then $L \oplus L' \simeq M \oplus L'$ implies $L \simeq M$.

(c') If $L \oplus nL \simeq M \oplus nL$ for some $n \geq 0$, then $L \simeq M$.

As was pointed out in [3], putting $\Gamma := \text{End}_\Lambda L$ and $B := K\Gamma$, there is an intimate connection between cancellation property and the approximation property of the group of Vaserstein $\tilde{E}(\hat{B})$ in the idele topology of $\hat{B}^\times$, of which precise definitions will be recalled in §1.

Here we only indicate, $\hat{R} := \prod R_p$, the direct product of p -adic completions over all

maximal ideals of R , $\widehat{M} := M \otimes_R \widehat{R}$ for any R -algebra M , and $\widetilde{E}(C) := \langle (1 + xy)(1 + yx)^{-1} \mid x, y \in C, 1 + xy \in C^\times \rangle$ for any ring $C \ni 1$. Our first remark is

Proposition 1 (proof in 1.5) *The property (c') for L is equivalent with the following property (c'') of Γ*

$$(c'') \quad \widetilde{E}(\widehat{B}) \subset \widehat{\Gamma}^\times B^\times \text{ as subsets of } \widehat{B}^\times$$

0.1

We shall consider, for any finite dimensional K -algebra B , the following three *approximation properties over R* , in the idele topology of $\widehat{B}^\times$

(a) Strong approximation property:

$$\widetilde{E}(B) \text{ is dense in } \widetilde{E}(\widehat{B})$$

(a') $B^\times$ -approximation property:

$$\widetilde{E}(\widehat{B}) \text{ is contained in the closure of } B^\times$$

(a'') $\widehat{R}^\times B^\times$ -approximation property:

$$\widetilde{E}(\widehat{B}) \text{ is contained in the closure of } \widehat{R}^\times B^\times$$

There are the obvious implications (a) $\Rightarrow$ (a') $\Rightarrow$ (a''). Our second (rather obvious) remark is

Proposition 2 (proof in 1.2) *The property (a'') for B is equivalent with the (validity of) property (c'') for any Λ -lattice L such that $K\text{End}_\Lambda L \simeq B$.*

In the following cases, the property (a) always holds.

- (1) B is commutative (since $\widetilde{E}(B) = \widetilde{E}(\widehat{B}) = 1$, by definitions).
- (2) R is semi-local (by the Chinese Remainder Theorem).
- (3) $B = M_n(C)$ by some K -algebra C ($n \geq 2$) (cf [3]).

0.2

We shall give the following reduction to division algebras.

Theorem 1 (proof in 2.3) *Writing as $B/J(B) = \bigoplus_{i=1}^m M_{n_i}(D_i)$, with the Jacobson radical $J(B)$ and the division algebras D_i , in such an ordering that $n_i = 1$ ($1 \leq i \leq r$) and $n_i \geq 2$ ($r < i \leq m$), we have*

(i) $(a') \text{ for } B \Leftrightarrow (a') \text{ for } D_i (1 \leq i \leq r).$

(ii) $(a) \text{ (resp. } (a'')) \text{ for } B \Rightarrow (a) \text{ (resp. } (a'')) \text{ for } D_i (1 \leq i \leq r).$

Thus the approximation properties of general B can be reduced, more or less, to that of non-commutative division algebras over non-semi-local R , and then under a reasonable restriction, to that of central division ones, by 1.6.

Since PF-fields are the most familiar and important source of non semi-local Dedekind domains, now we restrict our attention to central division algebras over PF-fields and recall some basic facts and known results.

0.3

Assume that K is a PF-field in the sense of Artin [1], Chap.12, and let D be a finite dimensional non-commutative central division algebra over K .

In particular, there is given a set of valuations $\mathfrak{V}$ of K , satisfying the product formula $\prod_{\mathfrak{v}} |x|_{\mathfrak{v}} = 1$ for any $x \in K^\times$. In fact K is either a number field or a function field (of one variable) over the constant field $K_0 := \{x \in K \mid |x|_{\mathfrak{v}} \leq 1 \text{ for any } \mathfrak{v} \in \mathfrak{V}\}$. K is called a global field if either it is a number field or a function field with $\#K_0 < \infty$.

(i) Let P be a proper non-empty subset of $\mathfrak{V}$ consisting of non-archimedean valuations. Then $R(P) := \{x \in K \mid |x|_p \leq 1 \text{ for any } p \in P\}$ is a Dedekind domain (with an additional requirement $R(P) \supset K_0$, if K is a function field) having K as its quotient field. Conversely, any such Dedekind domain R in K is obtained as $R = R(P)$ by some P .

Consider the following condition (EC) for D over $R = R(P)$, which is known as Eichler's condition when K is a global field.

(EC) There is at least one $v \in \mathfrak{V} \setminus P$, such that the completion $D_v = D \otimes_K K_v$ is not a division algebra.

(ii) If K is a global field, by Wang-Platonov Theorem (cf. [6]), $[D^\times, D^\times] = \tilde{E}(D) =$ the kernel of the reduced norm. Hence the well known Eichler-Kneser Strong Approximation Theorem [2],[4] (and its analog due independently to Morita [8] and Swan [9], when K is a function field with $\#K_0 < \infty$) implies

(SAT) $(a) \text{ for } D \text{ over } R(P) \Leftrightarrow (\text{EC}) \text{ for } D \text{ over } R(P).$

0.4

Apart from global fields, we shall prove;

Theorem 2 (proof in 3.4) *For any PF-field K ,*

(a'') for D over $R(P) \Rightarrow (EC)$ for D over $R(P)$.

