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**Studies on Sampled-Data Control Systems
— the H_∞ Problem of Discrete Linear
Periodically Time-Varying Systems and
Nonuniform Sampling Problems**

Shunji Tanaka

November, 1999

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Abstract

In this thesis two sorts of topics concerning sampled-data control systems are treated. In Chapters 2 and 3 the H_∞ problem of discrete Linear Periodically Time-Varying (LPTV) systems is studied, and in Chapters 4 and 5 nonuniform sampling problems are.

In Chapters 2, we propose a new method to solve the H_∞ problem of discrete LPTV systems. This method is different from the other existing methods in the point that the infimum of the achievable norm of the closed-loop system is given by those of the ordinary H_∞ problems, and what is called “the causality constraint” does not appear explicitly. We also show some properties of a class of the discrete H_∞ problem by applying this method.

In Chapters 3, we compare our method proposed in Chapter 2 and the other methods to solve the discrete LPTV H_∞ problem in the state-space domain. More specifically, we compare the solvability conditions for the discrete LPTV H_∞ problem derived by applying these methods. We show the explicit equivalence among these solvability conditions, and investigate how the causality constraint appears in them. We also construct algorithms to calculate the infimum of the achievable norm of the closed-loop system based on these solvability conditions, and compare them through numerical examples.

In Chapters 4, we investigate, as one type of nonuniform sampling, sampling with periodically time-varying rates. Our main concern in this chapter is to examine the relation between the timing of sampling and robust stability. We focus especially on whether uniform sampling yields the best robust stability. Through numerical examples we examine when uniform sampling yields the best robust stability and when uniform sampling does not yield the best robust stability.

In Chapters 5, we investigate unreliable sampling as another type of nonuniform sampling. Unreliable sampling is such a situation in which the measurement of the outputs fails occasionally at some sampling instants. We design a minimum-variance linear estimate filter under such a situation, and, derive a stability condition in connection with the “unreliability” of sampling. We show through numerical examples the effectiveness of this filter in the context of control.

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Chapter 1

Introduction

In this thesis two sorts of topics concerning sampled-data control systems are treated. Namely, in Chapters 2 and 3 the H_∞ problem of discrete Linear Periodically Time-Varying (LPTV) systems is studied, and in Chapters 4 and 5 nonuniform sampling problems are.

First, the background, the purpose and the scope of the study on the H_∞ problem of discrete LPTV systems shall be explained.

LPTV systems appear in many practical control problems, e.g. those of the multirate sampled-data control systems, the attitude control problem of satellites, certain control problems of rotating machines etc. For this reason, we have enough motivation to study the H_∞ problem of this class of systems. In addition, this class is evidently larger than that of the linear time-invariant systems, but is not so large as to include the whole time-varying systems, and thus offers topics with appropriate difficulty for the development of theory. Namely, we can expect that a study on the H_∞ problem of this class of systems brings us a deeper understanding about the nature of the H_∞ problem.

Up to now several methods of solution have been proposed for the H_∞ problem of discrete LPTV systems, which can be categorized, roughly speaking, into the following two approaches:

- (1) time-varying approach,
- (2) time-invariant (lifting) approach.

In the time-varying approach, the problem of LPTV systems is directly solved in the framework of general discrete Linear Time-Varying (LTV) systems. As for the H_∞ problem of general discrete LTV systems, Feintuch and Francis [13] first derived a complete solution in 1986. It was based on function space analysis, and the solution was not

in such a form that engineers can easily apply it to their practical problems. In fact, the optimal cost, i.e., the infimum of the achievable H_∞ norm of the closed-loop system, was given in terms of infinite number of operators in the function space. After the research of Feintuch and Francis [13] not much was reported on this topic for a while. But recently, new advancement emerged inspired by developments in the H_∞ theory of time-invariant systems. Actually, Dragan et al. [10], Katayama and Ichikawa [27], and Scherpen and Verhaegen [44] gave similar expressions of the solution in terms of Algebraic Riccati Equations (AREs), while the employed approaches were mutually different. Although numerical algorithms to solve time-varying AREs are still under development, we can, in principle, obtain the solution numerically by applying these results.

In the time-invariant approach, the solution for linear time-invariant systems is directly utilized to solve the problem of LPTV systems. Here, the lifting technique [35, 29] plays a key role. This technique associates a class of linear periodic systems with an equivalent class of linear time-invariant systems. More specifically, the class of m -input, p -output, discrete, linear, N -periodic systems can be shown to be equivalent to the class of mN -input, pN -output discrete linear time-invariant systems with the transfer matrix $P(\lambda)$ satisfying the condition that $P(0)$ be block lower triangular. Note that $P(0)$ gives the throughput term in the state space expression and the above condition corresponds to the causality requirement. For this reason, the above condition on $P(0)$ is referred to as the “causality constraint.” This lifting technique enables us to translate the solution for time-invariant systems to that of periodic systems, and reduces the difficulties of dealing with time-varying systems to one point: “how to secure the causality constraint, i.e., how to make the controller $K(\lambda)$ satisfy the condition that $K(0)$ be block lower triangular.” The methods belonging to this “time-invariant (lifting) approach” category can be classified into several sub-categories according to the ways how to cope with this causality constraint.

Feintuch and Francis [14] solved the sensitivity minimization problem of periodic systems based on the result of [13]. Georgiou and Khargonekar [17] proposed a constructive algorithm for the same problem. Voulgaris, et al. [52] showed another algorithm for both H_∞ and H_2 problems of general multirate systems including LPTV systems. However, these three methods focus on the so-called one-block H_∞ problem and are not easy to be extended to the four-block H_∞ problem.

A method to solve the general four-block H_∞ problem was proposed by Chen and Qiu [4, 39]. They treated the problem in the framework of multirate sampled-data systems, and introduced the notion of “nest operators.” Sågfors, et al. [41, 42] also proposed another method using the game theoretic consideration and formulated the solution in

terms of AREs.

In this thesis we propose an alternative way of solving the four-block H_∞ problem of discrete LPTV systems, which can be categorized into the time-invariant approach. We will investigate the relations of our method to other methods. More specifically, we compare the solvability conditions of the discrete LPTV H_∞ problem given by the following four methods:

- (1) the methods belonging to the time-invariant (lifting) approach:
 - (A) our method,
 - (B) the method by Qiu and Chen [4, 39],
 - (C) the method by Sgfors, et al. [41, 42],
- (2) the method belonging to the time-varying approach [10, 27, 44].

Our method is based on the result of [13], and thus bears a certain similarity to the results of [14, 17] that the optimal cost of the N -periodic H_∞ problem is given in terms of the maximum of “ N values,” where the “ N values” are the norms of N infinite dimensional matrices in the method of [14, 17], but are the optimal costs of the N LTI H_∞ problems without the causality constraint in our method. Our method is advantageous over other methods in that the optimal cost is given by solving ordinary (in the sense that no causality constraint is imposed on them) LTI H_∞ problems, but it possesses the disadvantage that it only gives the optimal cost of the LPTV H_∞ problem and does not give direct knowledge about the structure of optimal controllers.

Derivation of our result is done via function space analysis based on the result of [13]. On the other hand, comparisons with other methods are made in the state-space domain. It is well known that the solvability condition as well as the class of all admissible controllers of the discrete H_∞ problem can be given in terms of AREs in the state-space domain. The solvability conditions of the discrete LPTV H_∞ problem given via the above four methods are also described in terms of AREs. Since these conditions are necessary and sufficient, they must be, as a matter of course, all equivalent. However, direct relations among these solvability conditions have not been clarified. In this thesis we show these relations explicitly by converting three types of solvability conditions obtained from the time-invariant approach into the solvability condition obtained from the time-varying approach. We also discuss how the causality constraint appears in the solvability conditions. Furthermore, we construct algorithms to calculate the optimal cost numerically based on the three types of conditions obtained from the time-invariant

approach, and show some numerical examples in which we examine efficiency of the algorithms.

Second, the background, the purpose and the scope of the study on nonuniform sampling problems are as follows.

During the last four or five decades, theories of sampled-data systems were developed rapidly and remarkably. In the first few decades, the major interests of researchers stayed in the case of uniform sampling, i.e., when the sampling is done with some fixed sampling rate (period). But in the last two decades, they extended the areas of research to the case of nonuniform sampling and clarified variety of interesting properties [1]. In this thesis we will investigate two types of nonuniform sampling mechanisms: sampling with periodically time-varying rates and unreliable sampling.

One of the typical systems in which sampling is nonuniform is multirate systems, and these systems are well studied. The systems that we are to investigate have some relationship with multirate systems: systems with periodically time-varying sampling rates can be regarded as a special case of multirate systems in the sense that theories of multirate systems can be applied to such systems with some modifications, and, on the other hand unreliable systems can be regarded as a generalization of multirate systems.

As said above, systems with periodically time-varying sampling rates are a special class of multirate systems, and, for this reason, it might be estimated that little are left to be investigated. However, a fundamental (from the viewpoint of control engineering) problem has not been solved completely. Namely, whether uniform sampling rates yield the best performances is not known completely. Many researchers have investigated the relationship between the sampling rate and performances of sampled-data systems, for example, [33], [16], [5], [32], and so on. In contrast, there are not so many researches that treat the timing of sampling itself. The existing researches [51], [45], [7], [8] tackled the finite horizon case, and the relationship between the timing of sampling and performances has not been clarified in the infinite horizon case, yet. In this thesis, we investigate a sampled-data robust stabilization problem, which is a natural formulation of infinite horizon problems, and examine how the timing of sampling affects on the system performances.

The sampled-data robust stabilization problem can be treated as the discrete H_∞ problem [20]. If the sampling rate is uniform, this problem reduces to the discrete LTI H_∞ problem. Thus in this case, time-varying (possibly nonlinear) controllers are known to have no advantage over LTI controllers [13, 30, 54]. However, whether nonuniform sampling has any advantages over uniform sampling stays outside the scope of the above research and should be studied separately. To clarify this point, we consider a

sampled-data robust stabilization problem focusing more upon the point whether uniform sampling gives the best performance in respect to robust stability. In studying this problem, the contents of Chapters 2 and 3 are exploited.

Unreliable sampling is such a situation in which the measurement of the outputs fails occasionally at some sampling instants. This situation can occur, for example, in the control of chemical plants. Namely, complex analysis is often needed to measure outputs in chemical plants, and it may not be finished during the prescribed interval, which results in the loss of measurement data. We will design a minimum-variance linear estimate filter under such a situation, and, derive a stability condition in connection with the “unreliability” of sampling. To be concrete, it will be clarified how often measurement of the output should be successful in order to guarantee stability of the filter. This result offers us a guideline for the design of the filter. Some numerical examples of sampled-data regulator systems with this filter will show the effectiveness of our design method.

Chapter 2

A new method for obtaining the optimal cost of the discrete LPTV H_∞ problem

In this chapter, we will give a new method to obtain the optimal cost of the discrete LPTV H_∞ problem, i.e., the H_∞ problem of discrete LPTV systems. We first explain some notations and definitions used in this chapter. Next, we formulate the discrete LPTV H_∞ problem and convert it into a four block model matching problem. Then, we give our main result that gives the optimal cost of this H_∞ problem based on the result in [13]. Last, we show some result obtained by applying our result to LTI systems.

2.1 Preliminaries

In this section, we introduce the notations and definitions used in this chapter.

2.1.1 Spaces and norms

The space of complex $n \times 1$ vector valued sequences $x = \{x_k : k \geq 0\}$ is denoted by s^n , or, simply s . The subspace of s^n of square-summable sequences is denoted by l_2^n , or, simply l_2 .

The norm on l_2 , denoted by $\|\cdot\|_{l_2}$, is defined to be

$$\|x\|_{l_2}^2 = \sum_{k=0}^{\infty} x_k^* x_k \quad (2.1)$$

where $*$ denotes complex conjugate transpose.

The space of bounded linear operators from l_2^n to l_2^m is denoted by $\mathcal{B}^{m \times n}$ or, just $\mathcal{B}$. The norm on $\mathcal{B}$, denoted by $\|\cdot\|$, is defined to be

$$\|F\| = \sup_{x \in l_2} \frac{\|Fx\|_{l_2}}{\|x\|_{l_2}}. \quad (2.2)$$

Any operator F in $\mathcal{B}$ can be expressed by a matrix representation

$$\begin{bmatrix} F_{00} & F_{01} & F_{02} & \cdots \\ F_{10} & F_{11} & F_{12} & \cdots \\ F_{20} & F_{21} & F_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \quad (2.3)$$

The subspace of $\mathcal{B}^{m \times n}$ of causal operators is denoted by $\mathcal{C}^{m \times n}$, or, simply $\mathcal{C}$ and the matrix representation of such operators has a block lower triangular form. The subspace of $\mathcal{B}^{m \times n}$ of time-invariant operators is denoted by $\mathcal{T}^{m \times n}$, or, just $\mathcal{T}$ and the matrix representation of such operators has a block Toeplitz form.

The space of essentially bounded, matrix valued functions defined on the unit circle is denoted by L_∞ . The norm on L_∞ , denoted by $\|\cdot\|_\infty$, is defined to be

$$\|f\|_\infty = \text{ess sup}_{\theta \in [0, 2\pi]} \bar{\sigma}(f(e^{j\theta})) \quad (2.4)$$

where $\bar{\sigma}(\cdot)$ denotes the maximum singular value. The subspace of L_∞ whose element has analytic continuation into the open unit disc is denoted by H_∞ .

Let F be a time-invariant operator in $\mathcal{T}$. From the matrix representation of F

$$\begin{bmatrix} F_0 & F_{-1} & F_{-2} & \cdots \\ F_1 & F_0 & F_{-1} & \cdots \\ F_2 & F_1 & F_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (2.5)$$

define the transfer function $\hat{F}(\lambda)$ of F by

$$\hat{F}(\lambda) = \sum_{k=-\infty}^{\infty} F_k \lambda^k. \quad (2.6)$$

Then, $\hat{F}(e^{j\theta}) \in L_\infty$ and $\|F\| = \|\hat{F}\|_\infty$. If $F \in \mathcal{C} \cap \mathcal{T}$,

$$\hat{F}(\lambda) = \sum_{k=0}^{\infty} F_k \lambda^k, \quad (2.7)$$

and $\hat{F}(e^{j\theta}) \in H_\infty$.

2.1.2 Shift and truncation operators

The k -th shift operator Λ_k is defined by

$$\begin{aligned} \Lambda_k : \{x_0, x_1, \dots\} \\ \rightarrow \begin{cases} \underbrace{\{0, \dots, 0\}}_{k \text{ times}}, x_0, x_1, \dots, & \text{if } k \geq 0 \\ \{x_{-k}, x_{-k+1}, \dots\}, & \text{if } k < 0. \end{cases} \end{aligned} \quad (2.8)$$

For any F from s to s , k -th input/output-shift operator $S_k(F)$ ($k = 0, 1, \dots$) is defined by

$$S_k(F) = \Lambda_{-k} F \Lambda_k. \quad (2.9)$$

The k -th truncation operator Π_k ($k = -1, 0, \dots$) is defined by

$$\begin{aligned} \Pi_k : \{x_0, x_1, \dots\} \\ \rightarrow \begin{cases} \{0, 0, \dots\} & \text{if } k = -1 \\ \{x_0, x_1, \dots, x_k, 0, \dots\} & \text{if } k \geq 0. \end{cases} \end{aligned} \quad (2.10)$$

2.1.3 Periodic operator and lifting operator

Periodic operators are defined as follows via the input/output-shift operator.

Definition 2.1 The operator F is called N -periodic if

$$F = S_N(F) = \Lambda_{-N} F \Lambda_N. \quad (2.11)$$

The subspace consisting of N -periodic operators in $\mathcal{B}^n$ is denoted by $\mathcal{P}_N^n$, or, just $\mathcal{P}_N$.

Let Ξ_N be an isomorphism defined by

$$\begin{aligned} \Xi_N : \{x_0, x_1, \dots\} \in s^n \\ \rightarrow \left\{ \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix}, \begin{bmatrix} x_N \\ x_{N+1} \\ \vdots \\ x_{2N-1} \end{bmatrix}, \dots \right\} \in s^{nN} \end{aligned} \quad (2.12)$$

and let $L_N(\cdot)$ be the lifting operator defined by

$$L_N(F) = \Xi_N F \Xi_N^{-1} \quad F : s^n \rightarrow s^n. \quad (2.13)$$

Then, for any F in $\mathcal{C}^{n \times n} \cap \mathcal{P}_N^{n \times n}$, $L_N(F)$ belongs to $\mathcal{C}^{nN \times nN} \cap \mathcal{T}^{nN \times nN}$, and

$$\|L_N(F)\| = \|F\|. \quad (2.14)$$

However, for an operator F^L in $\mathcal{C}^{nN \times nN} \cap \mathcal{T}^{nN \times nN}$, $L_N^{-1}(F^L)$ might not belong to $\mathcal{C}^{n \times n} \cap \mathcal{P}_N^{n \times n}$ because of the *causality constraint*. Namely, the transfer function $\hat{F}^L$ of F^L should have such a structure that the throughput term $\hat{F}^L(0)$ is block lower triangular. Hence we define the subspace $\mathcal{W}_N$ of $\mathcal{C}^{nN \times nN} \cap \mathcal{T}^{nN \times nN}$ by

$$\mathcal{W}_N = \{F^L : L_N^{-1}(F^L) \in \mathcal{C}^{n \times n} \cap \mathcal{T}^{n \times n}\}. \quad (2.15)$$

Then any function in $\mathcal{W}_N$ can be associated with a function in $\mathcal{C}^{n \times n} \cap \mathcal{P}_N^{n \times n}$. We also define the subspace $\hat{\mathcal{W}}_N$ of H_∞ , the space of transfer functions $\hat{F}^L$ whose throughput term $\hat{F}^L(0)$ is block lower triangular. $\hat{\mathcal{W}}_N$ in H_∞ corresponds to $\mathcal{W}_N$ in $\mathcal{T}$.

2.2 The discrete LPTV H_∞ problem

In this section, we formulate the discrete LPTV H_∞ problem and convert it into a model matching problem.

Consider the discrete system shown in Fig. 2.1. In Fig. 2.1, w is an exogenous input vector, u is a control input vector, z is a measured output vector, and y is a control output vector, whose dimensions are m_1 , m_2 , p_1 , and p_2 , respectively. P denotes

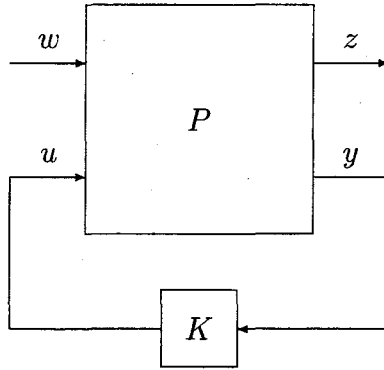


Figure 2.1: The block diagram of the discrete-time system

a discrete, linear, N -periodic, causal, finite-dimensional generalized plant that can be partitioned according to w , u , z , y into

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}. \quad (2.16)$$

K denotes a discrete linear causal controller.

Let us denote by $\mathcal{F}_l(P, K)$ the linear fractional transformation (LFT) of K on P , namely,

$$\mathcal{F}_l(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}. \quad (2.17)$$

The discrete LPTV H_∞ problem of P is to find K such that

- the closed-loop system is internally stable,
- the norm of $\mathcal{F}_l(P, K)$ (the operator from w to z) is minimized.

In other words, this H_∞ problem is the following optimization problem:

$$\nu = \inf_{K: \text{causal}} \|\mathcal{F}_l(P, K)\|. \quad (2.18)$$

In the next section, we will present our method to solve this problem.

2.3 Main result: a new method of solving the discrete LPTV H_∞ problem

In this section, we give a new type of method to obtain the optimal cost of the discrete LPTV H_∞ problem: a method to calculate ν in (2.18).

As mentioned in Introduction, there are several methods to solve the discrete LPTV H_∞ problem that can be categorized into two types of approaches: the time-varying approach and the time-invariant (lifting) approach. Our method is categorized into the time-invariant approach as the methods by Feintuch and Francis [14], Georgiou and Khargonekar [17], Voulgaris, et al. [52], Chen and Qiu [4, 39], and Sgfors, et al. [41, 42]. In particular, our method has a similarity to [14, 17] in the sense that all these methods are based on the results in [13], which solved the H_∞ problem of discrete LTV systems via function space analysis. Indeed, the optimal cost of the N -periodic H_∞ problem is given in terms of the maximum of N values both in our method and in [14, 17]. However, the significant difference is that in our method it is given by the maximum of N LTI H_∞ problems without the causality constraint that the throughput term of the controller should be block lower triangular, while in [14, 17] it is given by the norms of N infinite dimensional matrices. It is the primary advantage of our method since the optimal cost can be obtained by solving ordinary LTI H_∞ problems, although our method only gives the optimal cost of the LPTV H_∞ problem, and therefore it does not give any explicit knowledge of the structure of optimal controllers. In the following, we will show our

result that gives the optimal cost ν in terms of the optimal costs of ordinary LTI H_∞ problems.

For preparation, we first transform (2.18) into a model-matching problem by introducing the parametrization of stabilizing controllers and the inner-outer factorization.

Let us assume that P_{22} admits a doubly coprime factorization. In [40], it is shown that a doubly coprime factorization of an N -periodic operator F can be obtained by a doubly coprime factorization of the LTI system $L_N(F)$, and that each factor which appears in the doubly coprime factorization of $L_N(F)$ satisfies the causality constraint. By applying this result to P_{22} in our problem, we obtain

$$P_{22} = N_r D_r^{-1} = D_l^{-1} N_l \quad (2.19)$$

$$\begin{bmatrix} X_l & -Y_l \\ -N_l & D_l \end{bmatrix} \begin{bmatrix} D_r & Y_r \\ N_r & X_r \end{bmatrix} = I \quad (2.20)$$

where $N_r, D_r, X_r, Y_r, N_l, D_l, X_l$, and Y_l all belong to $\mathcal{C} \cap \mathcal{P}_N$. From (2.20) all controllers that internally stabilize the closed-loop system are parametrized by

$$\begin{aligned} K &= (Y_r - D_r Q)(X_r - N_r Q)^{-1} \\ &= (X_l - Q N_l)^{-1}(Y_l - Q D_l) \end{aligned} \quad (2.21)$$

where $Q \in \mathcal{C}$. Substituting (2.21) into (2.17), we obtain

$$\mathcal{F}_l(P, K) = R - SQT \quad (2.22)$$

where

$$R = P_{11} + P_{12} D_r Y_l P_{21} \in \mathcal{C} \cap \mathcal{P}_N, \quad (2.23)$$

$$S = P_{12} D_r \in \mathcal{C} \cap \mathcal{P}_N, \quad (2.24)$$

$$T = D_l P_{21} \in \mathcal{C} \cap \mathcal{P}_N. \quad (2.25)$$

Therefore, our problem is equivalent to [13, 52]

$$\nu = \inf_{Q \in \mathcal{C}} \|R - SQT\|, \quad R, S, T \in \mathcal{C} \cap \mathcal{P}_N \quad (2.26)$$

where $Q \in \mathcal{C}$ is a free parameter in the parametrization of stabilizing controllers. Furthermore, in [3, 52] it is shown that if R, S and T are N -periodic, the infimum in the right-hand side of (2.26) remains the same if Q is restricted to $\mathcal{C} \cap \mathcal{P}_N$. Namely,

$$\begin{aligned} \nu &= \inf_{Q \in \mathcal{C}} \|R - SQT\| \\ &= \inf_{Q \in \mathcal{C} \cap \mathcal{P}_N} \|R - SQT\|. \end{aligned} \quad (2.27)$$

Thus, by applying the lifting technique to (2.27), we obtain

$$\nu = \inf_{Q^L \in \mathcal{W}_N} \|R^L - S^L Q^L T^L\| \quad (2.28)$$

$$\begin{aligned} R^L &= L_N(R) \in \mathcal{W}_N, \quad S^L = L_N(S) \in \mathcal{W}_N, \\ T^L &= L_N(T) \in \mathcal{W}_N \end{aligned} \quad (2.29)$$

or, equivalently

$$\nu = \inf_{\hat{Q}^L(\lambda) \in \hat{\mathcal{W}}_N} \|\hat{R}^L(\lambda) - \hat{S}^L(\lambda) \hat{Q}^L(\lambda) \hat{T}^L(\lambda)\|_\infty \quad (2.30)$$

$$\hat{R}^L(\lambda), \hat{S}^L(\lambda), \hat{T}^L(\lambda) \in \hat{\mathcal{W}}_N \quad (2.31)$$

where $\hat{R}^L(\lambda)$, $\hat{S}^L(\lambda)$ and $\hat{T}^L(\lambda)$ denote the transfer functions of R^L , S^L , and T^L , respectively.

Here, we make the following assumption.

Assumption A1

$\hat{S}^L(\lambda)$ and $\hat{T}^{L*}(\lambda)$ are injective for every λ on the unit circle.

This assumption is necessary for existence of inner-outer factorizations of $\hat{S}^L(\lambda)$ and $\hat{T}^L(\lambda)$ in (2.30), and thus is necessary for deriving our main result Theorem 2.1. However, since it can be removed when we consider the H_∞ sub-optimal problem, it is not necessary for Corollary 2.1. We will explain later how to remove this assumption.

Under Assumption A1, $\hat{S}^L(\lambda)$ and $\hat{T}^L(\lambda)$ can be factorized as follows:

$$\hat{S}^L(\lambda) = \hat{S}_i^L(\lambda) \hat{S}_o^L(\lambda) \quad (2.32)$$

$$\hat{T}^L(\lambda) = \hat{T}_o^L(\lambda) \hat{T}_i^L(\lambda) \quad (2.33)$$

$$\hat{S}_i^L(\lambda) \in H_\infty, \quad \hat{S}_i^{L*}(\lambda) \hat{S}_i^L(\lambda) = I \quad (2.34)$$

$$\hat{T}_i^L(\lambda) \in H_\infty, \quad \hat{T}_i^L(\lambda) \hat{T}_i^{L*}(\lambda) = I \quad (2.35)$$

$$\hat{S}_o^L(\lambda), \hat{S}_o^{L^{-1}}(\lambda), \hat{T}_o^L(\lambda), \hat{T}_o^{L^{-1}}(\lambda) \in H_\infty \quad (2.36)$$

where $\tilde{T}(\lambda) = T^T(\lambda^{-1})$.