All in all, the most optimistic speculation would be “ $(a) \Leftrightarrow (a') \Leftrightarrow (a'') \Leftrightarrow (EC)$ ” for any central division algebras over any PF-fields. In this direction we can extend our previous result [11] as,

Theorem 3 (proof in 4.4) *When K is an algebraic function field of one variable over the reals,*

(EC) for D over $R(P) \Rightarrow (a)$ for D over $R(P)$.

1. Idele Topology

Let R be a Dedekind domain with the quotient field K . A finitely generated R -module L is called an R -lattice, if it is torsion free (or equivalently projective) over R , then $K \otimes_R L$ is a finite dimensional K -vector space and by the natural embedding $L \rightarrow K \otimes_R L$, one can identify as $K \otimes_R L = KL$. An R -algebra Λ is called an R -order if it is an R -lattice, then $K\Lambda = K \otimes_R \Lambda$ is a finite dimensional K -algebra. When a finite dimensional K -algebra B is given, we call that Γ is an R -order of B , if Γ is an R -order and $B = K\Gamma$.

For a maximal ideal p of R , let R_p always denote the p -adic completion of R . Let $\hat{R} := \prod R_p$, the product over all maximal ideals of R . By the diagonal embedding $R \rightarrow \hat{R}$, $\hat{R}$ is an R -algebra which is faithfully flat as an R -module. For any R -module M , put

$$M_p := M \otimes_R R_p, \quad \hat{M} := M \otimes_R \hat{R}.$$

We shall be concerned with only the following two special cases.

1) Γ is an R -order: Then, since Γ is finitely generated projective R -module, $\hat{\Gamma} := \Gamma \otimes_R \prod R_p \simeq \prod (\Gamma \otimes_R R_p) = \prod \Gamma_p$.

2) B is a finite dimensional K -algebra: Then $\widehat{B} := B \odot_R \widehat{R} \simeq B \odot_K K \odot_R \widehat{R} \simeq B \odot_K \widehat{K}$, and since $\widehat{R}$ is faithfully flat over R , one may canonically view as $\widehat{B} \supset \widehat{\Gamma}$, B and $B \cap \widehat{\Gamma} = \Gamma$. Moreover, there is a natural identification $\widehat{B} \simeq \varinjlim \widehat{\Gamma}/r$ ($r \in R \setminus \{0\}$) $\simeq \prod' B_p$ (w.r.t. Γ_p), where the last term denote the restricted direct product i.e. $\prod' B_p$ (w.r.t. Γ_p) := $\{x = (x_p) \in \prod B_p | x_p \in \Gamma_p \text{ for almost all } p\}$. The *adele topology* on $\widehat{B}$ is defined as the unique topology which induces on $\widehat{\Gamma}$ the direct product of p -adic topology $\prod \Gamma_p$, for one (hence any) R -order Γ of B . The name comes from the fact that $\widehat{K}$ with this topology is called the (restricted) adele ring of K .

The *idele topology* in $\widehat{B}^\times$ is defined as the unique topology which induces on $\widehat{\Gamma}^\times$ the direct product of p -adic topology $\prod \Gamma_p^\times$, for one (hence any) R -order Γ of B . The following explicit description of the idele topology will be useful for us.

1.1

For any R -order Γ of B and non-zero $r \in R$, put

$$(0) \quad \begin{cases} U_p(\Gamma, r) := \Gamma_p^\times \cap (1 + r\Gamma_p) = \begin{cases} \Gamma_p^\times & \text{if } r \in R_p^\times \\ 1 + r\Gamma_p & \text{if } r \in pR_p. \end{cases} \\ U(\Gamma, r) := \prod_p U_p(\Gamma, r) = \widehat{\Gamma}^\times \cap (1 + r\widehat{\Gamma}), \\ \Gamma(r) := R + r\Gamma, \text{ which is an } R\text{-order of } B \text{ again.} \end{cases}$$

By definitions, we have

(1) $\{U(\Gamma, r) | r \in R \setminus \{0\}\}$ is a fundamental system of neighbourhoods of 1 in $\widehat{B}^\times$ in the idele topology (for any one fixed Γ).

(1') $\{r\widehat{\Gamma} | r \in R \setminus \{0\}\}$ is a fundamental system of neighbourhoods of 0 in $\widehat{B}$ in the adele topology.

Let H be a subgroup of $\widehat{B}^\times$ and $\overline{H}$ will denote the closure of H in $\widehat{B}^\times$

(2) If $H \cap (1 + r\widehat{\Gamma}) \subset \widehat{\Gamma}^\times$ for some Γ and $r \in R \setminus \{0\}$, (in particular if $H \cap \widehat{\Gamma} \subset \widehat{\Gamma}^\times$), then the idele topology of $\widehat{B}^\times$ and the adele topology of $\widehat{B}$ induce the same topology on H . Indeed, $H \cap U(\Gamma, rr') = H \cap (1 + rr'\widehat{\Gamma})$ for any $r' \in R \setminus \{0\}$.