In [4], it is shown that $\hat{S}_o^L(\lambda)$, $\hat{T}_o^L(\lambda)$ can always be chosen so as to belong to $\hat{\mathcal{W}}_N$. In this case, the mapping from $\hat{Q}^L(\lambda)$ to $\hat{S}_o^L(\lambda) \hat{Q}^L(\lambda) \hat{T}_o^L(\lambda)$ is surjective on $\hat{\mathcal{W}}_N$. Therefore, (2.30) becomes

$$\nu = \inf_{\widehat{Q}^L(\lambda) \in \widehat{\mathcal{W}}_N} \|\widehat{R}^L(\lambda) - \widehat{S}_i^L(\lambda)\widehat{Q}^L(\lambda)\widehat{T}_i^L(\lambda)\|_\infty. \quad (2.37)$$

By multiplying

$$\begin{bmatrix} \widehat{S}_i^{L^*}(\lambda) \\ I - \widehat{S}_i^L(\lambda)\widehat{S}_i^{L^*}(\lambda) \end{bmatrix} \quad (2.38)$$

from the left of (2.37), and

$$\begin{bmatrix} \widehat{T}_i^{L^*}(\lambda) & I - \widehat{T}_i^{L^*}(\lambda)\widehat{T}_i^L(\lambda) \end{bmatrix} \quad (2.39)$$

from the right of (2.37), we obtain

$$\|\widehat{R}^L(\lambda) - \widehat{S}_i^L(\lambda)\widehat{Q}^L(\lambda)\widehat{T}_i^L(\lambda)\|_\infty = \left\| \begin{bmatrix} \widehat{X}^L(\lambda) - \widehat{Q}^L(\lambda) & \widehat{Y}^L(\lambda) \\ \widehat{Z}^L(\lambda) & \widehat{U}^L(\lambda) \end{bmatrix} \right\|_\infty \quad (2.40)$$

where $\widehat{X}^L(\lambda)$, $\widehat{Y}^L(\lambda)$, $\widehat{Z}^L(\lambda)$, and $\widehat{U}^L(\lambda)$ are defined by

$$\begin{bmatrix} \widehat{X}^L(\lambda) & \widehat{Y}^L(\lambda) \\ \widehat{Z}^L(\lambda) & \widehat{U}^L(\lambda) \end{bmatrix} = \begin{bmatrix} \widehat{S}_i^{L^*}(\lambda) \\ I - \widehat{S}_i^L\widehat{S}_i^{L^*}(\lambda) \end{bmatrix} R^L \begin{bmatrix} \widehat{T}_i^{L^*}(\lambda) & I - \widehat{T}_i^{L^*}(\lambda)\widehat{T}_i^L(\lambda) \end{bmatrix}. \quad (2.41)$$

Thus our problem (2.30) can be rewritten as

$$\nu = \inf_{\widehat{Q}^L(\lambda) \in \widehat{\mathcal{W}}_N} \left\| \begin{bmatrix} \widehat{X}^L(\lambda) - \widehat{Q}^L(\lambda) & \widehat{Y}^L(\lambda) \\ \widehat{Z}^L(\lambda) & \widehat{U}^L(\lambda) \end{bmatrix} \right\|_\infty \quad (2.42)$$

or

$$\nu = \inf_{Q^L \in \mathcal{W}_N} \left\| \begin{bmatrix} X^L - Q^L & Y^L \\ Z^L & U^L \end{bmatrix} \right\|. \quad (2.43)$$

Let

$$X = L_N^{-1}(X^L), \quad Y = L_N^{-1}(Y^L) \quad (2.44)$$

$$Z = L_N^{-1}(Z^L), \quad U = L_N^{-1}(U^L) \quad (2.45)$$

then, $X, Y, Z, U \in \mathcal{P}_N$ and

$$\nu = \inf_{Q \in \mathcal{C} \cap \mathcal{P}_N} \left\| \begin{bmatrix} X - Q & Y \\ Z & U \end{bmatrix} \right\|. \quad (2.46)$$

Now, the optimal cost ν is obtained by applying the following lemma.

Lemma 2.1 [13] Suppose that X, Y, Z and U belong to $\mathcal{B}$ and μ be given by

$$\mu = \inf_{Q \in \mathcal{C}} \left\| \begin{bmatrix} X - Q & Y \\ Z & U \end{bmatrix} \right\|. \quad (2.47)$$

Then, the following equation holds:

$$\mu = \sup_{k \geq -1} \|\Gamma_k\| \quad (2.48)$$

where

$$\Gamma_k = \begin{bmatrix} \Pi_k & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} X & Y \\ Z & U \end{bmatrix} \begin{bmatrix} I - \Pi_k & 0 \\ 0 & I \end{bmatrix}. \quad (2.49)$$

One may think that this lemma cannot be applied directly to (2.46), because Q in (2.46) is taken over $\mathcal{C} \cap \mathcal{P}_N$, while Q in (2.47) is taken over $\mathcal{C}$. However, as mentioned before, the right-hand side of (2.47) remains the same even if Q is restricted to $\mathcal{C} \cap \mathcal{P}_N$ provided that X, Y, Z and U belong to $\mathcal{P}_N$, and thus ν in (2.46) is given by the right-hand side of (2.48).

Although ν is given by the above lemma, it is in general difficult to calculate the right-hand side of (2.48). In contrast, our main result given in the following gives the optimal cost in a more explicit form.

Theorem 2.1 For $k = 0, 1, \dots, N-1$ let us define

$$X_k^L = L_N(S_k(X)), \quad Y_k^L = L_N(S_k(Y)), \quad (2.50)$$

$$Z_k^L = L_N(S_k(Z)), \quad U_k^L = L_N(S_k(U)), \quad (2.51)$$

and

$$\nu_k = \inf_{Q_k^L \in \mathcal{C} \cap \mathcal{T}} \left\| \begin{bmatrix} X_k^L - Q_k^L & Y_k^L \\ Z_k^L & U_k^L \end{bmatrix} \right\|. \quad (2.52)$$

Then,

$$\nu = \max(\nu_0, \nu_1, \dots, \nu_{N-1}). \quad (2.53)$$

Note that the infimum in (2.52) is not taken over $\mathcal{W}_N$, but over a larger class $\mathcal{C} \cap \mathcal{T}$.

Before giving the proof of this theorem, we will try to clarify its advantage. Theorem 2.1 shows that ν can be obtained by solving N model-matching problems. It is easy to check that the model-matching problem (2.52) is equivalent to the H_∞ problem

$$\nu_k = \inf_{K_k^L: \text{causal}} \|\mathcal{F}_l(P_k^L, K_k^L)\| \quad (2.54)$$

where P_k^L is defined by

$$\begin{aligned} P_k^L &= \begin{bmatrix} P_{k11}^L & P_{k12}^L \\ P_{k21}^L & P_{k22}^L \end{bmatrix} \\ &= \begin{bmatrix} L_N(S_k(P_{11})) & L_N(S_k(P_{12})) \\ L_N(S_k(P_{21})) & L_N(S_k(P_{22})) \end{bmatrix}. \end{aligned} \quad (2.55)$$

Thus we are led to the following corollary, in which the advantage of Theorem 2.1 is exploited more explicitly.

Corollary 2.1 Define P_{kij}^L ($i, j = 1, 2, \quad k = 0, 1, \dots, N-1$) by

$$P_{kij}^L = L_N(S_k(P_{ij})) \quad (2.56)$$

Then, the optimal cost ν is given by

$$\nu = \max(\nu_0, \nu_1, \dots, \nu_{N-1}) \quad (2.57)$$

where ν_k ($k = 0, 1, \dots, N-1$) is

$$\nu_k = \inf_{K_k^L: \text{causal}} \|\mathcal{F}_l(P_k^L, K_k^L)\|. \quad (2.58)$$

Since the H_∞ problem (2.58) is of the LTI system P_k^L and no causality constraint is imposed on it, ν_k is easily calculated by existing algorithms for the LTI H_∞ problem, and so is ν . As previously mentioned, this point is the primary advantage of our result and the difference from the results in [14] and [17].

Although Corollary 2.1 holds even for general four-block H_∞ problems, it is difficult to calculate the optimal cost of such problems analytically. For this reason, in many practical situations, we consider the sub-optimal H_∞ problem to find controllers that makes $\mathcal{F}_l(P, K) < \gamma$ for a given γ . Thus we will rewrite Corollary 2.1 to meet this situation.

Corollary 2.2 There exists stabilizing controller K such that

$$\|\mathcal{F}_l(P, K)\| < \gamma, \quad K: \text{causal} \quad (2.59)$$

if and only if there exists K_k^L ($k = 0, 1, \dots, N-1$) such that

$$\|\mathcal{F}_l(P_k^L, K_k^L)\| < \gamma, \quad K_k^L: \text{causal}. \quad (2.60)$$

Remark 2.1 Note that this corollary holds *without* Assumption A1. It is because Assumption A1 corresponds to the conditions of invariant zeros on the unit circle in the standard H_∞ problem, and therefore it can be avoided in the case of a sub-optimal problem as (2.59), by taking a similar method to [43, 31, 36].

Theorem 2.1 (or, its explicit forms Corollary 2.1 and Corollary 2.2) is also true when we apply our result to LTI systems, not to LPTV systems. From this, we can show some interesting properties of the discrete LTI H_∞ problem and the related problem. We will state such properties in the next section.

Now, we are to prove Theorem 2.1. To prove Theorem 2.1, we use the following lemma.

Lemma 2.2 Assume that X, Y, Z and U are any operators belonging to $\mathcal{P}_N$. Then, the following equation holds for any $i, j = 0, 1, \dots, N-1$:

$$\inf_{Q \in \mathcal{C} \cap \mathcal{P}_N} \left\| \begin{bmatrix} S_i(X) - Q & S_i(Y) \\ S_i(Z) & S_i(U) \end{bmatrix} \right\| = \inf_{Q \in \mathcal{C} \cap \mathcal{P}_N} \left\| \begin{bmatrix} S_j(X) - Q & S_j(Y) \\ S_j(Z) & S_j(U) \end{bmatrix} \right\|. \quad (2.61)$$

Proof of Lemma 2.2. Without loss of generality, we assume $i > j$. Since the mapping $S_k(\cdot)$ is bijective on $\mathcal{C} \cap \mathcal{P}_N$,

$$\begin{aligned} & \inf_{Q \in \mathcal{C} \cap \mathcal{P}_N} \left\| \begin{bmatrix} S_k(X) - Q & S_k(Y) \\ S_k(Z) & S_k(U) \end{bmatrix} \right\| \\ &= \inf_{Q \in \mathcal{C} \cap \mathcal{P}_N} \left\| \begin{bmatrix} S_k(X) - S_k(Q) & S_k(Y) \\ S_k(Z) & S_k(U) \end{bmatrix} \right\| \\ &= \inf_{Q \in \mathcal{C} \cap \mathcal{P}_N} \left\| \begin{bmatrix} S_k(X - Q) & S_k(Y) \\ S_k(Z) & S_k(U) \end{bmatrix} \right\|. \end{aligned} \quad (2.62)$$

Therefore, it suffices to show that for all $i, j = 0, 1, \dots, N-1$

$$\|S_i(F)\| = \|S_j(F)\|, \quad \forall F \in \mathcal{P}_N \quad (2.63)$$

holds. From the definition of $S_k(\cdot)$,

$$\begin{aligned} \|S_i(F)\| &= \|S_{i-j}(S_j(F))\| \\ &= \|\Lambda_{-i+j} S_j(F) \Lambda_{i-j}\| \\ &\leq \|S_j(F) \Lambda_{i-j}\| \\ &\leq \|S_j(F)\|. \end{aligned} \quad (2.64)$$

Since (2.64) is also true for $S_j(F)$ and $S_{N+i}(F)$,

$$\|S_j(F)\| \leq \|S_{N+i}(F)\| = \|S_i(F)\|. \quad (2.65)$$

This completes the proof.

Q.E.D.

Proof of Theorem 2.1. For simplicity, we only consider the 2-periodic case ($N = 2$), that is, we are to prove that

$$\nu = \max(\nu_0, \nu_1) \quad (2.66)$$

$$\begin{aligned} \nu_0 &= \inf_{Q_0^L \in \mathcal{C} \cap \mathcal{T}} \left\| \begin{bmatrix} X_0^L - Q_0^L & Y_0^L \\ Z_0^L & U_0^L \end{bmatrix} \right\| \\ &= \inf_{Q^L \in \mathcal{C} \cap \mathcal{T}} \left\| \begin{bmatrix} X^L - Q^L & Y^L \\ Z^L & U^L \end{bmatrix} \right\| \end{aligned} \quad (2.67)$$

$$\nu_1 = \inf_{Q_1^L \in \mathcal{C} \cap \mathcal{T}} \left\| \begin{bmatrix} X_1^L - Q_1^L & Y_1^L \\ Z_1^L & U_1^L \end{bmatrix} \right\|. \quad (2.68)$$

It would be evident that the following arguments can be extended to the general case.

Applying Lemma 2.1 to (2.67), ν_0 is given by

$$\nu_0 = \sup_{k \geq -1} \|\Gamma_k^L\| \quad (2.69)$$

where

$$\Gamma_k^L = \begin{bmatrix} \Pi_k & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} X^L & Y^L \\ Z^L & U^L \end{bmatrix} \begin{bmatrix} I - \Pi_k & 0 \\ 0 & I \end{bmatrix}. \quad (2.70)$$

From (2.49), $\|\Gamma_k^L\|$ can be expressed in terms of Γ_k as

$$\|\Gamma_k^L\| = \|\Gamma_{2k+1}\|. \quad (2.71)$$

Thus, from (2.69) and (2.71),

$$\nu_0 = \sup_{k=-1,1,\dots} \|\Gamma_k\|. \quad (2.72)$$

Similarly,

$$\nu_1 = \sup_{k=-1,1,\dots} \|\Gamma'_k\| \quad (2.73)$$

where

$$\Gamma'_k = \begin{bmatrix} \Pi_k & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} S_1(X) & S_1(Y) \\ S_1(Z) & S_1(U) \end{bmatrix} \begin{bmatrix} I - \Pi_k & 0 \\ 0 & I \end{bmatrix}. \quad (2.74)$$

Let the matrix representation of

$$\begin{bmatrix} X & Y \\ Z & U \end{bmatrix} \quad (2.75)$$

be

$$\left[\begin{array}{ccccc|ccccc}
 X_{00} & X_{1,-1} & X_{0,-2} & X_{1,-3} & \cdots & Y_{00} & Y_{1,-1} & Y_{0,-2} & Y_{1,-3} & \cdots \\
 X_{01} & X_{10} & X_{0,-1} & X_{1,-2} & \cdots & Y_{01} & Y_{10} & Y_{0,-1} & Y_{1,-2} & \cdots \\
 X_{02} & X_{11} & X_{00} & X_{1,-1} & \cdots & Y_{02} & Y_{11} & Y_{00} & Y_{1,-1} & \cdots \\
 X_{03} & X_{12} & X_{01} & X_{10} & \cdots & Y_{03} & Y_{12} & Y_{01} & Y_{10} & \cdots \\
 \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \\
 \hline
 Z_{00} & Z_{1,-1} & Z_{0,-2} & Z_{1,-3} & \cdots & U_{00} & U_{1,-1} & U_{0,-2} & U_{1,-3} & \cdots \\
 Z_{01} & Z_{10} & Z_{0,-1} & Z_{1,-2} & \cdots & U_{01} & U_{10} & U_{0,-1} & U_{1,-2} & \cdots \\
 Z_{02} & Z_{11} & Z_{00} & Z_{1,-1} & \cdots & U_{02} & U_{11} & U_{00} & U_{1,-1} & \cdots \\
 Z_{03} & Z_{12} & Z_{01} & Z_{10} & \cdots & U_{03} & U_{12} & U_{01} & U_{10} & \cdots \\
 \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots &
 \end{array} \right] \quad (2.76)$$

Then the matrix representation of Γ_k ($k = 1, 3, \dots$) is

$$\left[\begin{array}{c|ccc|cccc}
 & X_{0,-k-1} & X_{1,-k-2} & \cdots & Y_{00} & \cdots & Y_{0,-k-1} & \cdots \\
 0 & \vdots & \vdots & & \vdots & & \vdots & \\
 & X_{0,-1} & X_{1,-2} & \cdots & Y_{0k} & \cdots & Y_{0,-1} & \cdots \\
 \hline
 0 & \cdots & 0 & & \cdots & \cdots & 0 & \cdots \\
 \hline
 & Z_{0,-k-1} & Z_{1,-k-2} & \cdots & U_{00} & \cdots & U_{0,-k-1} & \cdots \\
 & \vdots & \vdots & & \vdots & & \vdots & \\
 0 & Z_{0,-1} & Z_{1,-2} & \cdots & U_{0k} & \cdots & U_{0,-1} & \cdots \\
 & \vdots & \vdots & & \vdots & & \vdots &
 \end{array} \right] \quad (2.77)$$

Also, the matrix representation of Γ'_k ($k = 1, 3, \dots$) is

$$\left[\begin{array}{c|ccc|cccc}
 & X_{1,-k-1} & X_{0,-k-2} & \cdots & Y_{10} & \cdots & Y_{1,-k-1} & \cdots \\
 0 & \vdots & \vdots & & \vdots & & \vdots & \\
 & X_{1,-1} & X_{0,-2} & \cdots & Y_{1k} & \cdots & Y_{1,-1} & \cdots \\
 \hline
 0 & \cdots & 0 & & \cdots & \cdots & 0 & \cdots \\
 \hline
 & Z_{1,-k-1} & Z_{0,-k-2} & \cdots & U_{10} & \cdots & U_{1,-k-1} & \cdots \\
 & \vdots & \vdots & & \vdots & & \vdots & \\
 0 & Z_{1,-1} & Z_{0,-2} & \cdots & U_{1k} & \cdots & U_{1,-1} & \cdots \\
 & \vdots & \vdots & & \vdots & & \vdots &
 \end{array} \right] \quad (2.78)$$

Since the matrix representation of Γ_{k-1} ($k = 1, 3, \dots$) is

$$\begin{array}{c|ccc|ccc}
0 & X_{1,-k} & X_{0,-k-1} & \cdots & Y_{00} & \cdots & Y_{1,-k} & \cdots \\
& \vdots & \vdots & & \vdots & & \vdots & \\
& X_{1,-1} & X_{0,-2} & \cdots & Y_{0,k-1} & \cdots & Y_{1,-1} & \cdots \\
0 & \cdots & 0 & \cdots & \cdots & \cdots & 0 & \cdots \\
\hline
& Z_{1,-k} & Z_{0,-k-1} & \cdots & U_{00} & \cdots & U_{1,-k} & \cdots \\
& \vdots & \vdots & & \vdots & & \vdots & \\
0 & Z_{1,-1} & Z_{0,-2} & \cdots & U_{0,k-1} & \cdots & U_{1,-1} & \cdots \\
& \vdots & \vdots & & \vdots & & \vdots &
\end{array} \quad (2.79)$$

Γ_{k-1} can be expressed by

$$\Gamma_{k-1} = \begin{bmatrix} \Lambda_{-1} & 0 \\ 0 & \Lambda_{-1} \end{bmatrix} \Gamma'_k \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_1 \end{bmatrix}. \quad (2.80)$$

Therefore,

$$\|\Gamma'_k\| \geq \|\Gamma_{k-1}\|, \quad \forall k = 1, 3, \dots, \quad (2.81)$$

and hence

$$\begin{aligned}
\nu &= \max \left(\sup_{k=-1,1,\dots} \|\Gamma_k\|, \sup_{k=0,2,\dots} \|\Gamma_k\| \right) \\
&\leq \max \left(\sup_{k=-1,1,\dots} \|\Gamma_k\|, \sup_{k=-1,1,\dots} \|\Gamma'_k\| \right) \\
&= \max(\nu_0, \nu_1). \quad (2.82)
\end{aligned}$$

Furthermore, from Lemma 2.1, Lemma 2.2, and (2.74)

$$\begin{aligned}
\nu &= \inf_{Q \in \mathcal{C} \cap \mathcal{P}_2} \left\| \begin{bmatrix} X - Q & Y \\ Z & U \end{bmatrix} \right\| \\
&= \inf_{Q \in \mathcal{C} \cap \mathcal{P}_2} \left\| \begin{bmatrix} S_1(X) - Q & S_1(Y) \\ S_1(Z) & S_1(U) \end{bmatrix} \right\| \\
&= \sup_{k \geq -1} \|\Gamma'_k\| \\
&\geq \sup_{k=-1,1,\dots} \|\Gamma'_k\| = \nu_1. \quad (2.83)
\end{aligned}$$

Since $\nu \geq \nu_0$ by (2.48) and (2.72), it follows from (2.82) and (2.83) that

$$\nu = \max(\nu_0, \nu_1). \quad (2.84)$$

Q.E.D.

2.4 Some application to LTI systems

In the preceding section, we showed Theorem 2.1 that gives the optimal cost of the discrete LPTV H_∞ problem in terms of the optimal costs of discrete LTI H_∞ problems. Although this theorem originally aims at the discrete LPTV H_∞ problem, it can be used to derive some property of the discrete LTI H_∞ problem. In this section, we show such properties that can be derived by applying Theorem 2.1 (or, Corollary 2.1 and Corollary 2.2) to LTI systems.

If P is an LTI system, P_k^L defined by (2.55) satisfies

$$P_0^L = P_1^L = \cdots = P_{N-1}^L \quad (2.85)$$

for any integer N . Thus Corollary 2.1 becomes as follows.

Corollary 2.3 Define P^L by

$$P^L = \begin{bmatrix} L_N(P_{11}) & L_N(P_{12}) \\ L_N(P_{21}) & L_N(P_{22}) \end{bmatrix} \quad (2.86)$$

where N is arbitrary positive integer. Then, the optimal cost ν satisfies

$$\nu = \inf_{K:\text{causal}} \|\mathcal{F}_l(P, K)\| = \inf_{K^L:\text{causal}} \|\mathcal{F}_l(P^L, K^L)\|. \quad (2.87)$$

This corollary claims that the optimal cost of the LPTV H_∞ problem of P and that of the LTI H_∞ problem of P^L , the lifted system of P , are identical. This fact implies that the optimal cost cannot be improved even if we use any N -periodic controller K such that the lifted system K^L of K is causal, but K itself is possibly noncausal. It is shown in [13, 30, 54] that we cannot improve the optimal cost even if we consider causal time-varying controllers in the case of the LTI H_∞ problem. Thus it is a matter of course that *causal* N -periodic controllers do not improve the optimal cost. However, Corollary 2.3 claims that even if we use such controllers that belong to some class of *noncausal* N -periodic systems, the optimal cost cannot be improved. Note that it does not lead that the optimal cost cannot be improved by any noncausal controllers, although N can be taken arbitrarily large. Indeed, in most cases the optimal cost improves with such noncausal controllers that can use information of the one step future. It is because the noncausal controllers which Corollary 2.3 assures that the optimal cost does not improve are not completely noncausal but are causal if they are lifted, and therefore these controllers cannot use no future information at $t = Nk - 1$. Although there is no qualitative explanation yet about the difference between these two types of noncausal

controllers, we can show some interesting property on the H_∞ problem of a certain system.

Consider a discrete LTI system P and let us consider the following two situations:

- (a) The control outputs are periodically delayed by one step. That is, for given N and for some q ($q = 0, 1, \dots, N-1$) the control outputs y at $t = Np + q$ ($p = 0, 1, \dots$) are not available by the controller until the next step $t = Np + q + 1$ because of, for example, periodic delay in the circuit of the sampler. More specifically, we consider a sampler whose input sequence is the sequence of the control outputs $y_0, y_1, \dots$, and whose output sequence is

$$y_0, y_1, \dots, y_{q-1}, \phi, \begin{bmatrix} y_q \\ y_{q+1} \end{bmatrix}, \dots, y_N, y_{N+1}, \dots, y_{N+q-1}, \phi, \begin{bmatrix} y_{N+q} \\ y_{N+q+1} \end{bmatrix}, \dots \quad (2.88)$$

where ϕ denotes the case that no outputs are obtained.

- (b) The control outputs are all delayed by one step. In other words, we consider a sampler whose input sequence is the sequence of the control outputs $y_0, y_1, \dots$, and whose output sequence is

$$\phi, y_0, y_1, \dots \quad (2.89)$$

In these two situations, we are to solve the H_∞ problems and to compare the optimal costs.

Intuitively, the optimal cost in (a) seems smaller than that in (b), because more information is available by controllers in (a) than in (b). However, in reality, the optimal costs in both situations are identical. It can be shown by applying Corollary 2.3, as we do in the following.

First, we consider the situation (a). If we regard delay of the control outputs as a constraint on the controller, we are to consider K such that the elements K_{ij} ($i, j = 0, 1, 2, \dots$) of its matrix representation satisfy

$$K_{ij} = 0 \quad (i < j, \text{ or } i = j = Np + q). \quad (2.90)$$

On the other hand, if we regard P together with the sampler (2.88) as a plant, it becomes an N -periodic system. Therefore, we can assume without loss of generality that $q = N-1$ from Lemma 2.2, and, as mentioned in the preceding section, we can restrict the class of the controllers to that of N -periodic systems. As a consequence, the problem that we should consider here is to find K^L for $P^L = L_N(P)$ such that the throughput term $\widehat{K}^L(0)$ of the transfer function of K^L has the form

$$\widehat{K}^L(0) = \begin{bmatrix} k_{00} & 0 & & 0 \\ k_{10} & k_{11} & & \\ \vdots & & \ddots & \\ k_{N-2,0} & k_{N-2,1} & \cdots & k_{N-2,N-2} & 0 \\ k_{N-1,0} & k_{N-1,1} & \cdots & k_{N-1,N-2} & 0 \end{bmatrix}. \quad (2.91)$$

If we define P' by

$$P' = \begin{bmatrix} P'_{11} & P'_{12} \\ P'_{21} & P'_{22} \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ \Lambda_1 P_{21} & \Lambda_1 P_{22} \end{bmatrix} \quad (2.92)$$

this problem can be converted into the problem to find $K'^L = L_N(K\Lambda_1^{-1})$ for $P'^L = L_N(P')$ such that the throughput term $\widehat{K}'^L(0)$ of the transfer function of K'^L has the form

$$\widehat{K}'^L(0) = \begin{bmatrix} k'_{00} & k'_{01} & & 0 \\ k'_{10} & k'_{11} & k'_{12} & \\ \vdots & \vdots & & \ddots \\ k'_{N-2,0} & k'_{N-2,1} & \cdots & k'_{N-2,N-2} \\ k'_{N-1,0} & k'_{N-1,1} & \cdots & k'_{N-1,N-1} \end{bmatrix}. \quad (2.93)$$

Next, we consider the situation (b). Since P together with the sampler (2.89) can be modeled as P' given by (2.92), in this situation we are to consider the LTI H_∞ problem of P' . In the lifted space, we are to consider the problem to find K'^L for P'^L such that $\widehat{K}'^L(0)$ satisfies the causality constraint, i.e., $\widehat{K}'^L(0)$ has the form

$$\widehat{K}'^L(0) = \begin{bmatrix} k'_{00} & 0 & & 0 \\ k'_{10} & k'_{11} & & \\ \vdots & \vdots & \ddots & \\ k'_{N-2,0} & k'_{N-2,1} & \cdots & k'_{N-2,N-2} & 0 \\ k'_{N-1,0} & k'_{N-1,1} & \cdots & k'_{N-1,N-2} & k'_{N-1,N-1} \end{bmatrix}. \quad (2.94)$$

As a matter of course, the class of K'^L satisfying (2.93) includes the class of K'^L satisfying (2.94), and, is included by the class of K'^L with no constraint on $\widehat{K}'^L(0)$. However, from Corollary 2.3, the optimal cost of the H_∞ problem of P'^L with K'^L satisfying (2.94) is identical to that of the H_∞ problem of P'^L with no constraint on $\widehat{K}'^L(0)$. Therefore, we can conclude that the optimal costs in both situations (a) and (b) are identical.

This example shows that when we consider the H_∞ problem of LTI systems, there is no advantage in trying to use the available current output if there exists even a *single*

sequence of the control outputs $y_p, y_{N+p}, \dots$ that is delayed by one step. What is worse, trying to do so rather becomes a disadvantage compared with the case that *all* the control outputs are delayed by one step, since in such a case we can restrict the class of controllers to that of LTI controllers and do not need to take time-varying controllers into account.

Chapter 3

Four types of solutions to the discrete LPTV H_∞ problem: state space consideration

In the preceding chapter, we showed a new method to obtain the optimal cost of the H_∞ problem of discrete LPTV systems via function space analysis. In this chapter we will compare our method and other methods in the state space domain. More specifically, we first show four types of solvability conditions of the discrete LPTV H_∞ problem based on the following methods:

- (1) the methods belonging to the time-invariant (lifting) approach:
 - (A) the method proposed in the preceding chapter,
 - (B) the method proposed by Qiu and Chen [4, 39],
 - (C) the method proposed by Sgfors, et al. [41, 42],
- (2) the method belonging to the time-varying approach [10, 27, 44].

We then clarify the equivalence among these solvability conditions through showing their explicit relationships. We also investigate how the causality constraint is treated in the three methods belonging to the time-invariant approach. Last, we show three types of algorithms to calculate the optimal cost based on the methods belonging to the time-invariant approach, and examine their efficiency through numerical examples.

3.1 The solutions to the discrete LPTV H_∞ problem — time-invariant approach

In this section, we introduce three types of solvability conditions based on three methods which can be categorized into the time-invariant approach:

Method A: the method proposed in the preceding chapter,

Method B: the method proposed by Qiu and Chen [4, 39],

Method C: the method proposed by Sgfors, et al. [41, 42]

As mentioned in the preceding chapter, in the time-invariant approach we use the results for LTI systems. Therefore, we will first state the solution to the discrete LTI H_∞ problem, and, based on it, we will show the results obtained by applying the three methods in the state space domain.

3.1.1 The solution to the discrete LTI H_∞ problem

Consider the discrete LTI H_∞ problem, namely, the H_∞ problem of the system shown in Fig. 2.1 where the generalized plant P is LTI.

We assume that a state space realization of P is given by

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array} \right]. \quad (3.1)$$

Next, we put the following assumptions on P .

Assumption AC1

The pair (A, B_2) is stabilizable.

Assumption AC2

D_{12} has full column rank.

Assumption AC3

The matrix $\begin{bmatrix} A - zI_n & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for any $z \in \partial\mathcal{D}$.

Assumption AO1

The pair (C_2, A) is detectable.

Assumption AO2

D_{21} has full row rank.

Assumption AO3

The matrix $\begin{bmatrix} A - zI_n & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for any $z \in \partial\mathcal{D}$.

Let us further define the following matrices:

$$D_X = \begin{bmatrix} D_{11} & D_{12} \end{bmatrix}, \quad D_Y = \begin{bmatrix} D_{11}^T & D_{21}^T \end{bmatrix}^T \quad (3.2)$$

$$R_X = D_X^T D_X - \begin{bmatrix} \gamma^2 I_{m_1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} D_{11}^T D_{11} - \gamma^2 I_{m_1} & D_{11}^T D_{12} \\ D_{12}^T D_{11} & D_{12}^T D_{12} \end{bmatrix} \quad (3.3)$$

$$R_Y = D_Y D_Y^T - \begin{bmatrix} \gamma^2 I_{p_1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} D_{11} D_{11}^T - \gamma^2 I_{p_1} & D_{11} D_{21}^T \\ D_{21} D_{11}^T & D_{21} D_{21}^T \end{bmatrix}. \quad (3.4)$$

Then, the solvability condition of the discrete LTI H_∞ problem is given by each of the following lemmas [23, 48, 34]. Here, an $(m_1 + m_2) \times (m_1 + m_2)$ matrix M is said to have a J_{m_1, m_2} -factorization if for some square nonsingular matrix T

$$M = T^T \begin{bmatrix} -\gamma^2 I_{m_1} & 0 \\ 0 & I_{m_2} \end{bmatrix} T \quad (3.5)$$

holds.

Lemma 3.1 (The solvability condition (A) of the discrete LTI H_∞ problem)

Let us assume that Assumptions AC1–AC3, AO1–AO3 hold on the system shown in Fig. 2.1. A causal stabilizing controller K for the generalized plant given by (3.1) that satisfies $\|\mathcal{F}_l(P, K)\| < \gamma$ exists if and only if the following conditions are satisfied:

C1 The Algebraic Riccati Equation (ARE)

$$\begin{aligned} X &= A^T X A + C_1^T C_1 - E_X^T M_X^{-1} E_X \\ E_X &= B^T X A + D_X^T C_1 \\ M_X &= B^T X B + R_X \end{aligned} \quad (3.6)$$

has a stabilizing solution $X \geq 0$ such that M_X has a J_{m_1, m_2} -factorization.

C2 The ARE

$$\begin{aligned} Y &= AY A^T + B_1 B_1^T - E_Y M_Y^{-1} E_Y^T \\ E_Y &= AY C_k^T + B_1 D_Y^T \\ M_Y &= CY C^T + R_Y \end{aligned} \quad (3.7)$$

has a stabilizing solution $Y \geq 0$ such that M_Y has a J_{p_1, p_2} -factorization.

C3 $\rho(XY) < \gamma^2$.

Lemma 3.2 (The solvability condition (B) of the discrete LTI H_∞ problem)

Let us assume that Assumptions AC1–AC3, AO1–AO3 hold on the system shown in Fig. 2.1. A causal stabilizing controller K for the generalized plant given by (3.1) that satisfies $\|\mathcal{F}_l(P, K)\| < \gamma$ exists if and only if the following conditions are satisfied:

C1 The same with Condition C1 in Lemma 3.1.

C2' The ARE

$$\begin{aligned} Z &= A_Z Z A_Z^T + B_{Z1} B_{Z1}^T - E_Z M_Z^{-1} E_Z^T \\ E_Z &= A_Z Z C_Z^T + B_{Z1} D_Z^T \\ M_Z &= C_Z Z C_Z^T + R_Z \end{aligned} \quad (3.8)$$

has a stabilizing solution $Z \geq 0$ such that M_Z has a J_{m_2, p_2} -factorization.

The matrices A_Z , B_{Z1} , C_Z , D_Z , and R_Z , appearing in Condition C2', are defined by

$$A_Z = A - B_1 F_1, \quad B_{Z1} = B_1 V_{11}^{-1} \quad (3.9)$$

$$C_Z = \begin{bmatrix} C_{Z1} \\ C_{Z2} \end{bmatrix} = \begin{bmatrix} V_{22} F_2 \\ C_2 - D_{21} F_1 \end{bmatrix}, \quad D_Z = \begin{bmatrix} D_{Z11} \\ D_{Z21} \end{bmatrix} = \begin{bmatrix} V_{21} V_{11}^{-1} \\ D_{21} V_{11}^{-1} \end{bmatrix} \quad (3.10)$$

$$R_Z = D_Z D_Z^T - \begin{bmatrix} \gamma^2 I_{m_2} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} D_{Z11} D_{Z11}^T - \gamma^2 I_{m_2} & D_{Z11} D_{Z21}^T \\ D_{Z21} D_{Z11}^T & D_{Z21} D_{Z21}^T \end{bmatrix} \quad (3.11)$$

where F_i is given by

$$F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = -M_X^{-1} E_X \quad (3.12)$$

and V_{ij} are arbitrary matrices (V_{11} and V_{22} are nonsingular) which satisfy

$$M_X = \begin{bmatrix} V_{11}^T & V_{21}^T \\ 0 & V_{22}^T \end{bmatrix} \begin{bmatrix} -\gamma^2 I_{m_1} & 0 \\ 0 & I_{m_2} \end{bmatrix} \begin{bmatrix} V_{11} & 0 \\ V_{21} & V_{22} \end{bmatrix}. \quad (3.13)$$

Remark 3.1 In [53], it is shown that Y in Condition C2 and Z in Condition C2' are related by

$$Z = Y(I - \gamma^{-2}XY)^{-1}. \quad (3.14)$$

If Conditions C1–C3 in Lemma 3.1 or Conditions C1 and C2' in Lemma 3.2 are satisfied, there exist many stabilizing controllers which satisfy $\|\mathcal{F}_l(P, K)\| < \gamma$. As is a well-known fact, all such controllers can be parametrized via a free parameter. The following lemma gives this parametrization [48, 18].

Lemma 3.3 (Controller parametrization for the discrete LTI H_∞ problem)

Let us assume that Assumptions AC1–AC3, AO1–AO3 hold on the system shown in Fig. 2.1. Let us further assume that Conditions C1–C3 or Conditions C1 and C2' are satisfied. Then we can define the system K_c whose state space realization is expressed by

$$K_c = \left[\begin{array}{c|cc} A_c & B_{c1} & B_{c2} \\ \hline C_{c1} & D_{c11} & D_{c12} \\ C_{c2} & D_{c21} & 0 \end{array} \right] \quad (3.15)$$

$$D_{c11} = -V_{22}^{-1}W_{12}W_{22}^{-1}, \quad D_{c12} = V_{22}^{-1}W_{11}, \quad D_{c21} = W_{22}^{-1} \quad (3.16)$$

$$C_{c1} = -V_{22}^{-1}C_{Z1} - D_{c11}C_{Z2}, \quad C_{c2} = -D_{c21}C_{Z2} \quad (3.17)$$

$$B_{c1} = E_{Z2}D_{c21}^T D_{c21} + B_2 D_{c11}, \quad B_{c2} = \gamma^{-2}E_{Z1}V_{22}^{-T} + \gamma^{-2}E_{Z2}D_{c11}^T + B_2 D_{c12} \quad (3.18)$$

$$A_c = A_Z + B_2 C_{k1} + E_{Z2}D_{c21}^T C_{c2} \quad (3.19)$$

where E_{Z1} , E_{Z2} are given by

$$E_Z = \begin{bmatrix} E_{Z1} & E_{Z2} \end{bmatrix} = \begin{bmatrix} A_Z Z C_{Z1}^T + B_{Z1} D_{Z11}^T & A_Z Z C_{Z2}^T + B_{Z1} D_{Z21}^T \end{bmatrix} \quad (3.20)$$

and, W_{11} , W_{12} and W_{22} are any matrices (W_{11} and W_{22} are nonsingular) which satisfy

$$M_Z = \begin{bmatrix} W_{11} & W_{12} \\ 0 & W_{22} \end{bmatrix} \begin{bmatrix} -\gamma^2 I_{m_2} & 0 \\ 0 & I_{p_2} \end{bmatrix} \begin{bmatrix} W_{11}^T & 0 \\ W_{12}^T & W_{22}^T \end{bmatrix}. \quad (3.21)$$

All the stabilizing controllers K for P that satisfy $\|\mathcal{F}_l(P, K)\| < \gamma$ are given by

$$K = \mathcal{F}(K_c, Q) \quad (3.22)$$

where Q is any causal LTI stable system which satisfies

$$\|Q\| < \gamma. \quad (3.23)$$

In the next section, we show three types of solvability conditions of the discrete LPTV H_∞ problem based on Lemma 3.1, Lemma 3.2, and Lemma 3.3, the results for the discrete LTI H_∞ problem.

3.1.2 Three types of solvability conditons for the discrete LPTV H_∞ problem

Consider the discrete LPTV H_∞ problem, namely, the H_∞ problem of the system shown in Fig. 2.1 where the generalized plant P is LPTV.

We assume that P is N -periodic and its space realization is given by

$$\left[\begin{array}{c|c} A(\cdot) & B(\cdot) \\ \hline C(\cdot) & D(\cdot) \end{array} \right] = \left[\begin{array}{c|cc} A(\cdot) & B_1(\cdot) & B_2(\cdot) \\ \hline C_1(\cdot) & D_{11}(\cdot) & D_{12}(\cdot) \\ C_2(\cdot) & D_{21}(\cdot) & 0 \end{array} \right] \quad (3.24)$$

where A , B , C , and D satisfy

$$\begin{aligned} A(k+N) &= A(k), & B(k+N) &= B(k), \\ C(k+N) &= C(k), & D(k+N) &= D(k), \quad \forall k. \end{aligned} \quad (3.25)$$

Let us denote the LTI systems $L_N(S_k(P_{ij}))$ by P_{kij}^L , and let us define P_k^L by

$$P_k^L = \left[\begin{array}{cc} P_{k11}^L & P_{k12}^L \\ P_{k21}^L & P_{k22}^L \end{array} \right] \quad (3.26)$$

In other words, a state space realization of P_k^L is given by

$$\left[\begin{array}{c|cc} \bar{A}_k & \bar{B}_{k1} & \bar{B}_{k2} \\ \hline \bar{C}_{k1} & \bar{D}_{k11} & \bar{D}_{k12} \\ \bar{C}_{k2} & \bar{D}_{k21} & \bar{D}_{k22} \end{array} \right] \quad (3.27)$$

where $\bar{A}_k$, $\bar{B}_{ki}$, $\bar{C}_{ki}$, and $\bar{D}_{kij}$ are defined by

$$\bar{A}_k = A(k+N-1) \cdots A(k), \quad (3.28)$$

$$\overline{B}_{ki} = \begin{bmatrix} A(k+N-1) \cdots A(k+1)B_i(k) & A(k+N-1) \cdots A(k+2)B_i(k+1) \\ & \dots\dots\dots B_i(k+N-1) \end{bmatrix} \quad (3.29)$$

$$\overline{C}_{ki} = \begin{bmatrix} C_i(k) \\ C_i(k+1)A(k) \\ \vdots \\ C_i(k+N-1)A(k+N-2) \cdots A(k) \end{bmatrix} \quad (3.30)$$

$$\begin{aligned} \overline{D}_{kij} &= \begin{bmatrix} D_{ij}(k) & & & 0 \\ \Psi_{ij}(k+1, k) & D_{ij}(k+1) & & \\ \vdots & \ddots & \ddots & \\ \Psi_{ij}(k+N-1, k) & \cdots & \Psi_{ij}(k+N-1, k+N-2) & D_{ij}(k+N-1) \end{bmatrix} \\ &\quad (3.31) \end{aligned}$$

$$\Psi_{ij}(l, m) = C_i(l)A(l-1) \cdots A(m+1)B_j(m). \quad (3.32)$$

Let us further define the following matrices:

$$D_X(k) = \begin{bmatrix} D_{11}(k) & D_{12}(k) \end{bmatrix}, \quad D_Y(k) = \begin{bmatrix} D_{11}^T(k) & D_{21}^T(k) \end{bmatrix}^T \quad (3.33)$$

$$\begin{aligned} R_X(k) &= D_X^T(k)D_X(k) - \begin{bmatrix} \gamma^2 I_{m_1} & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} D_{11}^T(k)D_{11}(k) - \gamma^2 I_{m_1} & D_{11}^T(k)D_{12}(k) \\ D_{12}^T(k)D_{11}(k) & D_{12}^T(k)D_{12}(k) \end{bmatrix} \end{aligned} \quad (3.34)$$

$$\begin{aligned} R_Y(k) &= D_Y(k)D_Y^T(k) - \begin{bmatrix} \gamma^2 I_{p_1} & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} D_{11}(k)D_{11}^T(k) - \gamma^2 I_{p_1} & D_{11}(k)D_{21}^T(k) \\ D_{21}(k)D_{11}^T(k) & D_{21}(k)D_{21}^T(k) \end{bmatrix} \end{aligned} \quad (3.35)$$

$$\overline{B}_k = \begin{bmatrix} \overline{B}_{k1} & \overline{B}_{k2} \end{bmatrix}, \quad \overline{C}_k = \begin{bmatrix} \overline{C}_{k1}^T & \overline{C}_{k2}^T \end{bmatrix}^T \quad (3.36)$$

$$\overline{D}_{Xk} = \begin{bmatrix} \overline{D}_{k11} & \overline{D}_{k12} \end{bmatrix}, \quad \overline{D}_Y = \begin{bmatrix} \overline{D}_{k11}^T & \overline{D}_{k21}^T \end{bmatrix}^T \quad (3.37)$$

$$\bar{R}_{Xk} = \bar{D}_{Xk}^T \bar{D}_{Xk} - \begin{bmatrix} \gamma^2 I_{Nm_1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \bar{D}_{k11}^T \bar{D}_{k11} - \gamma^2 I_{Nm_1} & \bar{D}_{k11}^T \bar{D}_{k12} \\ \bar{D}_{k12}^T \bar{D}_{k11} & \bar{D}_{k12}^T \bar{D}_{k12} \end{bmatrix} \quad (3.38)$$

$$\bar{R}_{Yk} = \bar{D}_{Yk} \bar{D}_{Yk}^T - \begin{bmatrix} \gamma^2 I_{Np_1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \bar{D}_{k11} \bar{D}_{k11}^T - \gamma^2 I_{Np_1} & \bar{D}_{k11} \bar{D}_{k21}^T \\ \bar{D}_{k21} \bar{D}_{k11}^T & \bar{D}_{k21} \bar{D}_{k21}^T \end{bmatrix}. \quad (3.39)$$

Next, we put the following assumptions on P , which correspond to Assumptions AC1–AC3 and AO1–AO3 on LTI P .

Assumption APC1

The pair $(\bar{A}_k, \bar{B}_{k2})$ is stabilizable for any k .

Assumption APC2

$\bar{D}_{k12}$ has full column rank for any k .

Assumption APC3

The matrix $\begin{bmatrix} \bar{A}_k - zI_{Nn} & \bar{B}_{k2} \\ \bar{C}_{k1} & \bar{D}_{k12} \end{bmatrix}$ has full column rank for any k and any $z \in \partial\mathcal{D}$.

Assumption APO1

The pair $(\bar{C}_{k2}, \bar{A}_k)$ is detectable for any k .

Assumption APO2

$\bar{D}_{k21}$ has full row rank for any k .

Assumption APO3

The matrix $\begin{bmatrix} \bar{A}_k - zI_{Nn} & \bar{B}_{k1} \\ \bar{C}_{k2} & \bar{D}_{k21} \end{bmatrix}$ has full row rank for any k and any $z \in \partial\mathcal{D}$.

Remark 3.2 Assumptions APC1 and APO1 are equivalent to the stabilizability and detectability of the triplet $(A(\cdot), B_2(\cdot), C_2(\cdot))$, respectively. Assumption APC2 is equivalent to the condition that $D_{12}(k)$ has full column rank for any k . Similarly, Assumption APO2 is equivalent to the condition that $D_{21}(k)$ has full row rank for any k .

Under these assumptions on P , we consider the discrete LPTV H_∞ problem.

3.1.3 Solvability condition A — based on the method in Chapter 2

First, we give the solvability condition of the discrete LPTV H_∞ problem based on Theorem 2.1.

Theorem 2.1 claims that the discrete LPTV H_∞ problem of P is solvable if and only if the discrete LTI H_∞ problem of all P_k^L ($k = 0, 1, \dots, N-1$) is solvable. Therefore, it is easy to derive the solvability condition by applying results for the LTI H_∞ problem. The only trouble is that Lemma 3.1 and 3.2, which give the solvability conditions of the discrete LTI H_∞ problem, are for systems whose throughput term from u to y is zero ($D_{22} = 0$). Since P_k^L has nonzero throughput term $\bar{D}_{k22}$, we cannot apply these lemmas directly to P_k^L . However, it can be shown by simple transformation of the system that the LTI H_∞ problem of P_k^L is solvable if and only if the LTI H_∞ problem of $P_k^{L'}$ is solvable, where $P_k^{L'}$ is defined by letting $\bar{D}_{k22}$ of P_k^L be zero. Therefore, from Lemma 3.1 and Lemma 3.2, it is direct to derive the following lemma.

Lemma 3.4 (Solvability condition via Method A) Let us assume that Assumptions APC1–APC3, APO1–APO3 hold on the system shown in Fig. 2.1. A causal, LPTV, stabilizing controller K for the generalized plant given by (3.24) that satisfies $\|\mathcal{F}_l(P, K)\| < \gamma$ exists if and only if the following conditions are satisfied.

CA For any $k = 0, 1, \dots, N-1$, the LTI H_∞ problem of P_k^L is solvable.

in other words, if and only if either Conditions CA1, CA2, and CA3, or Conditions CA1 and CA2' are satisfied:

CA1 For any $k = 0, 1, \dots, N-1$, the ARE

$$\begin{aligned}\bar{X}_k &= \bar{A}_k^T \bar{X}_k \bar{A}_k + \bar{C}_{k1}^T \bar{C}_{k1} - \bar{E}_{Xk}^T \bar{M}_{Xk}^{-1} \bar{E}_{Xk} \\ \bar{E}_{Xk} &= \bar{B}_k^T \bar{X}_k \bar{A}_k + \bar{D}_{Xk}^T \bar{C}_{k1} \\ \bar{M}_{Xk} &= \bar{B}_k^T \bar{X}_k \bar{B}_k + \bar{R}_{Xk}\end{aligned}\tag{3.40}$$

has a stabilizing solution $\bar{X}_k \geq 0$ such that $\bar{M}_{Xk}$ has a J_{Nm_1, Nm_2} -factorization.

CA2 For any $k = 0, 1, \dots, N-1$, the ARE

$$\begin{aligned}\bar{Y}_k &= \bar{A}_k \bar{Y}_k \bar{A}_k^T + \bar{B}_{k1} \bar{B}_{k1}^T - \bar{E}_{Yk} \bar{M}_{Yk}^{-1} \bar{E}_{Yk}^T \\ \bar{E}_{Yk} &= \bar{A}_k \bar{Y}_k \bar{C}_k^T + \bar{B}_{1k} \bar{D}_{Yk}^T \\ \bar{M}_{Yk}^T &= \bar{C}_k \bar{Y}_k \bar{C}_k^T + \bar{R}_{Yk}\end{aligned}\tag{3.41}$$

has a stabilizing solution $\bar{Y}_k \geq 0$ such that $\bar{M}_{Yk}$ has a J_{Np_1, Np_2} -factorization.

CA2' For any $k = 0, 1, \dots, N-1$, The ARE

$$\begin{aligned}\bar{Z}_k &= \bar{A}_{Zk} \bar{Z}_k \bar{A}_{Zk}^T + \bar{B}_{Zk1} \bar{B}_{Zk1}^T - \bar{E}_{Zk} \bar{M}_{Zk}^{-1} \bar{E}_{Zk}^T \\ \bar{E}_{Zk} &= \bar{A}_{Zk} \bar{Z}_k \bar{C}_{Zk}^T + \bar{B}_{Zk1} \bar{D}_{Zk}^T \\ \bar{M}_{Zk} &= \bar{C}_{Zk} \bar{Z}_k \bar{C}_{Zk}^T + \bar{R}_{Zk}\end{aligned}\tag{3.42}$$

has a stabilizing solution $\bar{Z}_k \geq 0$ such that $\bar{M}_{Zk}$ has a J_{Nm_2, Np_2} -factorization.

CA3 $\rho(\bar{X}_k \bar{Y}_k) < \gamma$ is satisfied for any $k = 0, 1, \dots, N-1$.