Since $\Gamma(r)_p^\times = R_p^\times U_p(\Gamma, r)$, we have

$$(3) \quad \widehat{R}^\times U(\Gamma, r) = \widehat{\Gamma(r)}^\times$$

$$(4) \quad \text{If } \widehat{R}^\times \subset \overline{H}, \text{ then } HU(\Gamma, r) \supset \widehat{R}^\times \text{ so that } \overline{H} = \bigcap_{r \neq 0} H\widehat{\Gamma(r)}^\times = \bigcap_{r \neq 0} \widehat{\Gamma(r)}^\times H.$$

1.2 Proof of Proposition 2 §0.

For any R -order Γ of B , put $\Lambda = \Gamma^{op}$, the opposite ring of Γ and $L := \Gamma$. then $\text{End}_\Lambda L = \Gamma$. Hence the condition (c'') for any L such that $K\text{End}_\Lambda L \simeq B$ is equivalent with the condition $\widetilde{E}(\widehat{B}) \subset \widehat{\Gamma}^\times B^\times$ for any Γ . But we have $\bigcap_{\Gamma} \widehat{\Gamma}^\times B^\times = \overline{\widehat{R}^\times B^\times}$. since $\widehat{\Gamma}^\times B^\times$ is closed and contains $\widehat{R}^\times B^\times$, so $\overline{\widehat{R}^\times B^\times} \subset \bigcap_{\Gamma} \widehat{\Gamma}^\times B^\times \subset \bigcap_{r \neq 0} \widehat{\Gamma(r)}^\times B^\times$ while we have $\overline{\widehat{R}^\times B^\times} = \bigcap_r \widehat{\Gamma(r)}^\times B^\times$, by (4).

1.3 Results of Vaserstein.

Let A be a ring with 1, and $E_n(A)$ be the elementary subgroup of $GL_n(A) := M_n(A)^\times$. By the usual embedding $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 1_{n-1} \end{pmatrix}$, we consider as $A^\times = GL_1(A) \subset GL_n(A) (n \geq 2)$. Let $\widetilde{E}(A)$ be the group of Vaserstein, i.e. the subgroup of $A^\times$ given by the generators as

$$\widetilde{E}(A) := \langle (1 + xy)(1 + yx)^{-1} \mid x, y \in A, 1 + xy \in A^\times \rangle$$

The commutator subgroup $[A^\times, A^\times]$ is always contained in $\widetilde{E}(A)$. Further, if A is local, $\widetilde{E}(A) = [A^\times, A^\times]$.

If A is semi-local, the well known Lemma of Bass and the fundamental results of Vaserstein ([10], Th.3.6) state:

$$(5) \quad GL_n(A) = A^\times E_n(A) (n \geq 2).$$

$$(6) \quad A^\times \cap E_n(A) = \widetilde{E}(A) (n \geq 2).$$

1.4

Lemma *Let B be a finite dimensional K -algebra and Γ be an R -order of B . Then the equality (5) of Bass (resp. (6) of Vaserstein) holds for $A = \widehat{B}$ or $\widehat{\Gamma}$. (where $\widehat{B}$ or $\widehat{\Gamma}$ is*

not semi-local if R is not semi-local.)

Proof In the proof of [10] Th.3.6 (a), where semi-locality of A is assumed, it is in fact proved that

(i) If the ring A satisfies the following condition (5'), then (5) holds.

(5') For any finitely generated left ideal L and $x \in A$,

$$Ax + L = A \Rightarrow (x + L) \cap A^\times \neq \emptyset.$$

(ii) If A satisfies (5') and moreover the following (6'), then (6) holds.

(6') $Ax_1 + Ax_2 = A \Rightarrow \forall y \in A, \exists v, q, u \in A$ such that $x_1 + vx_2 \in A^\times, 1 - yqv \in A^\times, x_1 + u(x_2 + yx_1) \in A^\times, x_1 + u(x_2 + yqx_1) \in A^\times$

Now, let $A = \prod' A_p$ (w.r.t C_p) be the restricted direct product of A_p with respect to its subring C_p , over some index p 's. If each A_p, C_p satisfies (5') and (6'), it is easy to see that A itself satisfies (5') and (6'). This applies for $\widehat{B}$ or $\widehat{\Gamma}$, since B_p and Γ_p are semi-local.

1.5 Proof of Proposition 1 §0.

As is well known (cf [3] §2 and §3), the property (c') is equivalent with the following

(c''') $\widehat{B}^\times \cap GL_n(B)GL_n(\widehat{\Gamma}) = B^\times \widehat{\Gamma}^\times$ for any $n \geq 2$.

By 1.4, we have

$$1) \quad GL_n(B) = B^\times E_n(B) \quad 2) \quad GL_n(\widehat{\Gamma}) = E_n(\widehat{\Gamma}) \widehat{\Gamma}^\times$$

Since $E_n(B)$ is dense in $E_n(\widehat{B})$ in the idele topology of $\widehat{B}^\times$ (cf [3] 1.2.1)

$$3) \quad E_n(B)GL_n(\widehat{\Gamma}) = E_n(\widehat{B})GL_n(\widehat{\Gamma}).$$

Using 1), 3), 2) in this order, we have: $GL_n(B)GL_n(\widehat{\Gamma}) = B^\times E_n(B)GL_n(\widehat{\Gamma}) = B^\times E_n(\widehat{B})GL_n(\widehat{\Gamma}) = B^\times E_n(\widehat{B})E_n(\widehat{\Gamma})\widehat{\Gamma}^\times = B^\times E_n(\widehat{B})\widehat{\Gamma}^\times$

Hence, the left hand side of (c''') = $\widehat{B}^\times \cap B^\times E_n(\widehat{B})\widehat{\Gamma}^\times = B^\times (\widehat{B}^\times \cap E_n(\widehat{B}))\widehat{\Gamma}^\times = B^\times \widetilde{E}(\widehat{B})\widehat{\Gamma}^\times$, the last equality by 1.4 again. This implies that (c''') is equivalent with (c'').