The matrices $\bar{A}_{Zk}$, $\bar{B}_{Zk1}$, $\bar{C}_{Zk}$, $\bar{D}_{Zk}$, and $\bar{R}_{Zk}$ appearing in Condition CA2', are defined by

$$\bar{A}_{Zk} = \bar{A}_k - \bar{B}_{k1} \bar{F}_{k1},\tag{3.43}$$

$$\bar{B}_{Zk1} = \bar{B}_{k1} \bar{V}_{k11}^{-1}\tag{3.44}$$

$$\bar{C}_{Zk} = \begin{bmatrix} \bar{C}_{Zk1} \\ \bar{C}_{Zk2} \end{bmatrix} = \begin{bmatrix} \bar{V}_{k22} \bar{F}_{k2} \\ \bar{C}_{k2} - \bar{D}_{k21} \bar{F}_{k1} \end{bmatrix}\tag{3.45}$$

$$\bar{D}_{Zk} = \begin{bmatrix} \bar{D}_{Zk11} \\ \bar{D}_{Zk21} \end{bmatrix} = \begin{bmatrix} \bar{V}_{k21} \bar{V}_{k11}^{-1} \\ \bar{D}_{k21} \bar{V}_{k11}^{-1} \end{bmatrix}\tag{3.46}$$

$$\bar{R}_{Zk} = \bar{D}_{Zk} \bar{D}_{Zk}^T - \begin{bmatrix} \gamma^2 I_{Nm_2} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \bar{D}_{Zk11} \bar{D}_{Zk11}^T - \gamma^2 I_{Nm_2} & \bar{D}_{Zk11} \bar{D}_{Zk21}^T \\ \bar{D}_{Zk21} \bar{D}_{Zk11}^T & \bar{D}_{Zk21} \bar{D}_{Zk21}^T \end{bmatrix}\tag{3.47}$$

where $\bar{F}_{ki}$ are given by

$$\bar{F}_k = \begin{bmatrix} \bar{F}_{k1} \\ \bar{F}_{k2} \end{bmatrix} = -\bar{M}_{Xk}^{-1} \bar{E}_{Xk}\tag{3.48}$$

and $\bar{V}_{kij}$ are arbitrary matrices ($\bar{V}_{k11}$ and $\bar{V}_{k22}$ are nonsingular) satisfying

$$\bar{M}_{Xk} = \begin{bmatrix} \bar{V}_{k11}^T & \bar{V}_{k21}^T \\ 0 & \bar{V}_{k22}^T \end{bmatrix} \begin{bmatrix} -\gamma^2 I_{Nm_1} & 0 \\ 0 & I_{Nm_2} \end{bmatrix} \begin{bmatrix} \bar{V}_{k11} & 0 \\ \bar{V}_{k21} & \bar{V}_{k22} \end{bmatrix}\tag{3.49}$$

Remark 3.3 Conditions CA2 and CA3 are derived by applying Lemma 3.1, and Conditions CA2' are derived by applying Lemma 3.2.

3.1.4 Solvability condition B — based on the method proposed by Qiu and Chen

Next, we state the solvability condition for the discrete LPTV H_∞ problem based on the method by Qiu and Chen [4, 39].

The LPTV H_∞ problem of P is solvable if and only if there exists a stabilizing controller K^L for P_0^L such that $\|\mathcal{F}_l(P_0^L, K^L)\| < \gamma$ and that it satisfies the causality constraint, i.e., its throughput term is block lower triangular. The authors in [4, 39] imposed the causality constraint on the free parameter Q in the controller parametrization and derive the solvability condition in terms of Q . Although it is a very intuitive way to cope with the causality constraint, equations in the solvability condition obtained are not so simple because the controller parametrization itself is complicated (see Lemma 3.3). The following lemma gives the solvability condition based on the method by Qiu and Chen (Method B).

Lemma 3.5 (Solvability condition via Method B) Let us assume that Assumptions APC1–APC3, APO1–APO3 hold on the system shown in Fig. 2.1. A causal, LPTV, stabilizing controller K for the generalized plant given by (3.24) that satisfies $\|\mathcal{F}_l(P, K)\| < \gamma$ exists if and only if either Conditions CB1, CB2, CB3 and CB4, or Conditions CB1, CB2' and CB4 are satisfied.

CB1 The ARE

$$\begin{aligned}\bar{X}_0 &= \bar{A}_0^T \bar{X}_0 \bar{A}_0 + \bar{C}_{01}^T \bar{C}_{01} - \bar{E}_{X0}^T \bar{M}_{X0}^{-1} \bar{E}_{X0} \\ \bar{E}_{X0} &= \bar{B}_0^T \bar{X}_0 \bar{A}_0 + \bar{D}_{X0}^T \bar{C}_{01} \\ \bar{M}_{X0} &= \bar{B}_0^T \bar{X}_0 \bar{B}_0 + \bar{R}_{X0}\end{aligned}\tag{3.50}$$

has a stabilizing solution $\bar{X}_0 \geq 0$ such that $\bar{M}_{X0}$ has a J_{Nm_1, Nm_2} -factorization.

CB2 The ARE

$$\begin{aligned}\bar{Y}_0 &= \bar{A}_0 \bar{Y}_0 \bar{A}_0^T + \bar{B}_{01} \bar{B}_{01}^T - \bar{E}_{Y0} \bar{M}_{Y0}^{-1} \bar{E}_{Y0}^T \\ \bar{E}_{Y0} &= \bar{A}_0 \bar{Y}_0 \bar{C}_0^T + \bar{B}_{01} \bar{D}_{Y0}^T \\ \bar{M}_{Y0} &= \bar{C}_0 \bar{Y}_0 \bar{C}_0^T + \bar{R}_{Y0}\end{aligned}\tag{3.51}$$

has a stabilizing solution $\bar{Y}_0 \geq 0$ such that $\bar{M}_{Y0}$ has a J_{Np_1, Np_2} -factorization.

CB2' The ARE

$$\begin{aligned}\bar{Z}_0 &= \bar{A}_{Z0} \bar{Z}_0 \bar{A}_{Z0}^T + \bar{B}_{Z01} \bar{B}_{Z01}^T - \bar{E}_{Z0} \bar{M}_{Z0}^{-1} \bar{E}_{Z0}^T \\ \bar{E}_{Z0} &= \bar{A}_{Z0} \bar{Z}_0 \bar{C}_{Z0}^T + \bar{B}_{Z01} \bar{D}_{Z0}^T \\ \bar{M}_{Z0} &= \bar{C}_{Z0} \bar{Z}_0 \bar{C}_{Z0}^T + \bar{R}_{Z0}\end{aligned}\tag{3.52}$$

has a stabilizing solution $\bar{Z}_0 \geq 0$ such that $\bar{M}_{Z_0}$ has a J_{Nm_2, Np_2} -factorization.

CB3 $\rho(\bar{X}_0 \bar{Y}_0) < \gamma$ is satisfied.

CB4 There exist a matrix D_Q such that

$$\|D_Q\| < \gamma \quad (3.53)$$

and

$$\bar{D}_K = \bar{D}_{c11} + \bar{D}_{c12} D_Q \bar{D}_{c21} \quad (3.54)$$

is block lower triangular. Here, $\bar{D}_{c11}$, $\bar{D}_{c12}$ and $\bar{D}_{c22}$ are defined by

$$\bar{D}_{c11} = -\bar{V}_{022}^{-1} \bar{W}_{012} \bar{W}_{022}^{-1}, \quad \bar{D}_{c12} = \bar{V}_{022}^{-1} \bar{W}_{011}, \quad \bar{D}_{c21} = \bar{W}_{022}^{-1} \quad (3.55)$$

where $\bar{V}_{0ij}$ are given by (3.49), and $\bar{W}_{011}$, $\bar{W}_{012}$, $\bar{W}_{022}$ are any matrices ($\bar{W}_{011}$ and $\bar{W}_{022}$ are nonsingular) satisfying

$$\bar{M}_{Z_0} = \begin{bmatrix} \bar{W}_{011} & \bar{W}_{012} \\ 0 & \bar{W}_{022} \end{bmatrix} \begin{bmatrix} -\gamma^2 I_{Nm_2} & 0 \\ 0 & I_{Np_2} \end{bmatrix} \begin{bmatrix} \bar{W}_{011}^T & 0 \\ \bar{W}_{012}^T & \bar{W}_{022}^T \end{bmatrix}. \quad (3.56)$$

Remark 3.4 As in Lemma 3.5, the LTI H_∞ problem considered in Lemma 3.5 is not for P_0^L , but for $P_0^{L'}$, which is defined by setting the throughput term D_{22}^L of P_0^L to zero. It is because there exists a stabilizing controller $K_0^{L'}$ for $P_0^{L'}$ satisfying $\|\mathcal{F}_l(P_0^{L'}, K_0^{L'})\| < \gamma$ if and only if there exists a stabilizing controller K_0^L for P_0^L satisfying $\|\mathcal{F}_l(P_0^L, K_0^L)\| < \gamma$, and,

$$K_0^L = K_0^{L'} (I + D_{22}^L K_0^{L'})^{-1}. \quad (3.57)$$

Moreover, from (3.57) the throughput term of K_0^L becomes block lower triangular if and only if the throughput term of $K_0^{L'}$ is block lower triangular.

3.1.5 Solvability condition C — based on the method proposed by Sgfors, et al.

Finally, we state the result by Sgfors, et al. [41, 42]. They treated the causality constraint as a min-max problem, and derived the following solvability condition. As it will be stated in Section 3.4, it is quite similar to the solvability condition derived via the time-varying approach.

Lemma 3.6 (Solvability condition via Method C) Let us assume that Assumptions APC1–APC3, APO1–APO3 hold on the system shown in Fig. 2.1. A causal, LPTV, stabilizing controller K for the generalized plant given by (3.24) that satisfies $\|\mathcal{F}_l(P, K)\| < \gamma$ exists if and only if the following conditions are satisfied:

CC1 The ARE (3.50) has a stabilizing solution $\bar{X}_0 \geq 0$.

CC2 The ARE (3.51) has a stabilizing solution $\bar{Y}_0 \geq 0$.

CC3 $\rho(\bar{X}_0 \bar{Y}_0) < \gamma$ is satisfied.

CC4 For any Ω_k ($k = 1, \dots, N-1$) defined recursively by

$$\begin{aligned}\Omega_k &= A^T(k)\Omega_{k+1}A(k) + C_1^T(k)C_1(k) - \Gamma_{Xk}^T \Sigma_{Xk}^{-1} \Gamma_{Xk} \\ \Gamma_{Xk} &= B^T(k)\Omega_{k+1}A(k) + D_X^T(k)C_1(k) \\ \Sigma_{Xk} &= B^T(k)\Omega_{k+1}B(k) + R_X(k) \\ \Omega_N &= \bar{X}_0\end{aligned}\tag{3.58}$$

Σ_{Xk} has a J_{m_1, m_2} -factorization.

CC5 For any Θ_k ($k = 1, 2, \dots, N-1$) defined recursively by

$$\begin{aligned}\Theta_k &= A(k-1)\Theta_{k-1}A^T(k-1) + B_1(k-1)B_1^T(k-1) - \Gamma_{Yk-1} \Sigma_{Yk-1}^{-1} \Gamma_{Yk-1}^T \\ \Gamma_{Yk-1} &= A(k-1)\Theta_{k-1}C^T(k-1) + B_1(k-1)D_Y^T(k-1) \\ \Sigma_{Yk-1} &= C(k-1)\Theta_{k-1}C^T(k-1) + R_Y(k-1) \\ \Theta_0 &= Y_0\end{aligned}\tag{3.59}$$

Σ_{Yk} has a J_{p_1, p_2} -factorization.

CC6 For any $k = 1, 2, \dots, N-1$, $\rho(\Omega_k \Theta_k) < \gamma$.

3.2 The solution to the discrete LPTV H_∞ problem — time-varying approach

In this section, we show the results that can be derived by applying the time-varying approach.

In late 1980's, some researchers developed the state space solutions to the H_∞ problem of discrete linear time-varying (LTV) systems [34, 2], but they were in the finite-horizon settings. However, from the middle of 90's, several researchers have solved the H_∞

problem of discrete LTV systems in the infinite-horizon settings [10, 27, 44]. By applying these results (mainly the results in [27] and [44]) to our problem, we can obtain the solvability condition for the discrete LPTV problem.

Although the solutions of the AREs that appear in the solvability condition in these results are not stationary ones, in the case of LPTV systems the solutions converge to stationary solutions of Periodic Riccati Equations (PREs) given in the following lemma. Their convergence can be shown directly by applying the results for the periodic Riccati equations in the continuous H_∞ problem [22], and the nondecreasing property of AREs in the discrete LTI H_∞ problem [49]. After all, the solvability condition for our problem is given by the following lemma.

Lemma 3.7 (Solvability condition via the time-varying approach) Let us assume that Assumptions APC1–APC3, APO1–APO3 hold on the system shown in Fig. 2.1. A causal, LPTV, stabilizing controller K for the generalized plant given by (3.24) that satisfies $\|\mathcal{F}_l(P, K)\| < \gamma$ exists if and only if either Conditions CV1, CV2 and CV3, or Conditions CV1 and CV2' are satisfied.

CV1 The Periodic Riccati Equation (PRE)

$$\begin{aligned} X(k) &= A^T(k)X(k+1)A(k) + C_1^T(k)C_1(k) - E_X^T(k)M_X^{-1}(k)E_X(k) \\ E_X(k) &= B^T(k)X(k+1)A(k) + D_X^T(k)C_1(k) \\ M_X(k) &= B^T(k)X(k+1)B(k) + R_X(k) \\ X(k) &= X(k+N) \quad \forall k = 0, 1, \dots \end{aligned} \tag{3.60}$$

has a stabilizing solution $X(k) \geq 0$ such that $M_X(k)$ has a J_{m_1, m_2} -factorization for any k .

CV2 The PRE

$$\begin{aligned} Y(k+1) &= A(k)Y(k)A^T(k) + B_1(k)B_1^T(k) - E_Y(k)M_Y^{-1}(k)E_Y^T(k) \\ E_Y(k) &= A(k)Y(k)C^T(k) + B_1(k)D_Y^T(k) \\ M_Y(k) &= C(k)Y(k)C^T(k) + R_Y(k) \\ Y(k) &= Y(k+N) \quad \forall k = 0, 1, \dots \end{aligned} \tag{3.61}$$

has a stabilizing solution $Y(k) \geq 0$ such that $M_Y(k)$ has a J_{p_1, p_2} -factorization for any k .

CV2' The PRE

$$\begin{aligned}
Z(k+1) &= A_Z(k)Z(k)A_Z^T(k) + B_{Z1}(k)B_{Z1}^T(k) - E_Z(k)M_Z^{-1}(k)E_Z^T(k) \\
E_Z(k) &= A_Z(k)Z(k)C_Z^T(k) + B_{Z1}(k)D_Z^T(k) \\
M_Z(k) &= C_Z(k)Z(k)C_Z^T(k) + R_Z(k) \\
Z(k) &= Z(k+N) \quad \forall k = 0, 1, \dots
\end{aligned} \tag{3.62}$$

has a stabilizing solution $Z(k) \geq 0$ such that $M_Z(k)$ has a J_{m_2, p_2} -factorization for any k .

CV3 For any $k = 0, 1, \dots, N-1$, $\rho(X(k)Y(k)) < \gamma$ holds.

A solution $X(k)$ of (3.60) is called a stabilizing solution if $A_F(N-1)A_F(N-2)\cdots A_F(0)$ is stable where $A_F(k)$ is defined by

$$A_F(k) = A(k) - B(k)M_X^{-1}(k)E_X(k). \tag{3.63}$$

The matrices $A_Z(k)$, $B_{Z1}(k)$, $C_Z(k)$, $D_Z(k)$, and $R_Z(k)$, appearing in Condition CV2', are defined by

$$A_Z(k) = A(k) - B_1(k)F_1(k) \tag{3.64}$$

$$B_{Z1}(k) = B_1(k)V_{11}^{-1}(k) \tag{3.65}$$

$$C_Z(k) = \begin{bmatrix} C_{Z1}(k) \\ C_{Z2}(k) \end{bmatrix} = \begin{bmatrix} V_{22}(k)F_2(k) \\ C_2(k) - D_{21}(k)F_1(k) \end{bmatrix} \tag{3.66}$$

$$D_Z(k) = \begin{bmatrix} D_{Z11}(k) \\ D_{Z21}(k) \end{bmatrix} = \begin{bmatrix} V_{21}(k)V_{11}^{-1}(k) \\ D_{21}(k)V_{11}^{-1}(k) \end{bmatrix} \tag{3.67}$$

$$\begin{aligned}
R_Z(k) &= D_Z(k)D_Z^T(k) - \begin{bmatrix} \gamma^2 I_{m_2} & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} D_{Z11}(k)D_{Z11}^T(k) - \gamma^2 I_{m_2} & D_{Z11}(k)D_{Z21}^T(k) \\ D_{Z21}(k)D_{Z11}^T(k) & D_{Z21}(k)D_{Z21}^T(k) \end{bmatrix}
\end{aligned} \tag{3.68}$$

where $F_i(k)$ is given by

$$F(k) = \begin{bmatrix} F_1(k) \\ F_2(k) \end{bmatrix} = -M_X^{-1}(k)E_X(k) \tag{3.69}$$

and $V_{ij}(k)$ is an arbitrary matrix ($V_{11}(k)$ and $V_{22}(k)$ are nonsingular) that satisfies

$$M_X(k) = \begin{bmatrix} V_{11}^T(k) & V_{21}^T(k) \\ 0 & V_{22}^T(k) \end{bmatrix} \begin{bmatrix} -\gamma^2 I_{m_1} & 0 \\ 0 & I_{m_2} \end{bmatrix} \begin{bmatrix} V_{11}(k) & 0 \\ V_{21}(k) & V_{22}(k) \end{bmatrix}. \tag{3.70}$$

3.3 Explicit relationships among four types of solvability conditions of LPTV H_∞ problems

In Sections 3.1 and 3.2, we showed four types of solvability conditions Lemmas 3.4–3.7 via the time-invariant and time-varying approaches. Although these conditions are necessary and sufficient conditions of the same problem, i.e., the discrete LPTV H_∞ problem, and hence they are all equivalent, the relationships among these conditions are not clear. Therefore, in this section we will show their equivalence explicitly. More specifically, we will show that three types of solvability conditions given in Lemmas 3.4–3.6 can be converted into the solvability condition given in Lemma 3.7.

From the viewpoint of how the causality constraint is treated, the condition based on Method B (Lemma 3.5) is the “best” compared with other two types of conditions because it is the most intuitive and straightforward way to cope with the causality constraint. However, from the viewpoint of how the solvability condition is related to that in Lemma 3.7, it is the “worst” condition because the condition in Lemma 3.5 differs most from that in Lemma 3.7 in the sense that it needs long and tedious calculation to prove the equivalence.

In contrast, the condition based on Method C (Lemma 3.6) has no clear relation to the causality constraint, but has a close relation to the condition in Lemma 3.7. As will be shown in the following, the equivalence can be easily shown from characteristics of AREs.

The condition via Method A (Lemma 3.4) is, so to speak, an intermediate of those of Method B and C. The relationship to the causality constraint is more explicit than Method C, but less explicit than Method B. On the other hand, the relationship to Lemma 3.7 is closer than Method B, but is not so close as Method D.

In this section we only show the explicit equivalence of these solvability conditions and in the next section we will examine how the causality constraint appears in the three types of solvability conditions based on the time-invariant approach.

3.3.1 The equivalence of Lemma 3.6 and Lemma 3.7

In this section, we show the equivalence of Lemma 3.6 and Lemma 3.7. We will only show that $\bar{X}_0$ in Condition CC1 is identical to $X(0)$ in Condition CV1 and that Ω_k ($k = 1, 2, \dots, N-1$) in Condition CC4 is identical to $X(k)$ in Condition CV1. More specifically, we will show that $\Omega_0 = \bar{X}_0$ and Ω_k ($k = 1, 2, \dots, N-1$) ≥ 0 . Then, it is a dual argument to show that $\bar{Y}_0$ in Condition CC2 is identical to $Y(0)$ in Condition CV2,

and that Θ_k ($k = 1, 2, \dots, N-1$) in Condition CC4 is identical to $Y(k)$ in Condition CV2. Moreover, it is obvious that Conditions CC3 and CC6 are identical to Condition CV3.

Here we only show that $\Omega_0 = \bar{X}_0$ and Ω_k ($k = 1, 2, \dots, N-1$) ≥ 0 under the assumption that Conditions CC1 and CC4 hold. (sufficiency of Conditions CC1 and CC4). The necessity part of the proof is direct from the argument in the proof of the sufficiency part.

Let us assume that Conditions CC1 and CC4 hold. Define $\tilde{B}_k$, $\tilde{D}_{Xk}$, and $\tilde{R}_{Xk}$ by

$$\tilde{B}_k = \begin{bmatrix} A(k+N-1) \cdots A(k+1)B(k) & A(k+N-1) \cdots A(k+2)B(k+1) \\ \cdots & B(k+N-1) \end{bmatrix} \quad (3.71)$$

$$\tilde{D}_{Xk} = \begin{bmatrix} D_X(k) & & & 0 \\ \tilde{\Psi}(k+1, k) & D_X(k+1) & & \\ \vdots & \ddots & \ddots & \\ \tilde{\Psi}(k+N-1, k) & \cdots & \tilde{\Psi}(k+N-1, k+N-2) & D_X(k+N-1) \end{bmatrix} \quad (3.72)$$

$$\tilde{\Psi}(l, m) = C_1(l)A(l-1) \cdots A(m+1)B(m) \quad (3.73)$$

$$\tilde{R}_{Xk} = \tilde{D}_{Xk}^T \tilde{D}_{Xk} - \text{diag} \left(\overbrace{\left[\begin{bmatrix} \gamma^2 I_{m_1} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \gamma^2 I_{m_1} & 0 \\ 0 & 0 \end{bmatrix}, \dots, \begin{bmatrix} \gamma^2 I_{m_1} & 0 \\ 0 & 0 \end{bmatrix} \right]}^N \right). \quad (3.74)$$

Since $\tilde{B}_k$ and $\tilde{D}_{Xk}$ are obtained by appropriately permuting the rows of $\bar{B}_k$ and $\bar{D}_{Xk}$, respectively, and $\tilde{R}_{Xk}$ is obtained by permuting the columns and rows according to the permutation of the rows of $\tilde{B}_k$ and $\tilde{D}_{Xk}$, (3.50) can be rewritten as

$$\begin{aligned} \bar{X}_0 &= \bar{A}_0^T \bar{X}_0 \bar{A}_0 + \bar{C}_{01}^T \bar{C}_{01} - \tilde{E}_{X0}^T \tilde{M}_{X0}^{-1} \tilde{E}_{X0} \\ \tilde{E}_{X0} &= \tilde{B}_0^T \bar{X}_0 \bar{A}_0 + \tilde{D}_{X0}^T \bar{C}_{01} \\ \tilde{M}_{X0} &= \tilde{B}_0^T \bar{X}_0 \tilde{B}_0 + \tilde{R}_{X0}. \end{aligned} \quad (3.75)$$

Because Σ_{Xk} ($k = 1, \dots, N-1$) has a J_{m_1, m_2} -factorization, and therefore is nonsingular, $\tilde{M}_{X0}$ can be factorized as

$$\tilde{M}_{X0} = \begin{bmatrix} I & & * \\ & I & \\ & & \ddots \\ 0 & & & I \end{bmatrix} \begin{bmatrix} \Sigma_{X0} & & 0 \\ & \Sigma_{X1} & \\ & & \ddots \\ 0 & & & \Sigma_{XN-1} \end{bmatrix} \begin{bmatrix} I & & 0 \\ & I & \\ & & \ddots \\ * & & & I \end{bmatrix} \quad (3.76)$$

where Σ_{X0} is defined by (3.58) with k set to zero. From (3.76) and nonsingularity of $\widetilde{M}_{X0}$ and Σ_{Xk} ($k = 1, \dots, N-1$), Σ_{X0} becomes nonsingular.

If, for example, $N = 2$, $\widetilde{M}_{X0}$ can be factorized as

$$\widetilde{M}_{X0} = \begin{bmatrix} I & B^T(0)\Gamma_{X1}^T\Sigma_{X1}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} \Sigma_{X0} & 0 \\ 0 & \Sigma_{X1} \end{bmatrix} \begin{bmatrix} I & 0 \\ \Sigma_{X1}^{-1}\Gamma_{X1}B(0) & I \end{bmatrix}. \quad (3.77)$$

Moreover, $\widetilde{E}_{X0}$ can be rewritten by

$$\widetilde{E}_{X0} = \begin{bmatrix} B^T(0)(\Omega_1 + C_1(1)^T C_1(1))A(0) + \widetilde{D}_X^T(0)C_1(0) \\ \Gamma_{X1}A(0) \end{bmatrix}. \quad (3.78)$$

Therefore, $\widetilde{E}_{X0}^T \widetilde{M}_{X0}^{-1} \widetilde{E}_{X0}$ becomes

$$\begin{aligned} \widetilde{E}_{X0}^T \widetilde{M}_{X0}^{-1} \widetilde{E}_{X0} &= \begin{bmatrix} A^T(0)(\Omega_1 + C_1(1)^T C_1(1))B(0) + C_1^T(0)D_X(0) & A^T(0)\Gamma_{X1} \end{bmatrix} \\ &\quad \times \begin{bmatrix} I & 0 \\ -\Sigma_{X1}^{-1}\Gamma_{X1}B(0) & I \end{bmatrix} \begin{bmatrix} \Sigma_{X0}^{-1} & 0 \\ 0 & \Sigma_{X1}^{-1} \end{bmatrix} \begin{bmatrix} I & B^T(0)\Gamma_{X1}^T\Sigma_{X1}^{-1} \\ 0 & I \end{bmatrix} \\ &\quad \times \begin{bmatrix} B^T(0)(\Omega_1 + C_1(1)^T C_1(1))A(0) + D_X^T(0)C_1(0) \\ \Gamma_{X1}A(0) \end{bmatrix} \\ &= \begin{bmatrix} \Gamma_{X0}^T & A^T(0)\Gamma_{X1}^T \end{bmatrix} \begin{bmatrix} \Sigma_{X0}^{-1} & 0 \\ 0 & \Sigma_{X1}^{-1} \end{bmatrix} \begin{bmatrix} \Gamma_{X0} \\ \Gamma_{X1}A(0) \end{bmatrix} \\ &= \Gamma_{X0}^T \Sigma_{X0}^{-1} \Gamma_{X0} + A^T(0)\Gamma_{X1}^T \Sigma_{X1}^{-1} \Gamma_{X1}A(0) \\ &= \Gamma_{X0}^T \Sigma_{X0}^{-1} \Gamma_{X0} - A^T(0)\Omega_1 A(0) + \overline{A}_0^T X_0 \overline{A}_0 + A^T(0)C_1^T(1)C_1(1)A(0) \end{aligned} \quad (3.79)$$

where Γ_{X0} is defined by (3.58) with k set to zero. From (3.75) and (3.79) it is direct to show that

$$\overline{X}_0 = A^T(0)\Omega_1 A(0) + C_1^T(0)C_1(0) - \Gamma_{X0}^T \Sigma_{X0}^{-1} \Gamma_{X0}. \quad (3.80)$$

Thus, the set $\{\overline{X}_0, \Omega_1\}$ becomes a solution of the PRE (3.60) in Condition CV1. Moreover, this set is a stabilizing solution of the PRE (3.60). It is because $\overline{X}_0$ is a stabilizing solution of the ARE (3.50) and hence

$$\begin{aligned} \overline{A}_{F0} &= \overline{A}_0 - \overline{B}_0 \overline{M}_{X0}^{-1} \overline{E}_{X0} \\ &= \overline{A}_0 - \widetilde{B}_0 \widetilde{M}_{X0}^{-1} \widetilde{E}_{X0} \\ &= A(1)A(0) - \begin{bmatrix} A(1)B(0) - B(1)\Sigma_{X1}^{-1}\Gamma_{X1}B(0) & B_1 \end{bmatrix} \begin{bmatrix} \Sigma_{X0}^{-1}\Gamma_{X0} \\ \Sigma_{X1}^{-1}\Gamma_{X1}A(0) \end{bmatrix} \\ &= (A(1) - B(1)\Sigma_{X1}^{-1}\Gamma_{X1})(A(0) - B(0)\Sigma_{X0}^{-1}\Gamma_{X0}) \\ &= A'_F(1)A'_F(0) \end{aligned} \quad (3.81)$$

$$A'_F(k) = A(k) - B(k)\Sigma_{Xk}^{-1}\Gamma_{Xk} \quad (3.82)$$

is stable.