1.6 Change of the base field.

Let K' be a finite extension field of K contained in the center of B , and let R' be the integral closure of R in K' . Then R' is a Dedekind domain with the quotient K' and B

is a finite dimensional K' -algebra. Assume the following condition

(f) R' is a finitely generated R -module.

Then there are canonical isomorphism $\widehat{R}' \simeq R' \otimes_R \widehat{R}$ and $K' \otimes_{R'} \widehat{R}' \simeq K' \otimes_R \widehat{R}$ (cf. [7] Th.1 and Prop.4 Chap. II §3), so that $B \otimes_{R'} \widehat{R}' \simeq B \otimes_R \widehat{R}$ including the topology. Hence the approximation property (a) (resp. (a')) of B over R is equivalent with that of B over R' and (a'') over R implies that over R'

(i) For a residually separable algebra B (i.e. $B/J(B)$ is separable) the $B^\times$ - approximation problem is reduced, by Theorem 1, to that of a central division algebra.

(ii) If K is a PF-field, the condition (f) always holds (cf. [5] Th.72), so that we get the reduction to a central division algebra even for residually inseparable case.

2. Reduction to a Division Algebra.

Let B be a finite dimensional K -algebra with the Jacobson radical $J = J(B)$, $\varphi : B \rightarrow B' := B/J$ be the canonical K -morphism and $\Gamma' := \varphi(\Gamma)$. Then Γ' is an R -order in B' , and φ induces the following surjective morphisms: $\varphi_0 : \Gamma \rightarrow \Gamma'$, $\widehat{\varphi} := \varphi \otimes 1 : \widehat{B} = B \otimes \widehat{R} \rightarrow B' \otimes \widehat{R} = \widehat{B}'$ and $\widehat{\varphi}_0 := \varphi_0 \otimes 1 : \widehat{\Gamma} = \Gamma \otimes \widehat{R} \rightarrow \Gamma' \otimes \widehat{R} = \widehat{\Gamma}'$

Since $\widehat{R}$ is faithfully flat over R ,

1) $\text{Ker} \varphi_0 = \Gamma \cap J \subset J(\Gamma)$, $\text{Ker} \widehat{\varphi} = J \otimes \widehat{R} = \widehat{J} \subset J(\widehat{B})$, $\text{Ker} \widehat{\varphi}_0 = \widehat{\Gamma} \cap \widehat{J} \subset J(\widehat{\Gamma})$.

2) Viewing as $\widehat{B} \supset \widehat{\Gamma}, B$ and $\widehat{\Gamma} \cap B = \Gamma$, $\varphi_0, \widehat{\varphi}_0, \varphi$ is the restriction of $\widehat{\varphi}$ to $\Gamma, \widehat{\Gamma}, B$ respectively.

By 1), $1 + \widehat{J} \subset \widehat{B}^\times$ so that $\widehat{\varphi}$ induces the exact sequence of groups:

3) $1 \rightarrow 1 + \widehat{J} \rightarrow \widehat{B}^\times \rightarrow \widehat{B}'^\times \rightarrow 1$, and $\widehat{\varphi}^{-1}(\widehat{B}'^\times) = \widehat{B}^\times$

Consequently, we have

4) $\widehat{\varphi}(\widetilde{E}(\widehat{B})) = \widetilde{E}(\widehat{B}')$.

By the same reasoning, we have

5) $\widehat{\varphi}(\widetilde{E}(B)) = \widetilde{E}(B')$.

Also we have

- 6) $\widehat{\Gamma}^\times = \widehat{\varphi}_0^{-1}(\widehat{\Gamma}'^\times)$, which in turn implies
 7) $\widehat{\varphi}(U(\Gamma, r)) = U(\Gamma', r)$, in the notation of 1.1.

2.1

Lemma *Let H be a subgroup of $\widehat{B}^\times$ and $\overline{H}$ be its closure in $\widehat{B}^\times$*

- (i) $\widetilde{E}(\widehat{B}) \subset \overline{H} \Rightarrow \widetilde{E}(\widehat{B}') \subset \overline{\widehat{\varphi}(H)}$
 (ii) *If $1 + \widehat{J} \subset \overline{H}$, then the converse implication ($\Leftarrow$) also holds.*
 (iii) $1 + \widehat{J} \subset \overline{B^\times}$

Proof (i) and (ii): $(\widetilde{E}(\widehat{B}) \subset \overline{H}) \stackrel{(1) \ 1,1}{\Longleftrightarrow} (\widetilde{E}(\widehat{B}) \subset HU(\Gamma, r) \text{ for any } r \in R \setminus \{0\})$
 $\stackrel{4) \ \& \ 7)}{\implies} (\widetilde{E}(\widehat{B}') \subset \widehat{\varphi}(H)U(\Gamma', r)) \text{ for any } r \in R \setminus \{0\} \stackrel{3), \ 4) \ \& \ 7)}{\implies} (\widetilde{E}(\widehat{B}) \subset (1 + \widehat{J})HU(\Gamma, r)$
 $(= HU(\Gamma, r) \text{ if } \overline{H} \supset 1 + \widehat{J}) \text{ for any } r \in R \setminus \{0\}).$