Although (3.80) and (3.81) are derived for $N = 2$, they are also true for $N \geq 3$ (it takes much more calculation, of course). Therefore, the proof is complete if we show that $\Omega_k \geq 0$ holds for any $k = 1, 2, \dots, N - 1$. This can be done by a similar argument taken in [49].

Let Γ_{Xk} and Σ_{Xk} in (3.58) be partitioned into

$$\Gamma_{Xk} = \begin{bmatrix} \Gamma_{Xk1} \\ \Gamma_{Xk2} \end{bmatrix} = \begin{bmatrix} B_1^T(k)\Omega_{k+1}A(k) + D_{11}^T(k)C_1(k) \\ B_2^T(k)\Omega_{k+1}A(k) + D_{12}^T(k)C_1(k) \end{bmatrix} \quad (3.83)$$

$$\begin{aligned} \Sigma_{Xk} &= \begin{bmatrix} \Sigma_{Xk1} & \Sigma_{Xk2} \\ \Sigma_{Xk2}^T & \Sigma_{Xk3} \end{bmatrix} \\ &= \begin{bmatrix} B_1^T(k)\Omega_{k+1}B_1(k) + D_{11}^T(k)D_{11}(k) - \gamma^2 I & B_1^T(k)\Omega_{k+1}B_2(k) + D_{11}^T(k)D_{12}(k) \\ B_2^T(k)\Omega_{k+1}B_1(k) + D_{12}^T(k)D_{11}(k) & B_2^T(k)\Omega_{k+1}B_2(k) + D_{12}^T(k)D_{12}(k) \end{bmatrix}. \end{aligned} \quad (3.84)$$

Define $A_U(k)$, $C_U(k)$, Γ_{Uk} , Σ_{Uk} by

$$A_U(k) = A(k) - B_2(k)\Sigma_{Xk3}^{-1}\Gamma_{Xk2} \quad (3.85)$$

$$C_U(k) = C_1(k) - D_{12}(k)\Sigma_{Xk3}^{-1}\Gamma_{Xk2} \quad (3.86)$$

$$\Gamma_{Uk} = \Gamma_{Xk1} - \Sigma_{Xk2}\Sigma_{Xk3}^{-1}\Gamma_{Xk2} \quad (3.87)$$

$$\Sigma_{Uk} = -\Sigma_{Xk1} + \Sigma_{Xk2}\Sigma_{Xk3}^{-1}\Sigma_{Xk2}^T. \quad (3.88)$$

Note that Σ_{Uk} becomes positive definite since Σ_{Xk} has a J_{m_1, m_2} -factorization. By using (3.85)–(3.88), Ω_k can be rewritten as

$$\Omega_k = A_U^T(k)\Omega_{k+1}A_U(k) + C_U^T(k)C_U(k) + \Gamma_{Uk}^T\Sigma_{Uk}^{-1}\Gamma_{Uk}. \quad (3.89)$$

Because $\Omega_N = \bar{X}_0$ is semi-positive definite from Condition CC1, Ω_k is also semi-positive definite for any $k = 1, 2, \dots, N - 1$.

3.3.2 The equivalence of Lemma 3.4 and Lemma 3.7

Next, we show the equivalence of Lemma 3.4 and Lemma 3.7. We only show the equivalence of Condition CA1 and Condition CV1.

First, we show that under Condition CA1 and an additional condition, Condition CV1 is satisfied. Let us assume that Condition CA1 holds and that $M'_X(k)$ defined by

$$M'_X(k) = B^T(k)\bar{X}_{k+1}B(k) + R_X(k) \quad (3.90)$$

is nonsingular for any $k = 0, 1, \dots, N-1$. Then, we can define $\bar{X}'_k$ ($k = 0, 1, \dots, N-1$) by

$$\begin{aligned} \bar{X}'_k &= A^T(k)\bar{X}_{k+1}A(k) + C_1^T(k)C_1(k) - E_X'^T(k)M_X'^{-1}(k)E_X'(k) \\ E_X'(k) &= B^T(k)\bar{X}_{k+1}A(k) + D_X^T(k)C_1(k), \end{aligned} \quad (3.91)$$

and $\bar{X}'_k$ becomes a solution of the ARE (3.40). If we define $\bar{E}'_{Xk}$ and $\bar{M}'_{Xk}$ by

$$\bar{E}'_{Xk} = \bar{B}_k^T \bar{X}'_k \bar{A}_k + \bar{D}_{Xk}^T \bar{C}_{k1} \quad (3.92)$$

$$\bar{M}'_{Xk} = \bar{B}_k^T \bar{X}'_k \bar{B}_k + \bar{R}_{Xk} \quad (3.93)$$

the matrices $(\bar{A}_{k+1} - \bar{B}_{k+1}\bar{M}_{k+1}^{-1}\bar{E}_{k+1})$ and $(\bar{A}_k - \bar{B}_k\bar{M}_k'^{-1}\bar{E}'_k)$ can be decomposed as (also refer to (3.81))

$$\bar{A}_{k+1} - \bar{B}_{k+1}\bar{M}_{k+1}^{-1}\bar{E}_{k+1} = (A(k) - B(k)\bar{M}'(k)\bar{E}'_k)\bar{A}_{Pk} \quad (3.94)$$

$$\bar{A}_k - \bar{B}_k\bar{M}_k'^{-1}\bar{E}'_k = \bar{A}_{Pk}(A(k) - B(k)\bar{M}'^{-1}(k)\bar{E}'(k)) \quad (3.95)$$

where $\bar{A}_{Pk}$ is some matrix that depends on k . Since the matrix $(\bar{A}_{k+1} - \bar{B}_{k+1}\bar{M}_{k+1}^{-1}\bar{E}_{k+1})$ is a stable matrix from Condition CA1, $(\bar{A}_k - \bar{B}_k\bar{M}_k'^{-1}\bar{E}'_k)$ is also a stable matrix. This means that $\bar{X}'_k$ is a stabilizing solution of the ARE (3.40), and from uniqueness of the stabilizing solution of the ARE [49],

$$\bar{X}'_k = \bar{X}_k \geq 0 \quad (3.96)$$

holds. Furthermore, $\bar{X}_0$ satisfies the rewritten ARE (3.75) and $\tilde{M}_{X0}$ can be factorized as

$$\tilde{M}_{X0} = \begin{bmatrix} I & & * \\ & I & \\ & & \ddots \\ 0 & & I \end{bmatrix} \begin{bmatrix} M'_X(0) & & 0 \\ & M'_X(1) & \\ & & \ddots \\ 0 & & M'_X(N-1) \end{bmatrix} \begin{bmatrix} I & & 0 \\ & I & \\ & & \ddots \\ * & & I \end{bmatrix} \quad (3.97)$$

by simple calculation (as is already shown in (3.76)). From the fact that $M'_X(k)$ can be partitioned into

$$\begin{aligned} M'_X(k) &= \begin{bmatrix} B_1^T(k)\bar{X}'_{k+1}B_1(k) + D_{11}^T(k)D_{11}(k) - \gamma^2 I & B_1^T(k)\bar{X}'_{k+1}B_2(k) + D_{11}^T(k)D_{12}(k) \\ B_2^T(k)\bar{X}'_{k+1}B_1(k) + D_{12}^T(k)D_{11}(k) & B_2^T(k)\bar{X}'_{k+1}B_2(k) + D_{12}^T(k)D_{12}(k) \end{bmatrix} \end{aligned} \quad (3.98)$$

and from (3.96), we can see that $M'_X(k)$ has at least m_2 positive eigenvalues. On the other hand, the number of positive eigenvalues of $\bar{M}_{X0}$, which has a J_{Nm_1, Nm_2} -factorization from Condition CA1, is Nm_2 . Therefore, from (3.97), $M'_X(k)$ has just m_2 positive eigenvalues and has a J_{m_1, m_2} -factorization. From (3.91), (3.96), and the fact that $M'_X(k)$ has a J_{m_1, m_2} -factorization, we can conclude that Condition CV1 is satisfied ($X(k) = \bar{X}_k$).

It has been shown that under Condition CA1 and the additional condition that $M'_X(k)$ is nonsingular for any k , Condition CV1 is satisfied. Because necessity of these conditions for Condition CV1 is obvious, we can say that Condition CA1 together with the condition that $M'_X(k)$ is nonsingular for any k is equivalent to Condition CV1.

Next, we show that $M'_X(k)$ is nonsingular if Condition CA1 holds. Let us assume that Condition CA1 holds and $M'_X(k)$ is singular for at least one k . Condition CA1 can be regarded as the solvability condition of LTI H_∞ Full-Information (FI) problems. Therefore, from sub-optimality of the H_∞ problem there exists $\epsilon > 0$ such that Condition CA1 holds for any γ' satisfying $\gamma - \epsilon < \gamma' \leq \gamma$.

Now, let $\bar{X}_k^0$ be the stabilizing solution of the ARE

$$\begin{aligned} \bar{X}_k^0 &= \bar{A}_k^T \bar{X}_k^0 \bar{A}_k + \bar{C}_{k1}^T \bar{C}_{k1} \\ &\quad - (\bar{A}_k^T \bar{X}_k^0 \bar{B}_{k2} + \bar{C}_{k1}^T \bar{D}_{k12})(\bar{B}_{k2}^T \bar{X}_k^0 \bar{B}_{k2} + I)^{-1}(\bar{B}_{k2}^T \bar{X}_k^0 \bar{A}_k + \bar{D}_{k12}^T \bar{C}_{k1}). \end{aligned} \quad (3.99)$$

Then, from [23], $\lim_{\gamma \rightarrow \infty} \bar{X}_k = \bar{X}_k^0$, and

$$\lim_{\gamma \rightarrow \infty} \begin{bmatrix} \gamma^{-1} & 0 \\ 0 & I \end{bmatrix} M'_X(k) \begin{bmatrix} \gamma^{-1} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} -I & 0 \\ 0 & B_2(k)^T \bar{X}_{k+1}^0 B_2(k) + D_{12}^T(k)D_{12}(k) \end{bmatrix} \quad (3.100)$$

holds. Since the righthand side of (3.100) is nonsingular, determinant of $M'_X(k)$ is not identically zero regardless of γ . Therefore, from the fact that a solution of an ARE is analytic [23, 38], and determinant of $M'_X(k)$ is also analytic, there exists γ'' ($\gamma' < \gamma'' < \gamma$) such that $M'_X(k)$ is nonsingular for any k . Hence Condition CV1 holds for γ'' . However,

again from the characteristic of the H_∞ problem, for any $\gamma \geq \gamma''$ the LPTV H_∞ FI problem is solvable and Condition CV1 holds. This implies that Condition CA1 and the condition that $M'_X(k)$ is nonsingular for any k hold. It contradicts the assumption.

3.3.3 The equivalence of Lemma 3.5 and Lemma 3.7

Last, we show the equivalence of Lemma 3.5 and Lemma 3.7. Although it requires long and tedious calculation, it is not theoretically difficult. We first treat the case of $N = 2$, the 2-periodic case, and then extend it to the general case.

In the case of $N = 2$, we will show the equivalence of the following four conditions:

- (1) Conditions CB1, CB2' and CB4 are satisfied.
- (2) Conditions CV1, CB2' and CB4 are satisfied.
- (3) Conditions CV1 and CB2' are satisfied, and, additionally, $M'_Z(0) = C_Z(0)\bar{Z}_0C_Z(0)^T + R_Z(0)$ has a J_{m_2,p_2} -factorization.
- (4) Conditions CV1 and CV2' are satisfied.

Thus the proof consists of three steps. In the following, we show the proof step by step. In each step, we only show the sufficiency part of the proof because the necessity part is obvious from the sufficiency part of the proof and so is omitted.

First step

First, we show the equivalence of (1) and (2).

Let us assume that (1) is satisfied. The matrix $\bar{D}_K$ in Condition CB4 can be rewritten by

$$\begin{aligned}\bar{D}_K &= \bar{D}_{c11} + \bar{D}_{c12}\bar{D}_Q\bar{D}_{c21} \\ &= -\bar{V}_{022}^{-1}\bar{W}_{012}\bar{W}_{022}^{-1} + \bar{V}_{022}^{-1}\bar{W}_{011}\bar{D}_Q\bar{W}_{022}^{-1} \\ &= \bar{V}_{022}^{-1}\bar{W}_{011}(\bar{D}_Q - \bar{W}_{011}^{-1}\bar{W}_{012})\bar{W}_{022}^{-1}.\end{aligned}\tag{3.101}$$

If we choose $\bar{V}_{022}$, $\bar{W}_{011}$, and $\bar{W}_{022}$ as block lower triangular (it is always possible), the condition that $\bar{D}_K$ is block lower triangular is equivalent to the condition that the matrix $\bar{D}'_K$ defined by

$$\bar{D}'_K = \bar{D}_Q - \bar{W}_{011}^{-1}\bar{W}_{012}\tag{3.102}$$

is block lower triangular.

Let us partition $\overline{W}_{011}$, $\overline{W}_{012}$, and $\overline{W}_{022}$ into

$$\overline{W}_{011} = \begin{bmatrix} S_{11} & 0 \\ S_{21} & S_{22} \end{bmatrix}, \quad \overline{W}_{012} = \begin{bmatrix} S_{13} & S_{14} \\ S_{23} & S_{24} \end{bmatrix}, \quad \overline{W}_{022} = \begin{bmatrix} S_{33} & 0 \\ S_{43} & S_{44} \end{bmatrix}. \quad (3.103)$$

Then, $\overline{W}_{011}^{-1} \overline{W}_{012}$ becomes

$$\overline{W}_{011}^{-1} \overline{W}_{012} = \begin{bmatrix} * & S_{11}^{-1} S_{14} \\ * & * \end{bmatrix}. \quad (3.104)$$

Thus, Condition CL4 is equivalent to

$$\|S_{11}^{-1} S_{14}\| < \gamma \quad (3.105)$$

or

$$\gamma^2 S_{11} S_{11}^T - S_{14} S_{14}^T > 0. \quad (3.106)$$

Now, let us define $\widetilde{M}_{Z01}$, $\widetilde{M}_{Z02}$, $\widetilde{M}_{Z03}$ and $\widetilde{M}_{Z0}$ by

$$\begin{bmatrix} \widetilde{M}_{Z01} & \widetilde{M}_{Z02} \\ \widetilde{M}_{Z02}^T & \widetilde{M}_{Z03} \end{bmatrix} = \widetilde{M}_{Z0} = U_Z \overline{M}_{Z0} U_Z^T \quad (3.107)$$

where U_Z is defined by

$$U_Z = \begin{bmatrix} I_{m_2} & 0 & 0 & 0 \\ 0 & 0 & I_{p_2} & 0 \\ 0 & I_{m_2} & 0 & 0 \\ 0 & 0 & 0 & I_{p_2} \end{bmatrix}. \quad (3.108)$$

Since the matrix $\overline{M}_{Z0}$ can be expressed by

$$\overline{M}_{Z0} = \begin{bmatrix} \overline{M}_{Z01} & \overline{M}_{Z02} \\ \overline{M}_{Z02}^T & \overline{M}_{Z03} \end{bmatrix} \quad (3.109)$$

$$\begin{aligned} \overline{M}_{Z01} &= -\gamma^2 \overline{W}_{011} \overline{W}_{011}^T + \overline{W}_{012} \overline{W}_{012}^T \\ &= \begin{bmatrix} -\gamma^2 S_{11} S_{11}^T + S_{13} S_{13}^T + S_{14} S_{14}^T & -\gamma^2 S_{11} S_{21}^T + S_{13} S_{23}^T + S_{14} S_{24}^T \\ -\gamma^2 S_{21}^T S_{11} + S_{23} S_{13}^T + S_{24} S_{14}^T & -\gamma^2 S_{21} S_{21}^T - \gamma^2 S_{22} S_{22}^T + S_{23} S_{23}^T + S_{24} S_{24}^T \end{bmatrix} \end{aligned} \quad (3.110)$$

$$\begin{aligned}\overline{M}_{Z02} &= \overline{W}_{012} \overline{W}_{022}^T \\ &= \begin{bmatrix} S_{13}S_{33}^T & S_{13}S_{43}^T + S_{14}S_{44}^T \\ S_{23}S_{33}^T & S_{23}S_{43}^T + S_{24}S_{44}^T \end{bmatrix}\end{aligned}\quad (3.111)$$

$$\begin{aligned}\overline{M}_{Z03} &= \overline{W}_{022} \overline{W}_{022}^T \\ &= \begin{bmatrix} S_{33}S_{33}^T & S_{33}S_{43}^T \\ S_{43}S_{33}^T & S_{43}S_{43}^T + S_{44}S_{44}^T \end{bmatrix}\end{aligned}\quad (3.112)$$

we obtain

$$\widetilde{M}_{Z01} = \begin{bmatrix} -\gamma^2 S_{11}S_{11}^T + S_{13}S_{13}^T + S_{14}S_{14}^T & S_{13}S_{33}^T \\ S_{33}S_{13}^T & S_{33}S_{33}^T \end{bmatrix}. \quad (3.113)$$

Therefore, (3.106) can be reduced to the condition that $\widetilde{M}_{Z01}$ has a J_{m_2, p_2} -factorization.

Let us partition $\overline{V}_{0ij}$ ($\overline{V}_{011}$ and $\overline{V}_{022}$ are chosen as block lower triangular) into

$$\overline{V}_{011} = \begin{bmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{bmatrix}, \quad \overline{V}_{21} = \begin{bmatrix} T_{31} & T_{32} \\ T_{41} & T_{42} \end{bmatrix}, \quad \overline{V}_{022} = \begin{bmatrix} T_{33} & 0 \\ T_{43} & T_{44} \end{bmatrix} \quad (3.114)$$

and define $\overline{F}_{0ij}$ by

$$\overline{F}_{01} = \begin{bmatrix} \overline{F}_{011} \\ \overline{F}_{012} \end{bmatrix} \}_{m_1}, \quad \overline{F}_{02} = \begin{bmatrix} \overline{F}_{021} \\ \overline{F}_{022} \end{bmatrix} \}_{m_2}. \quad (3.115)$$

Then, $\overline{C}_{Z01}$, $\overline{C}_{Z02}$, $\overline{D}_{Z011}$, and $\overline{D}_{Z021}$ can be expressed by

$$\begin{aligned}\overline{C}_{Z01} &= \overline{V}_{022} \overline{F}_{02} = \begin{bmatrix} T_{33} & 0 \\ T_{43} & T_{44} \end{bmatrix} \begin{bmatrix} \overline{F}_{021} \\ \overline{F}_{022} \end{bmatrix} \\ &= \begin{bmatrix} T_{33} \overline{F}_{021} \\ T_{43} \overline{F}_{021} + T_{44} \overline{F}_{022} \end{bmatrix}\end{aligned}\quad (3.116)$$

$$\begin{aligned}\overline{C}_{Z02} &= \overline{C}_{02} - \overline{D}_{021} \overline{F}_{011} = \begin{bmatrix} C_1(0) \\ C_2(1)A(0) \end{bmatrix} - \begin{bmatrix} D_{21}(0) & 0 \\ C_2(1)B_1(0) & D_{21}(1) \end{bmatrix} \begin{bmatrix} \overline{F}_{011} \\ \overline{F}_{012} \end{bmatrix} \\ &= \begin{bmatrix} C_2(0) - D_{21}(0) \overline{F}_{011} \\ C_2(1)(A(0) - B_1(0) \overline{F}_{011}) - D_{21}(1) \overline{F}_{012} \end{bmatrix}\end{aligned}\quad (3.117)$$

$$\begin{aligned}\overline{D}_{Z011} &= \overline{V}_{021} \overline{V}_{011}^{-1} = \begin{bmatrix} T_{31} & T_{32} \\ T_{41} & T_{42} \end{bmatrix} \begin{bmatrix} T_{11}^{-1} & 0 \\ T_{22}^{-1} T_{21} T_{11}^{-1} & T_{22}^{-1} \end{bmatrix} \\ &= \begin{bmatrix} (T_{31} - T_{32} T_{22}^{-1} T_{21}) T_{11}^{-1} & T_{32} T_{22}^{-1} \\ (T_{41} - T_{42} T_{22}^{-1} T_{21}) T_{11}^{-1} & T_{42} T_{22}^{-1} \end{bmatrix}\end{aligned}\quad (3.118)$$

$$\begin{aligned}
\bar{D}_{Z021} &= \bar{D}_{021} \bar{V}_{011}^{-1} = \begin{bmatrix} D_{21}(0) & 0 \\ C_2(1)B_1(0) & D_{21}(1) \end{bmatrix} \begin{bmatrix} T_{11}^{-1} & 0 \\ T_{22}^{-1}T_{21}T_{11}^{-1} & T_{22}^{-1} \end{bmatrix} \\
&= \begin{bmatrix} D_{21}(0)T_{11}^{-1} & 0 \\ (C_2(1)B_1(0) - D_{21}(1)T_{22}^{-1}T_{21})T_{11}^{-1} & D_{21}(1)T_{22}^{-1} \end{bmatrix}.
\end{aligned} \tag{3.119}$$

Since

$$\begin{bmatrix} \widetilde{M}_{Z01} & \widetilde{M}_{Z02} \\ \widetilde{M}_{Z02}^T & \widetilde{M}_{Z03} \end{bmatrix} = U_Z \left(\bar{C}_{Z0} \bar{Z}_0 \bar{C}_{Z0}^T + \bar{D}_{Z0} \bar{D}_{Z0}^T - \begin{bmatrix} \gamma^2 I_{Nm_2} & 0 \\ 0 & 0 \end{bmatrix} \right) U_Z^T \tag{3.120}$$

we obtain from (3.116)–(3.119)

$$\begin{aligned}
\widetilde{M}_{Z01} &= \begin{bmatrix} T_{33} \bar{F}_{021} \\ C_2(0) - D_{21}(0) \bar{F}_{011} \end{bmatrix} \bar{Z}_0 \begin{bmatrix} \bar{F}_{021}^T T_{33}^T & C_2^T(0) - D_{21}^T(0) \bar{F}_{011}^T \end{bmatrix} \\
&\quad + \begin{bmatrix} T_{31} - T_{32} T_{22}^{-1} T_{21} \\ D_{21}(0) \end{bmatrix} (T_{11}^T T_{11})^{-1} \begin{bmatrix} T_{31}^T - T_{21}^T T_{22}^{-T} T_{32}^T & D_{21}^T(0) \end{bmatrix} \\
&\quad - \begin{bmatrix} \gamma^2 I - T_{32} (T_{22}^T T_{22})^{-1} T_{32}^T & 0 \\ 0 & 0 \end{bmatrix}.
\end{aligned} \tag{3.121}$$

Let us define H by

$$\begin{aligned}
H &= \begin{bmatrix} H_1 & H_2^T \\ H_2 & H_3 \end{bmatrix} \\
&= \begin{bmatrix} T_{33} \bar{F}_{021} \\ C_2(0) - D_{21}(0) \bar{F}_{011} \end{bmatrix} \bar{Z}_0 \begin{bmatrix} \bar{F}_{021}^T T_{33}^T & C_2^T(0) - D_{21}^T(0) \bar{F}_{011}^T \end{bmatrix} \\
&\quad + \begin{bmatrix} T_{31} - T_{32} T_{22}^{-1} T_{21} \\ D_{21}(0) \end{bmatrix} (T_{11}^T T_{11})^{-1} \begin{bmatrix} T_{31}^T - T_{21}^T T_{22}^{-T} T_{32}^T & D_{21}^T(0) \end{bmatrix}
\end{aligned} \tag{3.122}$$

then, $\widetilde{M}_{Z01}$ can be written by

$$\widetilde{M}_{Z01} = \begin{bmatrix} H_1 & H_2^T \\ H_2 & H_3 \end{bmatrix} - \begin{bmatrix} \gamma^2 I - T_{32} (T_{22}^T T_{22})^{-1} T_{32}^T & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.123}$$

It is clear from (3.122) that $H \geq 0$. Furthermore, $H_3 > 0$ holds since $D_{21}(0)$ has full row rank from Assumption APO2 (also refer to Remark 3.2). Therefore,

$$H_1 - H_2^T H_3^{-1} H_2 \geq 0 \tag{3.124}$$

holds. However, since $\widetilde{M}_{Z01}$ should have a J_{m_2, p_2} -factorization from Condition CB2',

$$H_1 - H_2^T H_3^{-1} H_2 - (\gamma^2 I - T_{32}(T_{22}^T T_{22})^{-1} T_{32}^T) < 0 \quad (3.125)$$

should hold. Consequently,

$$\gamma^2 I - T_{32}(T_{22}^T T_{22})^{-1} T_{32}^T > 0 \quad (3.126)$$

should be satisfied, or, equivalently

$$\gamma^2 T_{22}^T T_{22} - T_{32}^T T_{32} > 0 \quad (3.127)$$

should be satisfied.

We are to prove that Condition CB1 and (3.127) imply Condition CV1. As the proof of the equivalence of Lemma 3.6 and Lemma 3.7, we rewrite the ARE (3.50) as (3.75). By using the matrix U_X defined by

$$U_X = \begin{bmatrix} I_{m_1} & 0 & 0 & 0 \\ 0 & 0 & I_{m_2} & 0 \\ 0 & I_{m_1} & 0 & 0 \\ 0 & 0 & 0 & I_{m_2} \end{bmatrix} \quad (3.128)$$

$\widetilde{M}_{X0}$ in (3.75) can be described by

$$\widetilde{M}_{X0} = U_X^T \overline{M}_{X0} U_X. \quad (3.129)$$

Moreover, $\widetilde{M}_{X0}$ can be partitioned into

$$\widetilde{M}_{X0} = \begin{bmatrix} * & * \\ * & M_X''(1) \end{bmatrix} \quad (3.130)$$

where $M_X''(1)$ is defined by

$$M_X''(1) = B^T(1) \overline{X}_0 B(1) + R_X(1). \quad (3.131)$$

From (3.130), (3.49) and (3.114), $M_X''(1)$ can be expressed by

$$M_X''(1) = \begin{bmatrix} -\gamma^2 T_{22}^T T_{22} + T_{32}^T T_{32} + T_{42}^T T_{42} & T_{42}^T T_{44} \\ T_{44}^T T_{42} & T_{44}^T T_{44} \end{bmatrix}. \quad (3.132)$$

Since $T_{44}^T T_{44}$ is positive definite, and, from (3.127)

$$-\gamma^2 T_{22}^T T_{22} + T_{32}^T T_{32} + T_{42}^T T_{42} - T_{42}^T T_{44} (T_{44}^T T_{44})^{-1} T_{44}^T T_{42} = -\gamma^2 T_{22}^T T_{22} + T_{32}^T T_{32} < 0 \quad (3.133)$$

holds, $M_X''(1)$ has a J_{m_1, m_2} -factorization. Therefore, we can define $\overline{X}_1''$ by

$$\begin{aligned} \overline{X}_1'' &= A^T(1) \overline{X}_0 A(1) + C_1^T(1) C_1(1) - E_X''^T(1) M_X''^{-1}(1) E_X''(1) \\ E_X''(1) &= B^T(1) \overline{X}_0 A(1) + D_X^T(1) C_1(1). \end{aligned} \quad (3.134)$$

Since (3.134) is no other than (3.58) in Condition CC4, we can conclude that Condition CV1 is satisfied from the proof of the equivalence of Lemma 3.6 and Lemma 3.7.

Second step

Next, we show the equivalence of (2) and (3).

Let us assume that (2) is satisfied. Since $M_X(0)$ and $M_X(1)$ have J_{m_1, m_2} -factorizations, and therefore they are nonsingular, $\bar{F}_{01}$ and $\bar{F}_{02}$ defined by (3.48) can be expressed by

$$\begin{aligned}
 \begin{bmatrix} \bar{F}_{01} \\ \bar{F}_{02} \end{bmatrix} &= \bar{M}_{X0}^{-1} \bar{E}_{X0} \\
 &= U_X \widetilde{M}_{X0}^{-1} \widetilde{E}_{X0} \\
 &= U_X \begin{bmatrix} I & 0 \\ -F(1)B(0) & I \end{bmatrix} \begin{bmatrix} M_X^{-1}(0) & 0 \\ 0 & M_X^{-1}(1) \end{bmatrix} \begin{bmatrix} I & -B^T(0)F^T(1) \\ 0 & I \end{bmatrix} \\
 &= U_X \begin{bmatrix} F(0) \\ F(1)(A(0) - B(0)F(0)) \end{bmatrix} \\
 &= \begin{bmatrix} F_1(0) \\ F_1(1)(A(0) - B(0)F(0)) \\ F_2(0) \\ F_2(1)(A(0) - B(0)F(0)) \end{bmatrix} \tag{3.135}
 \end{aligned}$$

where F_i are given by (3.12). Thus, $\bar{F}_{0ij}$ defined by (3.115) becomes

$$\begin{bmatrix} \bar{F}_{011} \\ \bar{F}_{012} \end{bmatrix} = \begin{bmatrix} F_1(0) \\ F_1(1)(A(0) - B(0)F(0)) \end{bmatrix} \tag{3.136}$$

$$\begin{bmatrix} \bar{F}_{021} \\ \bar{F}_{022} \end{bmatrix} = \begin{bmatrix} F_2(0) \\ F_2(1)(A(0) - B(0)F(0)) \end{bmatrix}. \tag{3.137}$$

By using $\bar{V}_{0ij}$ given by (3.114), $\widetilde{M}_{X0}$ can be expressed by

$$\begin{aligned}
 \widetilde{M}_{X0} &= \begin{bmatrix} M_X(0) + B^T(0)E_X^T(1)M_X^{-1}(1)E_X(1)B(0) & B^T(0)E_X^T(1) \\ E_X(1)B(0) & M_X(1) \end{bmatrix} \\
 &= U_X^T \begin{bmatrix} -\gamma^2 \bar{V}_{011}^T \bar{V}_{011} + \bar{V}_{021}^T \bar{V}_{021} & \bar{V}_{021}^T \bar{V}_{022} \\ \bar{V}_{022}^T \bar{V}_{021} & \bar{V}_{022}^T \bar{V}_{022} \end{bmatrix} U_X. \tag{3.138}
 \end{aligned}$$

Thus, from (3.114) $M_X(1)$ is expressed via T_{ij} by

$$M_X(1) = \begin{bmatrix} -\gamma^2 T_{22}^T T_{22} + T_{32}^T T_{32} + T_{42}^T T_{42} & T_{42}^T T_{44} \\ T_{44}^T T_{42} & T_{44}^T T_{44} \end{bmatrix}. \tag{3.139}$$

Moreover, from

$$B^T(0)E_X^T(1) = \begin{bmatrix} -\gamma^2 T_{21}^T T_{22} + T_{31}^T T_{32} + T_{41}^T T_{42} & T_{41}^T T_{44} \\ T_{33}^T T_{32} & T_{43}^T T_{44} \end{bmatrix} \quad (3.140)$$

and

$$\begin{aligned} M_X(0) + B^T(0)E_X^T(1)M_X^{-1}(1)E_X(1)B(0) \\ = \begin{bmatrix} -\gamma^2 T_{11}^T T_{11} - T_{21}^T T_{21} + T_{31}^T T_{31} + T_{41}^T T_{41} & T_{31}^T T_{33} + T_{41}^T T_{43} \\ T_{33}^T T_{31} + T_{43}^T T_{41} & T_{33}^T T_{33} + T_{43}^T T_{43} \end{bmatrix} \end{aligned} \quad (3.141)$$

$M_X(0)$ can be expressed via T_{ij} by

$$\begin{aligned} M_X(0) = & \begin{bmatrix} -\gamma^2 T_{11}^T T_{11} - T_{21}^T T_{21} + T_{31}^T T_{31} + T_{41}^T T_{41} & T_{31}^T T_{33} + T_{41}^T T_{43} \\ T_{33}^T T_{31} + T_{43}^T T_{41} & T_{33}^T T_{33} + T_{43}^T T_{43} \end{bmatrix} \\ & - \begin{bmatrix} -\gamma^2 T_{21}^T T_{22} + T_{31}^T T_{32} + T_{41}^T T_{42} & T_{41}^T T_{44} \\ T_{33}^T T_{32} & T_{43}^T T_{44} \end{bmatrix} \\ & \times \begin{bmatrix} -\gamma^2 T_{22}^T T_{22} + T_{32}^T T_{32} + T_{42}^T T_{42} & T_{42}^T T_{44} \\ T_{44}^T T_{42} & T_{44}^T T_{44} \end{bmatrix}^{-1} \\ & \times \begin{bmatrix} -\gamma^2 T_{21}^T T_{22} + T_{31}^T T_{32} + T_{41}^T T_{42} & T_{41}^T T_{44} \\ T_{33}^T T_{32} & T_{43}^T T_{44} \end{bmatrix}. \end{aligned} \quad (3.142)$$

Thus, if we partition $M_X(0)$ into

$$M_X(0) = \begin{bmatrix} M_{X1}(0) & M_{X2}^T(0) \\ M_{X2}(0) & M_{X3}(0) \end{bmatrix} \quad (3.143)$$

$M_{X1}(0)$, $M_{X2}(0)$ and $M_{X3}(0)$ become

$$\begin{aligned} M_{X1}(0) &= -\gamma^2 T_{11}^T T_{11} + T_{21}^T T_{21} + T_{31}^T T_{31} + (\gamma^2 T_{22}^T T_{21} - T_{32}^T T_{31})\Delta^{-1}(\gamma^2 T_{21}^T T_{22} - T_{31}^T T_{32}) \\ &= -\gamma^2 T_{11}^T T_{11} + (T_{31}^T - T_{21}^T T_{22}^{-T} T_{32}^T)(I + T_{32}\Delta^{-1}T_{32}^T)(T_{31} - T_{32}T_{22}^{-1}T_{21}) \end{aligned} \quad (3.144)$$

$$\begin{aligned} M_{X2}^T(0) &= (T_{31}^T - (\gamma^2 T_{21}^T T_{22} - T_{31}^T T_{32})\Delta^{-1}T_{32}^T)T_{33} \\ &= (T_{31}^T - T_{21}^T T_{22}^{-T} T_{32}^T)(I + T_{32}\Delta^{-1}T_{32}^T)T_{33} \end{aligned} \quad (3.145)$$

$$M_{X3}(0) = T_{33}^T(I + T_{32}\Delta^{-1}T_{32}^T)T_{33} \quad (3.146)$$

where Δ is defined by

$$\Delta = \gamma^2 T_{22}^T T_{22} - T_{32}^T T_{32}. \quad (3.147)$$

Note that $\Delta > 0$ from (3.127).

Now, let us define $V_{ij}(k)$ by

$$M_X(k) = \begin{bmatrix} V_{11}^T(k) & V_{21}^T(k) \\ 0 & V_{22}^T(k) \end{bmatrix} \begin{bmatrix} -\gamma^2 I_{m_1} & 0 \\ 0 & I_{m_2} \end{bmatrix} \begin{bmatrix} V_{11}(k) & 0 \\ V_{21}(k) & V_{22}(k) \end{bmatrix} \quad (3.148)$$

Then, $V_{ij}(0)$ and $V_{ij}(1)$ are expressed by

$$\begin{bmatrix} V_{11}(1) & 0 \\ V_{21}(1) & V_{22}(1) \end{bmatrix} = \begin{bmatrix} \Delta^{1/2} & 0 \\ T_{42} & T_{44} \end{bmatrix} \quad (3.149)$$

$$\begin{bmatrix} V_{11}(0) & 0 \\ V_{21}(0) & V_{22}(0) \end{bmatrix} = \begin{bmatrix} T_{11} & 0 \\ (I + T_{32}\Delta^{-1}T_{32}^T)^{1/2}(T_{31} - T_{32}T_{22}^{-1}T_{21}) & (I + T_{32}\Delta^{-1}T_{32}^T)^{1/2}T_{33} \end{bmatrix}. \quad (3.150)$$

Therefore, if we define Ξ by

$$\Xi = \begin{bmatrix} (I + T_{32}\Delta^{-1}T_{32}^T)^{-1/2} & 0 \\ 0 & I \end{bmatrix} \quad (3.151)$$

and if we substitute (3.136), (3.137), and (3.150) into (3.121), we obtain

$$\begin{aligned} \widetilde{M}_{Z01} &= \Xi \left(\begin{bmatrix} V_{22}(0)F_2(0) \\ C_2(0) - D_{21}(0)F_1(0) \end{bmatrix} \overline{Z}_0 \begin{bmatrix} F_2^T(0)V_{22}^T(0) & C_2^T(0) - D_{21}^T(0)F_1^T(0) \end{bmatrix} \right. \\ &\quad \left. + \begin{bmatrix} V_{21}(0)V_{11}^{-1}(0) \\ D_{21}(0)V_{11}^{-1}(0) \end{bmatrix} \begin{bmatrix} V_{11}^{-T}(0)V_{21}^T(0) & V_{11}^{-T}(0)D_{21}^T(0) \end{bmatrix} - \begin{bmatrix} \gamma^2 I & 0 \\ 0 & I \end{bmatrix} \right) \Xi \\ &= \Xi(C_Z(0)\overline{Z}_0C_Z(0)^T + R_Z(0))\Xi \\ &= \Xi M'_Z(0)\Xi \end{aligned} \quad (3.152)$$

where $M'_Z(0) = C_Z(0)\overline{Z}_0C_Z(0)^T + R_Z(0)$. Since $\widetilde{M}_{Z01}$ has a J_{m_2,p_2} -factorization as previously shown, if Condition CB4 is satisfied, we can conclude that $M'_Z(0)$ has a J_{m_2,p_2} -factorization.

Third step

Last, we show the equivalence of (3) and (4).

Assume that (3) is satisfied. Let us define $\underline{C}_{Z0}$, $\underline{D}_{Z0}$, $\underline{R}_{Z0}$, $\underline{M}_{Z0}$, and $\underline{E}_{Z0}$ by

$$\underline{C}_{Z0} = \begin{bmatrix} \overline{V}_{022}^{-1} & 0 \\ 0 & I \end{bmatrix} \overline{C}_{Z0} = \begin{bmatrix} \underline{C}_{Z01} \\ \underline{C}_{Z02} \end{bmatrix} \quad (3.153)$$

$$\underline{D}_{Z0} = \begin{bmatrix} \bar{V}_{022}^{-1} & 0 \\ 0 & I \end{bmatrix} \bar{D}_{Z0} = \begin{bmatrix} \underline{D}_{Z011} \\ \underline{\bar{D}}_{Z021} \end{bmatrix} \quad (3.154)$$

$$\underline{R}_{Z0} = \begin{bmatrix} \bar{V}_{022}^{-1} & 0 \\ 0 & I \end{bmatrix} \bar{R}_{Z0} \begin{bmatrix} \bar{V}_{022}^{-T} & 0 \\ 0 & I \end{bmatrix} = \underline{D}_{Z0} \underline{D}_{Z0}^T - \begin{bmatrix} \gamma^2 (\bar{V}_{022}^T \bar{V}_{022})^{-1} & 0 \\ 0 & 0 \end{bmatrix} \quad (3.155)$$

$$\underline{M}_{Z0} = \begin{bmatrix} \bar{V}_{022}^{-1} & 0 \\ 0 & I \end{bmatrix} \bar{M}_{Z0} \begin{bmatrix} \bar{V}_{022}^{-T} & 0 \\ 0 & I \end{bmatrix} = \underline{C}_{Z0} \bar{Z}_0 \underline{C}_{Z0}^T + \underline{R}_{Z0} \quad (3.156)$$

$$\underline{E}_{Z0} = \bar{E}_{Z0} \begin{bmatrix} \bar{V}_{022}^{-T} & 0 \\ 0 & I \end{bmatrix} = \bar{A}_{Z0} \bar{Z}_0 \underline{C}_{Z0}^T + \bar{B}_{Z01} \underline{D}_{Z0}^T \quad (3.157)$$

where $\underline{C}_{Z01}$ and $\underline{D}_{Z011}$ are defined by

$$\underline{C}_{Z01} = \bar{V}_{022}^{-1} \bar{C}_{Z01} = \bar{F}_{02} \quad (3.158)$$

$$\underline{D}_{Z011} = \bar{V}_{022}^{-1} \bar{D}_{Z011} = \bar{V}_{022}^{-1} \bar{V}_{021} \bar{V}_{011}^{-1}. \quad (3.159)$$

Then, The ARE (3.52) can be rewritten by

$$\begin{aligned} \bar{Z}_0 &= \bar{A}_{Z0} \bar{Z}_0 \bar{A}_{Z0}^T + \bar{B}_{Z01} \bar{B}_{Z01}^T - \underline{E}_{Z0} \underline{M}_{Z0}^{-1} \underline{E}_{Z0}^T \\ &= \bar{A}_{Z0} \bar{Z}_0 \bar{A}_{Z0}^T + \bar{B}_{Z01} \bar{B}_{Z01}^T - \underline{E}'_{Z0} \underline{M}'_{Z0}^{-1} \underline{E}'_{Z0}^T \end{aligned} \quad (3.160)$$

where

$$\underline{E}'_{Z0} = \underline{E}_{Z0} U_Z \quad (3.161)$$

$$\underline{M}'_{Z0} = U_Z \underline{M}_{Z0} U_Z^T. \quad (3.162)$$

In the following, we will calculate the right-hand side of (3.160).

First, we calculate $\underline{M}'_{Z0}^{-1}$. Let $C'_{Z1}(k)$, $D'_{Z11}(k)$, $C'_Z(k)$, and $D'_Z(k)$ be

$$C'_{Z1}(k) = V_{22}^{-1}(k) C_{Z1}(k) = F_2(k) \quad (3.163)$$

$$D'_{Z11}(k) = V_{22}^{-1}(k) D_{Z11}(k) = V_{22}^{-1}(k) V_{21}(k) V_{11}^{-1}(k) \quad (3.164)$$

$$C'_Z(k) = \begin{bmatrix} C'_{Z1}(k) \\ C_{Z2}(k) \end{bmatrix} \quad (3.165)$$

$$D'_Z(k) = \begin{bmatrix} D'_{Z11}(k) \\ D_{Z21}(k) \end{bmatrix}. \quad (3.166)$$

Then, from (3.45), (3.136), and (3.137), $U_Z \underline{C}_{Z0}$ can be expressed by

$$\begin{aligned}
U_Z \underline{C}_{Z0} &= U_Z \begin{bmatrix} \bar{F}_{02} \\ \bar{C}_{02} - \bar{D}_{021} \bar{F}_{01} \end{bmatrix} \\
&= \begin{bmatrix} F_2(0) \\ C_2(0) - D_{21}(0)F_1(0) \\ F_2(1)(A(0) - B_1(0)F_1(0)) - F_2(1)B_2(0)F_2(0) \\ (C_2(1) - D_{21}(1)F_2(1))(A(0) - B_1(0)F_1(0)) + D_{21}(1)F_2(1)B_2(0)F_2(0) \end{bmatrix} \\
&= \begin{bmatrix} C'_Z(0) \\ C'_Z(1)A_Z(0) + \Upsilon_1 C'_Z(0) \end{bmatrix} \tag{3.167}
\end{aligned}$$

where Υ_1 is defined by

$$\Upsilon_1 = \begin{bmatrix} -C_{Z2}(1)B_2(0) & 0 \\ D_{21}(1)F_1(1)B_2(0) & 0 \end{bmatrix}. \tag{3.168}$$

From (3.49) and (3.159), $\bar{M}_{X0}^{-1}$ becomes

$$\begin{aligned}
\bar{M}_{X0}^{-1} &= \left(\begin{bmatrix} \bar{V}_{011}^T & \bar{V}_{021}^T \\ 0 & \bar{V}_{022}^T \end{bmatrix} \begin{bmatrix} -\gamma^2 I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{V}_{011} & 0 \\ \bar{V}_{021} & \bar{V}_{022} \end{bmatrix} \right)^{-1} \\
&= \begin{bmatrix} \bar{V}_{011}^{-1} & 0 \\ -\underline{D}_{Z011} & \bar{V}_{022}^{-1} \end{bmatrix} \begin{bmatrix} -\gamma^{-2} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{V}_{011}^{-T} & -\underline{D}_{Z011}^T \\ 0 & \bar{V}_{022}^{-T} \end{bmatrix} \\
&= \begin{bmatrix} -\gamma^{-2}(\bar{V}_{011}^T \bar{V}_{011})^{-1} & \gamma^{-2} \bar{V}_{011}^{-1} \underline{D}_{Z011}^T \\ \gamma^{-2} \underline{D}_{Z011} \bar{V}_{011}^{-T} & (\bar{V}_{022}^T \bar{V}_{022})^{-1} - \gamma^{-2} \underline{D}_{Z011} \underline{D}_{Z011}^T \end{bmatrix}. \tag{3.169}
\end{aligned}$$

Hence $\underline{R}_{Z0}$ can be expressed by

$$\begin{aligned}
\underline{R}_{Z0} &= \begin{bmatrix} \underline{D}_{Z011} \underline{D}_{Z011}^T - \gamma^2 (\bar{V}_{022}^T \bar{V}_{022})^{-1} & \underline{D}_{Z011} \bar{V}_{011}^{-T} \bar{D}_{021}^T \\ \bar{D}_{021} \bar{V}_{011}^{-1} \underline{D}_{Z011}^T & \bar{D}_{021} (\bar{V}_{011}^T \bar{V}_{011})^{-1} \bar{D}_{021}^T \end{bmatrix} \\
&= -\gamma^2 \begin{bmatrix} 0 & -I \\ \bar{D}_{021} & 0 \end{bmatrix} \bar{M}_{X0}^{-1} \begin{bmatrix} 0 & \bar{D}_{021}^T \\ -I & 0 \end{bmatrix} \\
&= -\gamma^2 \begin{bmatrix} 0 & -I \\ \bar{D}_{021} & 0 \end{bmatrix} U_X \tilde{M}_{X0}^{-1} U_X^T \begin{bmatrix} 0 & \bar{D}_{021}^T \\ -I & 0 \end{bmatrix} \\
&= -\gamma^2 \begin{bmatrix} 0 & -I \\ \bar{D}_{021} & 0 \end{bmatrix} U_X \begin{bmatrix} I & 0 \\ -F(1)B(0) & I \end{bmatrix} \begin{bmatrix} M_X^{-1}(0) & 0 \\ 0 & M_X^{-1}(1) \end{bmatrix} \\
&\quad \times \begin{bmatrix} I & -B^T(0)F^T(1) \\ 0 & I \end{bmatrix} U_X^T \begin{bmatrix} 0 & \bar{D}_{021}^T \\ -I & 0 \end{bmatrix} \tag{3.170}
\end{aligned}$$

Substituting

$$\begin{aligned}
& U_Z \begin{bmatrix} 0 & -I \\ \bar{D}_{021} & 0 \end{bmatrix} U_X \begin{bmatrix} I & 0 \\ -F(1)B(0) & I \end{bmatrix} \\
&= \begin{bmatrix} 0 & -I & 0 & 0 \\ D_{21}(0) & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \\ C_2(1)B_1(0) & 0 & D_{21}(1) & 0 \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ -F_1(1)B_1(0) & -F_1(1)B_2(0) & I & 0 \\ -F_2(1)B_1(0) & -F_2(1)B_2(0) & 0 & I \end{bmatrix} \\
&= \begin{bmatrix} 0 & -I & 0 & 0 \\ D_{21}(0) & 0 & 0 & 0 \\ C'_{Z1}(1)B_1(0) & C'_{Z1}(1)B_2(0) & 0 & -I \\ C_{Z2}(1)B_1(0) & -D_{21}(1)F_1(1)B_2(0) & D_{21}(1) & 0 \end{bmatrix} \quad (3.171)
\end{aligned}$$

into (3.170), we obtain after a little calculation

$$\begin{aligned}
& U_Z \underline{R}_{Z0} U_Z^T \\
&= \begin{bmatrix} R'_Z(0) & D_Z(0)B_{Z1}^T(0)C_Z'^T(1) \\ C'_Z(1)B_{Z1}(0)D_Z^T(0) & R'_Z(1) + C'_Z(1)B_{Z1}(0)B_{Z1}^T(0)C_Z'^T(1) \end{bmatrix} \\
&\quad - \begin{bmatrix} 0 & R'_Z(0)\Upsilon_1^T \\ \Upsilon_1 R'_Z(0) & \Upsilon_1 D'_Z(0)B_{Z1}^T(0)C_Z'^T(1) + C'_Z(1)B_{Z1}(0)D_Z^T(0)\Upsilon_1^T + \Upsilon_1 R'_Z(0)\Upsilon_1^T \end{bmatrix} \quad (3.172)
\end{aligned}$$

where $R'_Z(0)$ is given by

$$\begin{aligned}
R'_Z(0) &= \begin{bmatrix} V_{22}^{-1}(0) & 0 \\ 0 & I \end{bmatrix} R_Z(0) \begin{bmatrix} V_{22}^{-T}(0) & 0 \\ 0 & I \end{bmatrix} \\
&= D'_Z(0)D_Z^T(0) - \begin{bmatrix} \gamma^2(V_{22}^T(0)V_{22}(0))^{-1} & 0 \\ 0 & 0 \end{bmatrix}. \quad (3.173)
\end{aligned}$$