(iii) Since any element of $\widehat{J}$ is nilpotent, $(1 + \widehat{J}) \cap (1 + r\widehat{\Gamma}) = 1 + (\widehat{J} \cap r\widehat{\Gamma}) \subset \widehat{\Gamma}^\times$, hence by (2) 1.1, the idele topology on $1 + \widehat{J}$ is induced from the adele topology. Since J is dense in $\widehat{J}$ in the adele topology, $1 + J$ is dense in $1 + \widehat{J}$ in the idele topology so that $1 + \widehat{J} \subset (1 + J)U(\Gamma, r) \subset B^\times U(\Gamma, r)$ for any $r \in R \setminus \{0\}$.

2.2

Lemma *Let $B = \bigoplus_{i=1}^m B_i$ be the ring direct sum of finite dimensional K -algebras.*

Then we have the following implications.

- (i) (a) (resp. (a')) for $B \Leftrightarrow$ (a) (resp. (a')) for any B_i ($1 \leq i \leq m$).
 (ii) (a'') for $B \Rightarrow$ (a'') for any B_i ($1 \leq i \leq m$).

Proof Let Γ_i be an R -order of B_i , then $\Gamma := \bigoplus \Gamma_i$ is an R -order of B . By the canonical isomorphism $\widehat{B} = B \odot \widehat{R} \simeq \bigoplus (B_i \odot \widehat{R}) = \bigoplus \widehat{B}_i$, $\widehat{B}^\times \simeq \prod \widehat{B}_i^\times$, $\widehat{\Gamma}^\times \simeq \prod \widehat{\Gamma}_i^\times$, $U(\Gamma, r) \simeq \prod U(\Gamma_i, r)$, $\widetilde{E}(B) \simeq \prod \widetilde{E}(B_i)$ and $\widetilde{E}(\widehat{B}) \simeq \prod \widetilde{E}(\widehat{B}_i)$, the claims are completely obvious.

2.3 Proof of Theorem 1 §0.

Put $B_i = M_{n_i}(D_i)$, $n_i = 1$ ($1 \leq i \leq r$), $n_i \geq 2$ ($r < i \leq m$). Recall that (a) holds for

$B_i (r < i \leq m)$ ((3) of 0.1) and apply 2.1 and 2.2, then we get the following implications which obviously prove Theorem 1.

$$(a) \text{ for } B \Rightarrow (a) \text{ for } B' \Leftrightarrow (a) \text{ for } D_i (1 \leq i \leq r)$$

$$(a') \text{ for } B \Leftrightarrow (a') \text{ for } B' \Leftrightarrow (a') \text{ for } D_i (1 \leq i \leq r)$$

$$(a'') \text{ for } B \Leftrightarrow (a'') \text{ for } B' \Rightarrow (a'') \text{ for } D_i (1 \leq i \leq r).$$

3. $(a'') \Rightarrow (\text{EC})$ for a PF-field.

Let K be a PF-field in the sense of [1], D be a central division K -algebra of dimension n^2 , $[D : K] = n^2$. Let $D_v := D \otimes_v K_v$ be the completion at $v \in \mathfrak{V}$. Let $\mathfrak{N} : D \rightarrow K$ be the reduced norm and $\mathfrak{N}_v : D_v \rightarrow K_v$ be its extension.

If D_v is a division algebra, $D_v \ni x \mapsto |\mathfrak{N}_v x|_v^{1/n}$ defines a norm of D_v as a K_v -vector space. While for any basis $\{\epsilon_i | 1 \leq i \leq n^2\}$ of D over K , writing $x = \sum \xi_i \epsilon_i \in D_v$, $x \mapsto \text{Max}_i |\xi_i|_v$ is also a norm of D_v . Hence there is a constant $c_v > 0$ such that

$$(1) \quad \text{Max}_i |\xi_i|_v \leq c_v |\mathfrak{N}_v x|_v^{1/n} \quad (x = \sum \xi_i \epsilon_i).$$

For almost all v , we have: v is non-archimedean; $\{\sum \xi_i \epsilon_i | |\xi_i|_v \leq 1\}$ is a maximal order of D_v ; $|\det \text{Tr}(\epsilon_i \epsilon_j)|_v = 1$. Hence for almost all v such that D_v is a division algebra, D_v/K_v is unramified and $|\mathfrak{N}_v x|_v^{1/n} = \text{Max}_i |\xi_i|_v$. Thus we can choose c_v as

$$(1') \quad c_v = 1 \text{ for almost all } v \text{ such that } D_v \text{ is a division algebra.}$$

Let R be a Dedekind domain with the quotient field K , so that it has the form $R = R(P) := \{\xi \in K | |\xi|_p \leq 1 \text{ for any } p \in P\}$ by some non-empty proper subset P consisting of non-archimedean valuation of $\mathfrak{V}$. For a fixed R , we can obviously choose a basis $\{\epsilon_i | 1 \leq i \leq n^2\}$ satisfying

$$(2) \quad \Gamma := \sum_{i=1}^{n^2} R \epsilon_i \text{ is an } R\text{-order of } D. \text{ and } \epsilon_1 = 1.$$

Then $\Gamma(r) := R + r\Gamma$ is also an R -order for any $r (\neq 0) \in R$.

3.1

Lemma Assume that D does not satisfy the Eichler's condition (EC) over $R = R(P)$, i.e. the following $\neg(\text{EC})$ is satisfied.

$\neg(\text{EC})$: D_v is a division algebra for any $v \in \mathfrak{V} \setminus P$

(i) Let $\{e_i\}$ be a basis of D satisfying (2), then there is a positive constant c depending only on $\{e_i\}$ but not on $r(\neq 0) \in R$ such that

$$\prod_P |r|_p < c \Rightarrow \Gamma(r)^\times = R^\times$$

(ii) $\widehat{R}^\times D^\times$ is closed in $\widehat{D}^\times$

Proof (i) It suffices to take $c := \prod_{\mathfrak{V} \setminus P} c_v^{-1}$ (which is well defined by (1')). Indeed, if $\Gamma(r)^\times \neq R^\times$, there is some $x = \sum \xi_i e_i \in \Gamma(r)^\times$ with $\xi := \xi_i \neq 0$ for some $i \geq 2$. At $p \in P$, since $x \in \Gamma(r)^\times$ so that $|\mathfrak{N}_p x|_p = 1$, we have

$$(3) \quad |\xi|_p \leq |r|_p = |r|_p |\mathfrak{N}_p x|_p^{1/n}$$

Using the product formula, (1) at $v \in \mathfrak{V} \setminus P$ and (3) at $p \in P$, the product formula again, in this order, we get

$$\begin{aligned} 1 &= \prod_{\mathfrak{V}} |\xi|_v = \prod_{\mathfrak{V} \setminus P} |\xi|_v \times \prod_P |\xi|_p \leq \prod_{\mathfrak{V} \setminus P} c_v |\mathfrak{N}_v x|_v^{1/n} \times \prod_P |r|_p |\mathfrak{N}_p x|_p^{1/n} \\ &= \prod_{\mathfrak{V} \setminus P} c_v \times \prod_P |r|_p = c^{-1} \prod_P |r|_p. \end{aligned}$$

(ii) Put $R(c) := \{r \in R \setminus \{0\} \mid \prod_P |r|_p < c\}$. If $r \in R(c)$, by (i), we have $\widehat{\Gamma}(r)^\times \cap D^\times = \Gamma(r)^\times = R^\times$. This obviously implies

$$(4) \quad \bigcap_{r \in R(c)} (D^\times \widehat{\Gamma(r)}^\times) = D^\times \left(\bigcap_{r \in R(c)} \widehat{\Gamma(r)}^\times \right).$$

Then together with (4) 1.1, we have

$$\overline{D^\times \widehat{R}^\times} = \bigcap_{r \neq 0} (D^\times \widehat{\Gamma(r)}^\times) \subset \bigcap_{r \in R(c)} (D^\times \widehat{\Gamma(r)}^\times) = D^\times \left(\bigcap_{r \in R(c)} \widehat{\Gamma(r)}^\times \right) = D^\times \widehat{R}^\times \subset \overline{D^\times \widehat{R}^\times}$$

3.2

As usual, we consider $D_p^\times$ as the subgroup of $\widehat{D}^\times$ consisting of the elements $x = (x_p) \in \widehat{D}^\times$ such that $x_q = 1$ for $q \in P \setminus \{p\}$. Under this convention, the following is obvious.

$$(5) \quad \#P \geq 2 \Rightarrow \widehat{R}^\times D^\times \cap D_p^\times \subset K_p^\times$$

If $\#P < \infty$, then R is semi-local and $\overline{D^\times} = \widehat{D}^\times$, hence 3.1 implies

$$(6) \quad 2 \leq \#P < \infty \Rightarrow (\text{EC}).$$

Indeed: $\neg(\text{EC})$ implies $\overline{\widehat{R}^\times D^\times} = \widehat{R}^\times D^\times$ so that $\widehat{D}^\times \subset \widehat{R}^\times D^\times$ hence $D_p^\times \subset D_p^\times \cap \widehat{R}^\times D^\times \subset K_p^\times$ a contradiction to the assumption that D is non-commutative.

3.3

Lemma *Let D be a central division algebra over a PF-field K . Then D_v is not a division algebra for infinitely many $v \in \mathfrak{V}$.*

Proof If $\mathfrak{V}$ contains at least one archimedean valuation (i.e. if K is a number field), as is well known, much stronger results are known. Assume that $\mathfrak{V}$ consists of non-archimedean valuations. If $\#\{v \in \mathfrak{V} \mid D_v \text{ is not a division algebra}\} < \infty$, then obviously we can choose a subset P of $\mathfrak{V}$ such that $2 \leq \#P < \infty$ and $\neg(\text{EC})$, a contradiction with (6) 3.2.