From (3.167) and (3.172), $\underline{M}'_{Z0}$ can be expressed by

$$\begin{aligned}
\underline{M}'_{Z0} &= \begin{bmatrix} M'_Z(0) & E_Z'^T(0)C_Z'^T(1) \\ C'_Z(1)E'_Z(0) & R'_Z(1) + C'_Z(1)(A_Z(0)\bar{Z}_0 A_Z(0)^T + B_{Z1}(0)B_{Z1}^T(0))C_Z'^T(1) \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 & M'_Z(0)\Upsilon_1^T \\ \Upsilon_1 M'_Z(0) & C'_Z(1)E'_Z(0)\Upsilon_1^T + \Upsilon_1 E_Z'^T(0)C_Z'^T(1) + \Upsilon_1 M'_Z(0)\Upsilon_1^T \end{bmatrix} \quad (3.174)
\end{aligned}$$

where $E'_Z(0)$ and $M'_Z(0)$ are defined by

$$E'_Z(0) = A_Z(0)\bar{Z}_0 C_Z'^T(0) + B_{Z1}(0)D_Z^T(0) \quad (3.175)$$

$$M'_Z(0) = C'_Z(0)\overline{Z}_0C'^T_Z(0) + R'_Z(0). \quad (3.176)$$

If we define $M'_Z(1)$ by

$$M'_Z(1) = C'_Z(1)\overline{Z}'C'^T_Z(1) + R'_Z(1) \quad (3.177)$$

$$\overline{Z}' = A_Z(0)\overline{Z}_0A^T_Z(0) + B_{Z1}(0)B^T_{Z1}(0) - E'_Z(0)M'^{-1}_Z(0)E'^T_Z(0) \quad (3.178)$$

(3.174) can be rewritten by

$$\begin{aligned} \underline{M}'_{Z0} = & \begin{bmatrix} I & 0 \\ C'_Z(1)E'_Z(0)M'^{-1}_Z(0) + \Upsilon_1 & I \end{bmatrix} \begin{bmatrix} M'_Z(0) & 0 \\ 0 & M'_Z(1) \end{bmatrix} \\ & \times \begin{bmatrix} I & M'^{-1}_Z(0)E'^T_Z(0)C'^T_Z(1) + \Upsilon_1^T \\ 0 & I \end{bmatrix}. \end{aligned} \quad (3.179)$$

Next, we calculate $\underline{E}'^T_{Z0} = U_Z \underline{C}_{Z0} \overline{Z}_0 \overline{A}^T_{Z0} + U_Z \underline{D}_{Z0} \overline{B}^T_{Z01}$. The matrix $U_Z \underline{D}_{Z0} \overline{B}^T_{Z01}$ can be expressed by

$$\begin{aligned} U_Z \underline{D}_{Z0} \overline{B}^T_{Z01} &= -\gamma^{-2} U_Z \underline{D}_{Z0} \overline{M}_X^{-1} \begin{bmatrix} \overline{B}^T_{01} \\ 0 \end{bmatrix} \\ &= -\gamma^{-2} U_Z \underline{D}_{Z0} U_X \widetilde{M}_X^{-1} U_X^T \begin{bmatrix} \overline{B}^T_{01} \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} D'_Z(0)B^T_{Z1}(0)A^T_Z(1) \\ C'_Z(1)B_{Z1}(0)B^T_{Z1}(0)A^T_Z(1) + D'_Z(1)B^T_{Z1}(1) \end{bmatrix} \\ &\quad + \begin{bmatrix} M'_Z(0)\Upsilon_2^T \\ \Upsilon_1 D'_{Z11}(0)B^T_{Z1}(0) + C'_Z(1)B_{Z1}(0)D'^T_{Z11}(0)\Upsilon_2^T + \Upsilon_1 M'_Z(0)\Upsilon_2^T \end{bmatrix}. \end{aligned} \quad (3.180)$$

where Υ_2 is defined by

$$\Upsilon_2 = \begin{bmatrix} B_1(1)F_1(1)B_2(0) & 0 \end{bmatrix}. \quad (3.181)$$

Furthermore,

$$\begin{aligned} \overline{A}_{Z0} &= \overline{A}_0 - \overline{B}_{01}\overline{F}_{01} \\ &= A(1)A(0) - A(1)B_1(0)F_1(0) - B_1(1)F_1(1)A_Z(0) - B_1(1)F_1(1)B_2(0)C_{Z1}(0) \\ &= A_Z(1)A_Z(0) - \Upsilon_2 C_Z(0) \end{aligned} \quad (3.182)$$

holds. From (3.167), (3.180), and (3.182), we obtain

$$\begin{aligned} \underline{E}'_Z = & \begin{bmatrix} E'_Z(0)A_Z^T(1) \\ C'_Z(1) \left(A_Z(0)\bar{Z}_0A_Z(0)^T + B_{Z1}(0)B_{Z1}^T(0) \right) A_Z^T(1) + D_Z(1)B_{Z1}^T(1) \end{bmatrix} \\ & + \begin{bmatrix} M'_Z(0)\Upsilon_2^T \\ \Upsilon_1 E'_Z(0)A_Z^T(1) + C'_Z(1)E'_Z(0)\Upsilon_2^T + \Upsilon_1 M'_Z(0)\Upsilon_2^T \end{bmatrix}. \end{aligned} \quad (3.183)$$

From (3.179) and (3.183), $\underline{E}'_{Z0}\underline{M}'_{Z0}{}^{-1}\underline{E}'_{Z0}^T$ can be rewritten by (it requires a little calculation)

$$\begin{aligned} \underline{E}'_{Z0}\underline{M}'_{Z0}{}^{-1}\underline{E}'_{Z0}^T &= \begin{bmatrix} A_Z(1)E'_Z(0) + \Upsilon_2 M'_Z(0) & E'_Z(1) \end{bmatrix} \begin{bmatrix} M'_Z{}^{-1}(0) & 0 \\ 0 & M'_Z{}^{-1}(1) \end{bmatrix} \\ &\quad \times \begin{bmatrix} E'_Z(0)A_Z^T(1) + M'_Z(0)\Upsilon_2^T \\ E'_Z(1) \end{bmatrix} \\ &= A_Z(1)E'_Z(0)M'_Z{}^{-1}(0)E'_Z(0)^T + E'_Z(1)E'_Z(1)^T \\ &\quad + \Upsilon_2 E'_Z(0)A_Z^T(1) + A_Z(1)E'_Z(0)\Upsilon_2^T + E'_Z(0)M'_Z(0)E'_Z(0)^T. \end{aligned} \quad (3.184)$$

Finally, we calculate $\bar{A}_{Z0}\bar{Z}_0\bar{A}_{Z0}^T + \bar{B}_{Z01}\bar{B}_{Z01}^T$. By noting that

$$\begin{aligned} \bar{B}_{Z01}\bar{B}_{Z01}^T &= -\gamma^{-2} \begin{bmatrix} \bar{B}_{01} & 0 \end{bmatrix} \bar{M}_X^{-1} \begin{bmatrix} \bar{B}_{01}^T \\ 0 \end{bmatrix} \\ &= -\gamma^{-2} \begin{bmatrix} \bar{B}_{01} & 0 \end{bmatrix} U_X \bar{M}_X^{-1} U_X^T \begin{bmatrix} \bar{B}_{01}^T \\ 0 \end{bmatrix} \\ &= A_Z(1)B_{Z1}(0)B_{Z1}^T(0)A_Z^T(1) + B_{Z1}(1)B_{Z1}^T(1) + \Upsilon_2 D'_{Z11}(0)B_{Z1}^T(0) \\ &\quad + B_{Z1}(0)D'_{Z11}(0)\Upsilon_2^T + \Upsilon_2 M'_Z{}^{-1}(0)\Upsilon_2^T \end{aligned} \quad (3.185)$$

$\bar{A}_{Z0}\bar{Z}_0\bar{A}_{Z0}^T + \bar{B}_{Z01}\bar{B}_{Z01}^T$ can be expressed by

$$\begin{aligned} \bar{A}_{Z0}\bar{Z}_0\bar{A}_{Z0}^T + \bar{B}_{Z01}\bar{B}_{Z01}^T &= A_Z(1) \left(A_Z(0)\bar{Z}_0A_Z^T(0) + B_{Z1}(0)B_{Z1}^T(0) \right) A_Z^T(1) \\ &\quad + B_{Z1}(1)B_{Z1}^T(1) + \Upsilon_2 E'_Z(0)A_Z^T(1) \\ &\quad + A_Z(1)E'_Z(0)\Upsilon_2^T + \Upsilon_2 M'_Z(0)\Upsilon_2^T. \end{aligned} \quad (3.186)$$

By substituting (3.184) and (3.186) into (3.162), we obtain

$$\begin{aligned} \bar{Z}_0 &= \bar{A}_{Z0}\bar{Z}_0\bar{A}_{Z0}^T + \bar{B}_{Z01}\bar{B}_{Z01}^T - \underline{E}'_{Z0}\underline{M}'_{Z0}{}^{-1}\underline{E}'_{Z0}^T \\ &= A_Z(1) \left(A_Z(0)\bar{Z}_0A_Z^T(0) + B_{Z1}(0)B_{Z1}^T(0) - E'_Z(0)M'_Z{}^{-1}(0)E'_Z(0)^T \right) A_Z^T(1) \\ &\quad + B_{Z1}(1)B_{Z1}^T(1) - E'_Z(1)M'_Z{}^{-1}(1)E'_Z(1)^T \\ &= A_Z(1)\bar{Z}'A_Z^T(1) + B_{Z1}(0)B_{Z1}^T(0) - E'_Z(1)M'_Z{}^{-1}(1)E'_Z(1)^T \\ &= A_Z(1)\bar{Z}'A_Z^T(1) + B_{Z1}(0)B_{Z1}^T(0) - E_Z(1)M_Z{}^{-1}(1)E_Z^T(1). \end{aligned} \quad (3.187)$$

It implies that the set $\bar{Z}_0$ and $\bar{Z}'$ is a solution of the PRE (3.62) in Condition CV2'. Since we can show that $\bar{Z}' \geq 0$ holds by the similar argument taken in Section 3.3.1, and that $M_Z'^{-1}(1)$ has a J_{m_2, p_2} -factorization from (3.179), the proof is complete if we show that this set is the (unique) stabilizing solution of the PRE (3.62). By noting that $\bar{E}_{Z0} \bar{M}_{Z0}^{-1} \bar{C}_{Z0}$ can be expressed by

$$\begin{aligned} \bar{E}_{Z0} \bar{M}_{Z0}^{-1} \bar{C}_{Z0} &= \underline{E}'_{Z0} \underline{M}'_{Z0}{}^{-1} \underline{C}'_{Z0} \\ &= \begin{bmatrix} A_Z(1)E'_Z(0) + \Upsilon_2 M'_Z(0) & E'_Z(1) \end{bmatrix} \begin{bmatrix} M_Z'^{-1}(0) & 0 \\ 0 & M_Z'^{-1}(1) \end{bmatrix} \\ &\quad \times \begin{bmatrix} C'_Z(0) \\ C'_Z(1) (A_Z(0) - E'_Z(0) M_Z'^{-1}(0) C'_Z(0)) \end{bmatrix} \\ &= A_Z(1)E'_Z(0)M_Z'^{-1}(0)C'_Z(0) - \Upsilon_2 C'_Z(0) \\ &\quad + E'_Z(1)M_Z'^{-1}(1)C'_Z(1)(A_Z(0) - E'_Z(0)M_Z'^{-1}(0)C'_Z(0)) \quad (3.188) \end{aligned}$$

we can express $\bar{A}_{Z0} - \bar{E}_{Z0} \bar{M}_{Z0}^{-1} \bar{C}_{Z0}$ by

$$\begin{aligned} \bar{A}_{Z0} - \bar{E}_{Z0} \bar{M}_{Z0}^{-1} \bar{C}_{Z0} &= (A_Z(1) - E'_Z(1)M_Z'^{-1}(1)C'_Z(1)) \\ &\quad \times (A_Z(0) - E'_Z(0)M_Z'^{-1}(0)C'_Z(0)). \quad (3.189) \end{aligned}$$

Consequently, the set of $\bar{Z}_0$ and $\bar{Z}'$ becomes the stabilizing solution of the PRE (3.62), and $Z(0) = \bar{Z}_0$, $Z(1) = \bar{Z}'$.

General case

It has been shown that Lemma 3.5 and Lemma 3.7 are equivalent in the case of $N = 2$. We now extend it to the general case.

Recall that Condition CB4 can be reduced to the condition that there exists $\|\bar{D}_Q\| < \gamma$ such that $\bar{D}'_K$ defined by (3.102) is lower triangular. Let us define V_U by

$$V_U = \bar{W}_{011}^{-1} \bar{W}_{012} \quad (3.190)$$

and partition it into $N \times N$ blocks (the ij -th block of V_U is denoted by V_{Uij}). Then, all $\bar{D}_Q$ that make $\bar{D}'_K$ block triangular are expressed by

$$\bar{D}_Q = \begin{bmatrix} V_{U11} + \chi_{11} & V_{U12} & V_{U13} & \cdots & V_{U1N} \\ V_{U21} + \chi_{21} & V_{U22} + \chi_{22} & V_{U23} & \cdots & V_{U2N} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ V_{UN1} + \chi_{N1} & V_{UN2} + \chi_{N2} & \cdots & V_{UN,N-1} & V_{UNN} + \chi_{NN} \end{bmatrix} \quad (3.191)$$

where χ_{ij} is arbitrary. By applying the well-known relation

$$\min_x \left\| \begin{bmatrix} y_{11} & y_{12} \\ y_{21} + x & y_{22} \end{bmatrix} \right\| = \max(\| [y_{11} \quad y_{12}] \|, \left\| \begin{bmatrix} y_{12} \\ y_{22} \end{bmatrix} \right\|) \quad (3.192)$$

to (3.191) recursively, we obtain

$$\min \|D_Q\| = \max\{\|\bar{V}_{U1}\|, \|\bar{V}_{U2}\|, \dots, \|\bar{V}_{UN-1}\|\} \quad (3.193)$$

where $\bar{V}_{Ui}$ is given by

$$\bar{V}_{Ui} = \begin{bmatrix} V_{U1,i+1} & V_{U1,i+2} & \cdots & V_{U1N} \\ V_{U2,i+1} & V_{U2,i+2} & \cdots & V_{U2N} \\ \vdots & \vdots & & \vdots \\ V_{Ui,i+1} & V_{Ui,i+2} & \cdots & V_{UiN} \end{bmatrix}. \quad (3.194)$$

If we partition $\bar{W}_{011}$ and $\bar{W}_{021}$ into

$$\begin{aligned} \bar{W}_{011} &= \left[\begin{array}{c|c} \overbrace{S_{i11}}^{im_2} & \overbrace{0}^{(N-i)m_2} \\ \hline S_{i21} & S_{i22} \end{array} \right] \begin{matrix} \} im_2 \\ \} (N-i)m_2 \end{matrix} \\ \bar{W}_{021} &= \left[\begin{array}{c|c} \overbrace{S_{i31}}^{im_2} & \overbrace{S_{i32}}^{(N-i)m_2} \\ \hline S_{i41} & S_{i42} \end{array} \right] \begin{matrix} \} ip_2 \\ \} (N-i)p_2 \end{matrix} \end{aligned} \quad (3.195)$$

$\bar{V}_{Ui}$ can be expressed by

$$\bar{V}_{Ui} = S_{i21} S_{i11}^{-1}. \quad (3.196)$$

Therefore, Condition CB4 reduces to the condition that for any $i = 1, 2, \dots, N-1$

$$\gamma^2 S_{i22} S_{i22} - S_{i32}^T S_{i32} > 0 \quad (3.197)$$

holds.

The condition (3.197) for each i can be interpreted as a condition for P rearranged as a 2-periodic system, which corresponds to (3.106). More specifically, (3.197) corresponds

to (3.106) in the case that we lift the signals as

$$u_0, u_1, \dots, u_{N-1}, u_N, \dots, u_{2N-1}, \dots$$

$$\rightarrow \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{i-1} \end{bmatrix}, \begin{bmatrix} u_i \\ u_{i+1} \\ \vdots \\ u_{N-1} \end{bmatrix}, \begin{bmatrix} u_N \\ u_{N+1} \\ \vdots \\ u_{N+i-1} \end{bmatrix}, \begin{bmatrix} u_{N+i} \\ u_{N+i+1} \\ \vdots \\ u_{2N-1} \end{bmatrix}, \dots \quad (3.198)$$

and rearrange P as a 2-periodic system. From the argument for the case of $N = 2$, we can show that (3.197) is equivalent to the condition that $\overline{M}_{\chi^i}$, the submatrix of $\widetilde{M}_{X0}$ defined by

$$\widetilde{M}_{X0} = \left[\begin{array}{c|c} & \overbrace{\hspace{1cm}}^{(N-i)(m_1+m_2)} \\ \hline * & * \\ \hline * & M_{\chi^i} \end{array} \right] \}_{(N-i)(m_1+m_2)} \quad (3.199)$$

has a $J_{(N-i)m_1, (N-i)m_2}$ -factorization and $\overline{M}_{\zeta^i}$ defined by

$$\overline{M}_{\zeta^i} = \overline{C}_{\zeta^i} \overline{Z}_0 \overline{C}_{\zeta^i}^T + \overline{R}_{\zeta^i} \quad (3.200)$$

has a J_{im_2, ip_2} -factorization, where $\overline{C}_{\zeta^k}$ and $\overline{R}_{\zeta^k}$ are defined as $\overline{C}_{Z^k}$ in (3.45) and $\overline{R}_{Z^k}$ in (3.47) by using $X(k)$ and

$$A_{Uk} = A(k-1)A(k-2)\cdots A(0) \quad (3.201)$$

$$B_{Uki} = \begin{bmatrix} A(k-1)\cdots A(1)B_i(0) & A(k-1)\cdots A(2)B_i(1) & \cdots & B_i(k-1) \end{bmatrix} \quad (3.202)$$

$$C_{Uki} = \begin{bmatrix} C_i(0) \\ C_i(1)A(0) \\ \vdots \\ C_i(k-1)A(k-2)\cdots A(0) \end{bmatrix} \quad (3.203)$$

$$D_{Ukij} = \begin{bmatrix} D_{ij}(0) & & & 0 \\ \Psi_{ij}(1,0) & D_{ij}(1) & & \\ \vdots & \ddots & \ddots & \\ \Psi_{ij}(k-1,0) & \cdots & \Psi_{ij}(k-1,k-2) & D_{ij}(k-1) \end{bmatrix} \quad (3.204)$$

in place of $\overline{X}_k$, $\overline{A}_k$, $\overline{B}_{ki}$, $\overline{C}_{ki}$, and $\overline{D}_{kij}$, respectively. It is a routine to show that Conditions CV1 and CV2' are satisfied.

3.4 The causality constraint in the methods belonging to the time-invariant approach

In this section, we compare three methods shown in Section 3.1 through investigating how the causality constraint appears in these methods.

First, we consider the method by Qiu and Chen (Method B). As mentioned in the preceding section, their method is an intuitive way to cope with the causality constraint: they imposed it on the free parameter of controller parametrization. Therefore, in Lemma 3.5, the causality constraint appears very clearly, namely, Condition CB4 is directly connected to it.

Next, we consider Method C. Roughly speaking, Conditions CC4–CC6 correspond to the causality constraint in Method C, i.e., in Lemma 3.6. However, these conditions have no qualitative explanation connected with the causality constraint. It is because their method is an intermediate of the time-varying and time-invariant approaches, in the sense that they converted the causality constraint into a min-max problem by applying the dynamic game theory of H_∞ control. It is sure that they treated the LPTV H_∞ problem by separating the problem into the lifted LTI H_∞ problem and the causality constraint, but, from a different viewpoint, it can be said that their method only separated the solvability condition obtained based on the time-varying approach into two parts: one corresponds to the lifted LTI H_∞ problem and the other part the causality constraint. This can also be verified from the fact that the solvability condition in Lemma 3.6 has a closer similarity to the condition based on the time-varying approach (Lemma 3.7) than those in Lemma 3.4 and Lemma 3.5, as shown in Section 3.3.

The method based on Theorem 2.1 (Method A) does not treat the causality constraint directly, and therefore we cannot discuss how the causality constraint appears in Lemma 3.4 as does in the other two methods. Indeed, the condition in Lemma 3.4 that corresponds to the causality constraint is that Condition CA is required for $k = 1, 2, \dots, N-1$, and the relationship with the causality constraint is apparently quite unclear. However, we can make an explanation of how that is connected to the causality constraint.

Let us consider, for example, the case that P is 3-periodic ($N = 3$). In this case, the solvability condition is that the LTI H_∞ problems of P_0^L , P_1^L and P_2^L are all solvable (that correspond to the case $k = 0$, $k = 1$, and $k = 2$ in the condition of Lemma 3.4, respectively).

First, we assume that the LTI H_∞ problem only of P_0^L is solvable. In this case, the controllers obtained are such that the causality constraint is not taken into account. In other words, the controllers are causal only in the weak sense that all noncausal paths

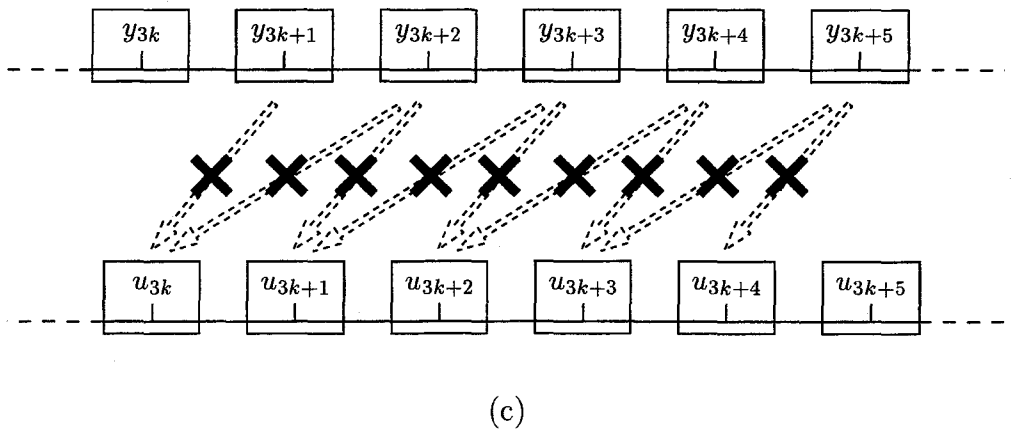
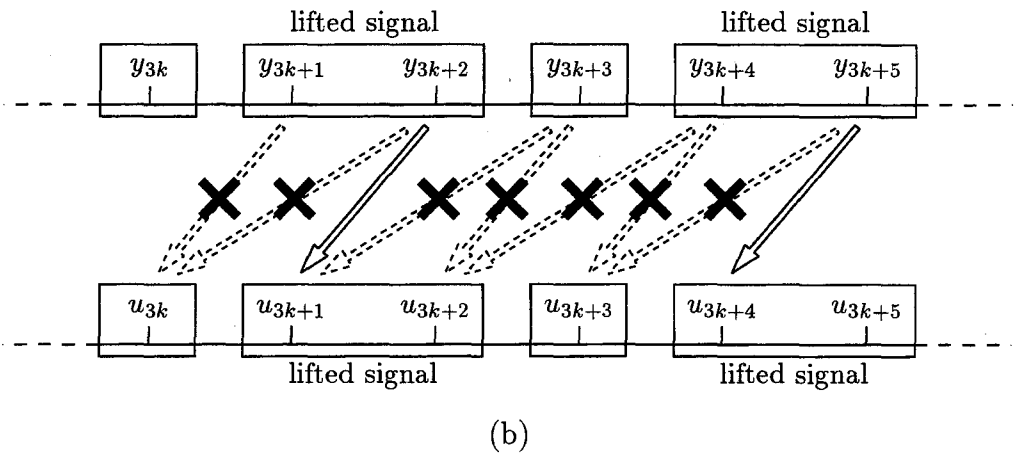
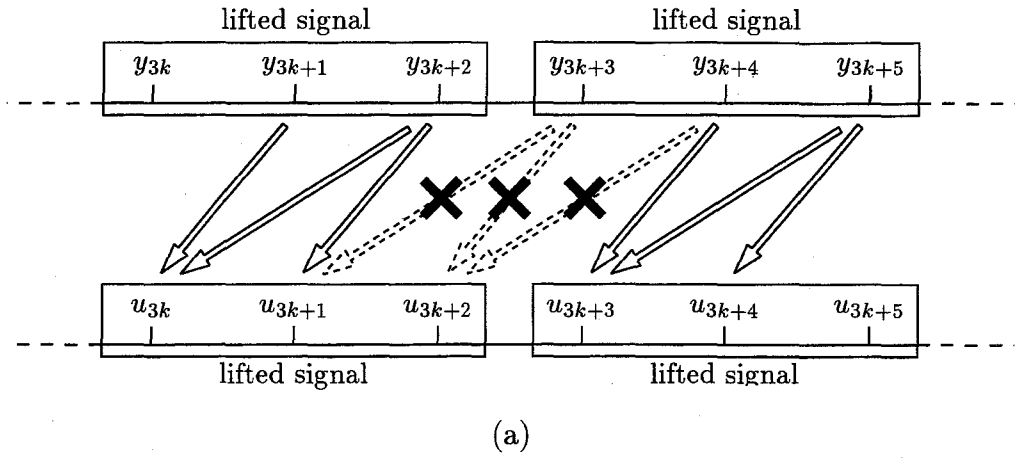


Figure 3.1: Noncausal paths allowed in controllers

between lifted signals are forbidden (in Fig. 3.1(a), the paths denoted by the arrows with dashed lines), but might be noncausal in the sense that noncausal paths between original (not lifted) signals are not forbidden (in Fig. 3.1(a) the paths denoted by the arrows with solid lines).

Next, we assume that the LTI H_∞ problem of P_1^L is also solvable in addition to the LTI H_∞ problem of P_0^L . These conditions can be regarded as the solvability condition to the LPTV H_∞ problem of P rearranged as a 2-periodic system. Here, “ P rearranged as a 2-periodic system” means the rearranged system by lifting the signals partially as

$$u_0, u_1, u_2, u_3, u_4, u_5, \dots \rightarrow u_0, \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, u_3, \begin{bmatrix} u_4 \\ u_5 \end{bmatrix}, \dots \quad (3.205)$$

In this case, although the controllers might be still noncausal, the noncausal paths are less than those of the case that the LTI H_∞ problem only of P_0^L is solvable (Fig. 3.1(b)).