3.4 Proof of Theorem 2

We shall prove:

$$\neg(\text{EC}) \Rightarrow [\widehat{D}^\times, \widehat{D}^\times] \not\subset \overline{\widehat{R}^\times D^\times}$$

Suppose not, then $[\widehat{D}^\times, \widehat{D}^\times] \subset \widehat{R}^\times D^\times$ by 3.1, so that $[D_p^\times, D_p^\times] = D_p^\times \cap [\widehat{D}^\times, \widehat{D}^\times] \subset D_p^\times \cap \widehat{R}^\times D^\times \subset K_p^\times$ for any $p \in P$. It is a contradiction, since if x, y do not commute in $D_p^\times$, then one of $[x, y]$ and $[x, 1 + y]$ does not belong to $K_p^\times$.

4. (EC) $\Rightarrow$ (a) for a Real Coefficient Case.

We shall derive our Theorem 3 from our previous result [11], where it is proved only

for a special case of $K = \mathbb{R}(X)$. For this purpose, we prepare a few lemmas, which are of quite general nature, but regrettably, effectively applicable only for a very restricted situation like in Theorem 3, so that we state them only for PF-fields.

4.1

Let D be a central division algebra over a PF-field K and $R = R(P)$ as in 0.3. For a fixed $p_0 \in P$, as usual, we identify $D_{p_0}^\times$ as the (closed normal) subgroup of $\widehat{D}^\times$, consisting of elements $x = (x_p) \in \widehat{D}^\times \subset \prod D_p^\times$ with $x_p = 1$ for $p \neq p_0$. Then $\{\tilde{E}(D_p) | p \in P\}$ generates a dense subgroup of $\tilde{E}(\widehat{D})$ in $\widehat{D}^\times$ (cf. [2] §51). Hence a closed subgroup H of $\widehat{D}^\times$ contains $\tilde{E}(\widehat{D})$ if and only if it contains $\tilde{E}(D_p) = [D_p^\times, D_p^\times]$ for all $p \in P$. By the Chinese Remainder Theorem, ‘all’ can be replaced by ‘almost all’. In particular we have:

$$(1) \quad (a) \text{ for } D \text{ over } R \Leftrightarrow [D_p^\times, D_p^\times] \subset \overline{\tilde{E}(D)} \text{ for almost all } p,$$

and the corresponding $(1')$ (resp. $(1'')$) for (a') (resp. (a'')).

Let K' be a finite extension field of K , and let P' be the set of all (non-equivalent) valuations of K' lying over P , $P' = \{p' | p' \supset p, p \in P\}$. The integral closure R' of R in K' is given by $R' = \{0\} \cup \{x \in K'^\times | |x|_{p'} \leq 1 \text{ for any } p' \in P'\}$.

Put $D' := D \otimes_K K'$. By 1.6, $\widehat{D'} := D' \otimes_{R'} \widehat{R'} \simeq D' \otimes_R \widehat{R} \supset D \otimes_R \widehat{R} = \widehat{D}$ as topological rings, and

$$(2) \quad \widehat{D'}^\times \supset \widehat{D}^\times, \widehat{D'}^\times \supset \prod_{p' \supset p} D_{p'}'^\times \simeq D_p'^\times \supset D_p^\times \text{ as topological groups.}$$

In the following $\overline{(\quad)}$ denotes the closure in $\widehat{D'}^\times$.

Let consider the following condition $(*)$.

$$(*) \quad \text{For almost all } p \in P \quad p' \supset p \Rightarrow [D_{p'}'^\times, [D_p^\times, D_p^\times]] = [D_{p'}'^\times, D_{p'}'^\times].$$

Lemma *Assume that the condition $(*)$ holds. Then*

$$(a'') \text{ for } D \text{ over } R \Rightarrow (a) \text{ for } D' \text{ over } R'$$

Proof By the Chinese Remainder Theorem, $D'^\times$ is dense in $\prod_{p' \supset p} D_{p'}'^\times$. Hence, by

$$(2), [D_{p'}'^\times, [D_p^\times, D_p^\times]] \subset \overline{[D'^\times, [D_p^\times, D_p^\times]]}, \text{ so that the assumption } (*) \text{ implies}$$

$$(3) \quad [D_{p'}'^\times, D_{p'}'^\times] \subset \overline{[D'^\times, [D_p^\times, D_p^\times]]} \text{ for almost all } p \in P$$

On the other hand we have

(a'') for D over $R \xLeftrightarrow{(1'')} [D_p^\times, D_p^\times] \subset \widehat{R}^\times D^\times$ for almost all $p \in P \Rightarrow [D'^\times, [D_p^\times, D_p^\times]] \subset [D'^\times, \widehat{R}^\times D^\times] \subset [\overline{D'^\times}, \overline{D^\times}] \subset [\overline{D'^\times}, \overline{D'^\times}] = \widetilde{E}(D')$.

Hence by (3), we have $[D_{p'}'^\times, D_{p'}'^\times] \subset \widetilde{E}(D')$ for almost all p , which is equivalent with ((a) for D' over R') by (1).

4.2

Now assume that the constant field $K_0 = \mathbb{R}$, i.e. K is an algebraic function field of one variable over the reals.

Recall from [11] that $Br(K) \simeq K^\times / \mathfrak{N}(K(\sqrt{-1})^\times) = K^\times / (K^2 + K^2) \cap K^\times$, so that any central division algebra D over K is a quaternion algebra of the form $D \simeq \{-1, f\}$ with $f \in K^\times$. D is trivial if and only if $f \in K^2 + K^2$.

We call a valuation $v \in \mathfrak{V}$ real (resp. imaginary) if the residue field is isomorphic to $\mathbb{R}$ (resp. $\mathbb{C}$). $K(\sqrt{-1})$ is an algebraic function field of one variable over $\mathbb{C}$, so the corresponding $\mathfrak{V}'$ is identified with the Riemann surface $\mathfrak{R}$, and $K(\sqrt{-1})$ with the field of all meromorphic functions on $\mathfrak{R}$. Since a real valuation v of K does not decompose on $K(\sqrt{-1})$, the set $RP(K)$ of all real valuations can be embedded in $\mathfrak{R}$ as a finite disjoint union of closed curves. Then we have

$$K = \{\varphi \in K(\sqrt{-1}) \mid \varphi(z) \in \mathbb{R} \text{ for } z \in RP(K)\}.$$

Furthermore, as shown in [11],

$$K^2 + K^2 = \{f \in K \mid f(z) \geq 0 \text{ for } z \in RP(K)\},$$

so $\{-1, f\}$ is trivial for such f .

Let P be a non-empty proper subset of $\mathfrak{V}$.

Lemma *If D satisfies (EC) over $R(P)$, then D can be written as $D = D_0 \odot_{\mathbb{R}(g)} K$, where $g \in R(P) \setminus \mathbb{R}$ and D_0 is a central division $\mathbb{R}(g)$ -algebra satisfying (a) over $\mathbb{R}[g]$.*

Proof (EC) for D means that D_{v_0} is trivial for some $v_0 \in \mathfrak{V} \setminus P$. From Riemann-Roch Theorem, for any $f \in K^\times$ we can find $h \in K^\times$ such that $g := h^2 f$ has the unique

pole at v_0 . Therefore D can be written as $D = \{-1, g\}$, where $g \in R(P)$ and has the unique pole at v_0 .

Since D_{v_0} is trivial, we have either (i) v_0 is imaginary or (ii) v_0 is real and g is positive around v_0 . In any case, g is bounded from below on $RP(K)$, since g has no pole other than v_0 . So, $g + c \in K^2 + K^2$ for some $c \in \mathbb{R}$, hence $D = \{-1, g\} = \{-1, g(g+c)\} \simeq D_0 \odot_{\mathbb{R}(g)} K$ where $D_0 = \{-1, g(g+c)\}$ over $\mathbb{R}(g)$ which satisfies (EC) over $\mathbb{R}[g]$ since $X(X+c)$ is monic and quadratic. From our previous result [11], D_0 satisfies (a) over $\mathbb{R}[g]$.

4.3

Lemma *If K is an algebraic function field of one variable over $\mathbb{R}$, then the condition (*) in 4.1 is satisfied for any D .*

Proof Note that D_p is unramified for almost all $p \in P$. If D_p is trivial, then $D_p^\times = GL(2, K_p)$ and $[D_p^\times, D_p^\times] = SL(2, K_p)$. In this case $[D_{p'}'^\times, [D_p^\times, D_p^\times]] = [GL(2, K_{p'}'), SL(2, K_p)]$ is a normal subgroup of $SL(2, K_{p'}')$ not contained in its center, so it must coincide with $SL(2, K_{p'}')$.

If D_p is an unramified quaternion algebra, then p is real so that $-1 \notin K_p^2$ and $K_p^2 + K_p^2 = K_p^2$. Thus the reduced norm $\mathfrak{N}_p: D_p^\times \rightarrow K_p^\times$ maps $D_p^\times$ onto $K_p^{\times 2}$ with the kernel $[D_p^\times, D_p^\times]$. This implies $D_p^\times = K_p^\times [D_p^\times, D_p^\times]$, so that $[D_{p'}'^\times, [D_p^\times, D_p^\times]] = [D_{p'}'^\times, D_p^\times] \supset [D_p^\times, D_p^\times]$, hence the left hand side is a normal subgroup of $[D_{p'}'^\times, D_{p'}'^\times]$ containing $i \in [D_p^\times, D_p^\times]$, and as such it coincides with $[D_{p'}'^\times, D_{p'}'^\times]$. (Proof for $D_{p'}' \simeq \{-1, -1\}$ is as follows: Let N be a normal subgroup of $[D_{p'}'^\times, D_{p'}'^\times]$ containing i , then $\{x \in D_{p'}' | x^2 + 1 = 0\} \subset N$ since such x is conjugate with i by Skolem-Noether Theorem. So for any $a \in K_{p'}'$ such that $1 - a^2 \in K_{p'}'^2$, we have $-ai + bj \in N$ (with $a^2 + b^2 = 1$), hence $y := i(-ai + bj) = a + bij \in N$ which satisfies $y^2 - 2ay + 1 = 0$. Thus, again from Skolem-Noether Theorem, every $y \in [D_{p'}'^\times, D_{p'}'^\times]$ belongs to N).

4.4 Proof of Theorem 3 §0

Assume that D satisfies (EC) over $R(P)$. Applying Lemmas 4.1 and 4.3 to the result of Lemma 4.2 (regarding $\mathbb{R}(g)$ as K and K as K'), we see that D satisfies (a) over $\mathbb{R}[g]_K$.

the integral closure of $\mathbb{R}[g]$ in K . Since $g \in R(P)$, we have $R(P) \supset \mathbb{R}[g]_K$ so that (a) over $\mathbb{R}[g]_K$ implies (a) over $R(P)$.

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