Last, we assume that the LTI H_∞ problem of P_2^L is also solvable, namely, the LTI H_∞ problems of P_0^L , P_1^L and P_2^L are all solvable. In this case, all the noncausal paths left in Fig. 3.1(b) are forbidden and the controllers obtained become causal (Fig. 3.1(c)).

As this example shows, if the LTI H_∞ problems of i lifted systems $P_{k_0}^L, P_{k_1}^L, \dots, P_{k_{i-1}}^L$ are solvable, the LPTV H_∞ problem of P rearranged as an $(N - i + 1)$ -periodic system is solvable. In this manner, the condition that the LTI H_∞ problems of $P_0^L, P_1^L, \dots, P_{N-1}^L$ are solvable, is connected to causality of the controllers.

3.5 Algorithms to calculate $\gamma_{\min}$ for the discrete LPTV H_∞ problem

In this section, we describe three algorithms to calculate $\gamma_{\min} = \inf \gamma$ based on Method A, B, and C, that is, Lemmas 3.4–3.6. As is well-known, the algorithm called “ γ -iteration” is applied to calculate $\gamma_{\min}$ for the standard (LTI) H_∞ problem. Along this line, we describe algorithms to calculate $\gamma_{\min}$ for the discrete LPTV H_∞ problem. Then, we compare these algorithms through some numerical examples.

Note that we only deal with the algorithms based on Lemmas 3.4–3.6 and do not treat Lemma 3.7. One reason is that to solve stationary AREs (not including PREs) is easier than to solve PREs directly, and an efficient algorithm to solve stationary AREs is well-known based on the deflating subspace, although there are some algorithms to solve PREs (for example, periodic Shur decomposition [47]). Another reason is that the algorithm based on Lemma 3.7 is almost identical to that based on Lemma 3.6.

3.5.1 The γ -iteration

Before we describe the algorithms for the discrete LPTV H_∞ problem, we first draw a brief sketch of the γ -iteration.

The algorithm to obtain $\gamma_{\min}$ analytically is not yet known except some special cases. Therefore, some numerical method is required. The γ -iteration is a simple algorithm to obtain $\gamma_{\min}$ numerically, which is no other than the bisection algorithm. The basic procedure is as follows:

- (1) Set the interval $[\gamma_l, \gamma_u]$ so that $\gamma_{\min}$ is included.
- (2) If the H_∞ problem at $\gamma = (\gamma_l + \gamma_u)/2$ is solvable, $\gamma_{\min}$ is included in the interval $[\gamma_l, \gamma]$. Otherwise, $\gamma_{\min}$ is included in the interval $[\gamma, \gamma_u]$.
- (3) Narrow the interval by repeating (1) and (2).

In practical situations, the following algorithm is often used.

- 0° Set the interval $[\gamma_l, \gamma_u]$, and the tolerance $\epsilon > 0$.
- 1° Check solvability of the H_∞ problem at $\gamma := \gamma_u$. If it is not solvable, terminate (the interval $[\gamma_l, \gamma_u]$ does not include $\gamma_{\min}$). Otherwise, go to 2°.
- 2° Check solvability of the H_∞ problem at $\gamma := (\gamma_l + \gamma_u)/2$. If it is solvable, set $\gamma_u := (\gamma_l + \gamma_u)/2$. Otherwise, $\gamma_l := (\gamma_l + \gamma_u)/2$.
- 3° If $\gamma_u - \gamma_l < \epsilon$, set $\gamma_{\min} = \gamma_u$ and terminate. Otherwise, go to 2°.

This algorithm is very simple in the sense that it terminates when the initial interval is not appropriate, i.e., $\gamma_{\min}$ is not included. Therefore, in such a case, we should change the interval manually until it includes $\gamma_{\min}$. Although we can extend this algorithm so that it automatically widen the interval until $\gamma_{\min}$ is included, in the following we assume that $\gamma_{\min} \in [\gamma_l, \gamma_u]$ and proceed arguments based on this algorithm to simplify the arguments.

3.5.2 An algorithm based on Lemma 3.4

First, we state an algorithm based on Lemma 3.4. We can do this by simply using the condition in Lemma 3.4 as solvability checking of the discrete LPTV H_∞ problem in 1° and 2°, in the algorithm of the γ -iteration previously mentioned. However, it is not so good an algorithm from the viewpoint of efficiency. Since if the LTI H_∞ problem of P_k^L at a certain γ_k is solvable, the LTI H_∞ problem of P_k^L for $\gamma \geq \gamma_k$ is always solvable,

the algorithm becomes more efficient by storing the latest γ_k . The following algorithm takes this point into account (let the initial value of γ_k be ∞).

0° Set $k := 0$.

1° If $\gamma \geq \gamma_k$, go to 2°. Otherwise, check whether the LTI H_∞ problem of P_k^L for γ is solvable by applying Conditions CA1, CA2, and CA3. If solvable, go to 2°. Otherwise, terminate (the original LPTV H_∞ problem is not solvable for this γ).

2° If $k = N - 1$, terminate (the original LPTV H_∞ problem is solvable for this γ). Otherwise, set $k := k + 1$ and go to 1°.

The reason why we use Condition CA1, CA2, and CA3 not Conditions CA1 and CA2' in 1° is that the former takes less calculation than the latter.

When we use this procedure as solvability checking in the γ -iteration, calculation time varies according to the order of solvability checking of the LTI H_∞ problems of P_k^L . For example, let us consider the case $N = 3$ (3-periodic) $[\gamma_l, \gamma_u] = [1, 17]$ and $\epsilon = 0.5$. Assume that $\gamma_{\min, k}$ of the LTI H_∞ problems of P_k^L are given by $\{\gamma_{\min, 0}, \gamma_{\min, 1}, \gamma_{\min, 2}\} = \{7.5, 4, 2\}$. We compare the case that we check solvability in the order of P_0^L, P_1^L, P_2^L and in the order of P_2^L, P_1^L, P_0^L . Table 3.1 shows how solvability checking is done in each step of the γ -iteration. In this table, "solvable" and "not solvable" denote that solvability checking is done and "skipped" denotes that checking is skipped since for that γ it is already known that the H_∞ problem is not solvable and checking is unnecessary. On the other hand, "-" denotes that checking is skipped since for that γ it is known that the LTI H_∞ problem of P_k^L is solvable from the stored value of γ_k . From this table, we can see that the number of solvability checks in the case of $P_0^L \rightarrow P_1^L \rightarrow P_2^L$ is 14, whereas in the case of $P_2^L \rightarrow P_1^L \rightarrow P_0^L$ the number is 12. This example shows that it takes less calculation time if we check the solvability conditions of P_k^L in the increasing order of $\gamma_{\min, k}$. Of course, as mentioned before, we cannot know $\gamma_{\min, k}$ apriori. However, we can guess the order of $\gamma_{\min, k}$ to some extent from that of $\rho(\overline{X}_k \overline{Y}_k)$, which appears in Condition CA4 and is calculated for the first time in the procedure 1° of the γ -iteration.

After all, the overall algorithm for obtaining $\gamma_{\min}$ for discrete LPTV H_∞ problems becomes:

0° Set the interval $[\gamma_l, \gamma_u]$, and the tolerance $\epsilon > 0$.

1° Set $\gamma := \gamma_u$ and $k := 0$.

2° Check if the H_∞ problem of P_k^L for γ is solvable by applying Conditions CB1–CB3. If solvable, go to 3°. Otherwise, set $\gamma_{\min} = \infty$ and terminate.

Table 3.1: Solvability checking in the γ -iteration based on Lemma 3.4

(a) The case of $P_0^L \rightarrow P_1^L \rightarrow P_2^L$

γ	check for P_0^L	check for P_1^L	check for P_2^L
17	solvable	solvable	solvable
9	solvable	solvable	solvable
5	not solvable	skipped	skipped
7	not solvable	skipped	skipped
8	solvable	solvable	solvable
7.5	solvable	solvable	solvable

(b) The case of $P_2^L \rightarrow P_1^L \rightarrow P_0^L$

γ	check for P_2^L	check for P_1^L	check for P_0^L
17	solvable	solvable	solvable
9	solvable	solvable	solvable
5	solvable	solvable	not solvable
7	—	—	not solvable
8	—	—	solvable
7.5	—	—	solvable

- 3° If $k = N - 1$, go to 4°. Otherwise, set $k := k + 1$ and go to 2°.
- 4° Sort P_k^L by the increasing order of $\rho(\overline{X}_k \overline{Y}_k)$. Set $g_k := \gamma_u$ ($k = 0, 1, \dots, N - 1$).
- 5° Set $\gamma := (\gamma_l + \gamma_u)/2$, $k := 0$, and $\text{flag} := 0$.
- 6° If $\gamma \geq g_k$, go to 7°. Otherwise, check if the H_∞ problem of P_k^L for γ is solvable by applying Conditions CB1–CB3. If solvable, set $g_k := \gamma$ and go to 7°. Otherwise, set $\text{flag} := 1$ and go to 8°.
- 7° If $k = N - 1$, go to 8°. Otherwise, set $k := k + 1$ and go to 6°.
- 8° If $\text{flag} = 0$, set $\gamma_u := \gamma$. Otherwise, set $\gamma_l := \gamma$.
- 9° If $\gamma_u - \gamma_l < \epsilon$, set $\gamma_{\min} = \gamma_u$ and terminate. Otherwise, go to 5°.

3.5.3 Algorithms based on Lemma 3.5 and Lemma 3.6

Unlike the algorithm based on Lemma 3.4, we cannot improve efficiency of algorithms based on Lemma 3.5 and Lemma 3.6. Therefore, in these cases, we construct the algorithms to obtain $\gamma_{\min}$ by simply using the conditions of the lemmas as solvability checking in the ordinary γ -iteration described in Section 3.5.1.

Although it is almost direct to construct an algorithm for checking solvability based on the condition of Lemma 3.5 or the condition of Lemma 3.6, how to check Condition CB4 in Lemma 3.5 is still unclear. To check Condition CB4, we make use of the following procedure [4, 39]:

- (1) Find the factorizations

$$\overline{D}_{c12} = L_{t1} L_{o1}, \quad \overline{D}_{c21} = -L_{o2} L_{t2} \quad (3.206)$$

where L_{t1} , L_{t2} , L_{o1} , L_{o2} are all invertible, and, L_{o1} , L_{o2} are orthogonal, and L_{t1} , L_{t2} are block lower triangular.

- (2) Define L by

$$L = L_{t1}^{-1} \overline{D}_{c11} L_{t2}^{-1}. \quad (3.207)$$

- (3) Define L_k ($k = 1, 2, \dots, N - 1$) by partitioning L as

$$L = \left[\begin{array}{c|c} \overbrace{\begin{matrix} * & * \end{matrix}}^{kp_2} & \overbrace{\begin{matrix} L_k \end{matrix}}^{(N-k)p_2} \\ \hline \begin{matrix} * \\ * \end{matrix} & * \end{array} \right] \begin{matrix} \} km_2 \\ \} (N-k)m_2 \end{matrix} \quad (3.208)$$

(4) Check if

$$\|L_k\| < \gamma \quad (3.209)$$

is satisfied for any k . If so, Condition CB4 holds. Otherwise, Condition CB4 does not hold for this γ .

3.5.4 Numerical examples

In the preceding sections, we showed how to construct the algorithms to obtain $\gamma_{\min}$ for the discrete LPTV H_∞ problem based on three types of solvability conditions. In this section, we apply these algorithms to some numerical examples, and examine effectiveness of each method.

Consider the N -periodic system P whose system matrices are given by

$$\begin{bmatrix} A(Ni+k) & B_1(Ni+k) & B_2(Ni+k) \\ C_1(Ni+k) & D_{11}(Ni+k) & D_{12}(Ni+k) \\ C_2(Ni+k) & D_{21}(Ni+k) & D_{22}(Ni+k) \end{bmatrix} = \begin{bmatrix} 0.8+0.1k & 1 & 1 & 0 & 2 \\ 0.2 & 0.7+0.2k & 1-0.1k & -0.2 & -1 \\ 0.1+0.1k & -0.1 & 0.5 & 0.2 & -0.2-0.1k \\ 1 & -0.2 & -0.2-0.4k & 1 & 0 \end{bmatrix} \quad (3.210)$$

where $k = 0, 1, \dots, N-1$. We make three systems, i.e., 2-periodic, 5-periodic and 8-periodic systems by setting $N = 2, 5, 8$, respectively, and apply the algorithms to these systems by changing ϵ , the tolerance in the γ -iteration. In all settings, the initial interval for the γ -iteration is set to $[1, 100]$. Calculation is done by using MATLAB (Linux on Pentium 133MHz). The results are shown in Table 3.2. In Table 3.2, total calculation time for each setting is shown. Note that $\gamma_{\min}$ obtained by each algorithm is identical (for example, in the setting of $N = 2$ and $\epsilon = 1.0^{-3}$, $\gamma_{\min} = 22.087$ is obtained by all the algorithms).

Table 3.2: Numerical examples of the algorithms for obtaining $\gamma_{\min}$

(a) 2 periodic case

Algorithm	$\epsilon = 10^{-3}$ ($\gamma_{\min} = 22.087$)	$\epsilon = 10^{-6}$ ($\gamma_{\min} = 22.086315$)
Method A	0.81sec	1.17sec
Method B	0.90sec	1.39sec
Method C	0.81sec	1.25sec

(b) 5 periodic case

Algorithm	$\epsilon = 10^{-3}$ ($\gamma_{\min} = 38.370$)	$\epsilon = 10^{-6}$ ($\gamma_{\min} = 38.370166$)
Method A	1.40sec	1.84sec
Method B	1.20sec	1.84sec
Method C	1.25sec	1.94sec

(c) 8 periodic case

Algorithm	$\epsilon = 10^{-3}$ ($\gamma_{\min} = 98.701$)	$\epsilon = 10^{-6}$ ($\gamma_{\min} = 98.703991$)
Method A	2.10sec	2.69sec
Method B	1.70sec	2.63sec
Method C	1.82sec	2.82sec

The algorithm based on Method A takes shorter calculation time in the 2-periodic case and the case of $\epsilon = 10^{-6}$. On the other hand, the algorithm based on Method B takes shorter calculation time especially in the 8-periodic case. The algorithm based on Method C is intermediate between these two algorithms. The reason why the algorithm based on Method A takes less calculation time in the case of $\epsilon = 10^{-6}$ is that when γ approaches to $\gamma_{\min}$, it requires to check solvability of only one lifted system not of all lifted systems (see Table 3.1(b)) and therefore it works more effectively if the number of iterations becomes larger and larger. However, this algorithm takes rather long time in initial iterations, since it takes longer time to solve AREs compared with the calculation of checking Condition CB4 stated in the preceding section. Thus, in the 8-periodic case, the performance of this algorithm is not good. Anyway, there is little difference among three types of algorithms and thus any one of these can be used as a practical method to obtain $\gamma_{\min}$.

Chapter 4

The relationship between robust stability of sampled-data systems and the timing of sampling

Our purpose in this chapter is to know how the timing of sampling affects on robust stability of sampled-data systems. Many researchers have investigated the relation between the sampling rate and performances of sampled-data systems, for example, [33], [16], [5], [32], and so on. In contrast, there are not so many researches which treat the timing of sampling itself [51], [45], [7], [8]. These researches tackle the finite horizon case, and the relation between the timing of sampling and performances is unclear in the infinite horizon case. In this chapter, we will investigate an infinite horizon problem of a sampled-data robust stabilization problem, and examine how the timing of sampling affects robust stability of sampled-data systems. We will mainly focus on whether uniform sampling rates give the best performances.

The reason why we treat a sampled-data robust stabilization problem is as follows. A sampled-data robust stabilization problem can be treated as a special class of the discrete H_∞ problem [20]. If the sampling rate is uniform, this problem reduces to the discrete LTI H_∞ problem. Thus in this case, as already mentioned in Chapter 2, time-varying controllers has no advantage over LTI controllers [13, 30, 54]. However, whether time-varying sampling rates have any advantages over uniform sampling rates under a more general setting is not yet known. To clarify this point, we consider a sampled-data robust stabilization problem.

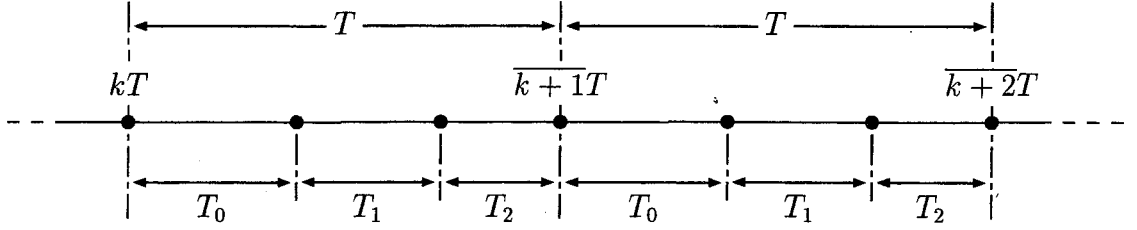


Figure 4.1: A periodically time-varying sampling rate

4.1 The periodically time-varying sampling rate

Since we are to treat an infinite horizon problem and should take into account the infinite number of sampling instants, it is practically impossible to treat the problem if the timing of sampling changes without any restriction. Hence we introduce a certain restriction on the change of the timing of sampling.

One of the simplest and easiest way to do this, which we adopt here, is to vary the sampling rate periodically, which is referred to by Jury, et al. [24] as “a periodically time-varying sampling rate.” Fig. 4.1 shows this situation, in which the sampling rate is 3-periodic. In this case, by changing T_1 and T_2 as parameters, we can study the relation between the timing of sampling and the performances of systems. In the following, assuming that the sampling rate is N -periodic, we denote each sampling interval by $T_0, T_1, \dots, T_{N-1}, T_0, \dots$. Furthermore, the timing of sampling in each frame is denoted by t_k ($k = 0, 1, \dots, N - 1$), which is defined by

$$t_k = \begin{cases} 0 & (k = 0) \\ \sum_{i=0}^{k-1} T_i & (k = 1, 2, \dots, N - 1). \end{cases} \quad (4.1)$$

We also define the frame period T by

$$T = \sum_{k=0}^{N-1} T_k \quad (4.2)$$

and we refer to the interval $[kT, (k+1)T)$ as the k -th frame.

4.2 The sampled-data robust stabilization problem with periodically time-varying sampling rates

In this section, we treat the sampled-data robust stabilization problem with periodically time-varying sampling rates and examine whether uniform sampling rates yield the

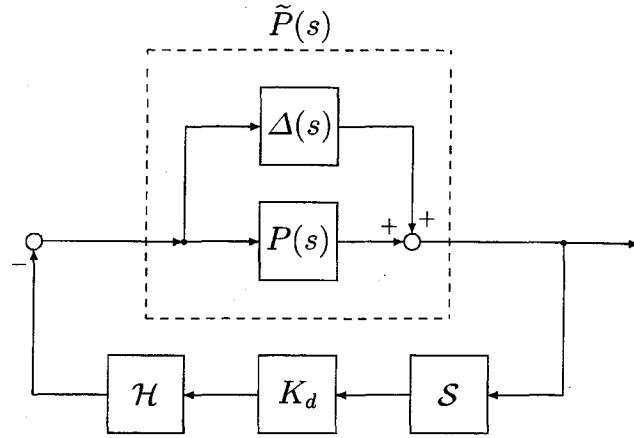


Figure 4.2: The perturbed sampled-data system

best robust stability through numerical examples.

4.2.1 The sampled-data robust stabilization problem and its description as the sampled-data H_∞ problem

In this subsection, we state a sampled-data robust stabilization problem considered throughout this section and describe it as the sampled-data H_∞ problem.

Consider the sampled-data system shown in Fig. 4.2. In this figure, $\tilde{P}(s)$ denotes the continuous-time plant described by

$$\tilde{P}(s) = P(s) + \Delta(s) \quad (4.3)$$

where $P(s)$ is the nominal plant that is strictly proper and has no poles on the imaginary axis, and $\Delta(s)$ is an additive uncertainty. $\mathcal{S}$ and $\mathcal{H}$ respectively denote an ideal sampler and a 0th-order hold circuit that function with periodically time-varying sampling rates, and K_d denotes a discrete LPTV controller. We assume that the uncertainty $\Delta(s)$ is stable and satisfies for some $\bar{\delta}(s)$

$$\bar{\sigma}(\Delta(j\omega)) < |\bar{\delta}(j\omega)|, \quad \forall \omega \quad (4.4)$$

where $\bar{\delta}(s)$ is a stable, strictly proper transfer function that has no unstable zeros, and $\bar{\sigma}(\cdot)$ denotes the maximal singular value.

The robust stabilization problem for the system given in Fig. 4.2 is to find an LPTV controller K_d for given $P(s)$ and $\bar{\delta}(s)$ such that the system is internally stable for any

stable $\Delta(s)$ satisfying (4.4). Here, to know the degree of robust stability of the system, we choose $\bar{\delta}(s)$ as

$$\bar{\delta}(s) = \epsilon \delta(s). \quad (4.5)$$

Then, for given $P(s)$ and $\delta(s)$ we consider the problem to find the largest ϵ ($\epsilon_{\max}$) such that a stabilizing controller K_d exists for $\tilde{P}(s)$ with stable $\Delta(s)$ satisfying (4.4). This $\epsilon_{\max}$ is an index for robust stability.

In order to obtain $\epsilon_{\max}$, we transform the system Fig. 4.2 into an equivalent system Fig. 4.3 [20]. By applying the small gain theorem to this system, we know that the sufficient condition for robust stability of the system is that

$$\|\mathcal{K}(1 + P\mathcal{K})^{-1}\delta\|_{L_2/L_2} < 1/\epsilon \quad (4.6)$$

is satisfied, where $\|\cdot\|_{L_2/L_2}$ denotes the L_2 induced norm, and $\mathcal{K}$ is defined by

$$\mathcal{K} = \mathcal{H}K_d\mathcal{S}. \quad (4.7)$$

By checking the existence of the controller K_d satisfying (4.6), $\epsilon_{\max}$ can be obtained.

Remark 4.1 Since the small gain theorem only gives a sufficient condition for robust stability, (4.6) is conservative. In the case of the uniform sampling rate, a necessary and sufficient condition for robust stability is derived [11, 12, 37]. However, due to theoretical difficulties in application of this condition, sampled-data robust control problems are only studied under some restricted settings [50]. Although (4.6) includes conservativeness as explained above, the following fact validates our study to a great extent. Namely, if we extend the class of uncertainties, (4.6) becomes also necessary. It is shown in [46] that in the case of uniform sampling (the sampling period is T_s), (4.6) is necessary and sufficient for linear, T_s -periodic, causal uncertainties. In our case of periodically time-varying sampling rates, (4.6) becomes a necessary and sufficient condition for robust stability if we extend the class of $\Delta(s)/\delta(s)$ ($\Delta(s)$) to the class of linear causal systems that are periodic with the frame period T .

In order to check existence of the controller K_d satisfying (4.6), we consider the sampled-data H_∞ problem with the generalized plant $G(s)$ chosen as shown in Fig. 4.4:

$$G(s) = \begin{bmatrix} 0 & I \\ \delta(s)I & -P(s) \end{bmatrix}. \quad (4.8)$$

It can be easily verified that if the controller K_d that internally stabilizes the system in Fig. 4.4, and that makes

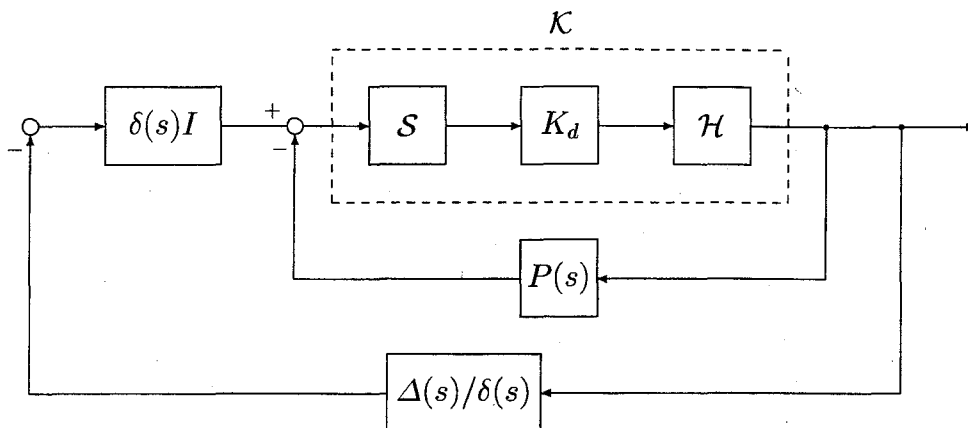


Figure 4.3: An equivalent system

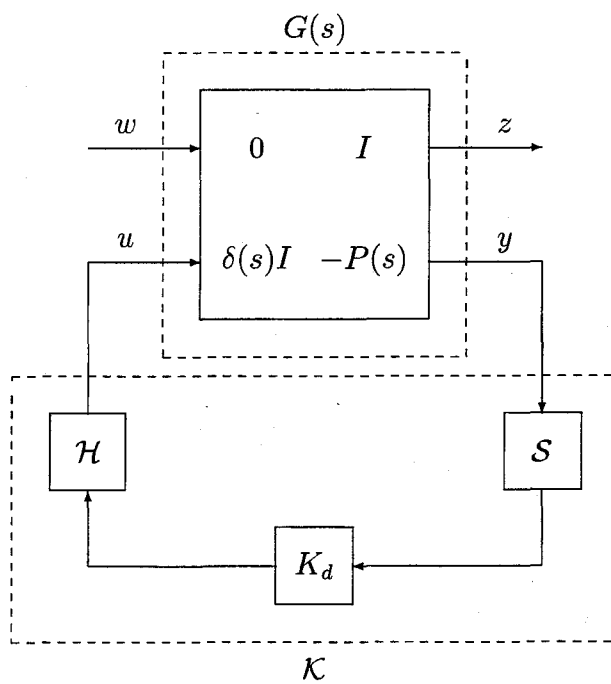


Figure 4.4: Description as the sampled-data H_∞ problem

$$\|\mathcal{F}_l(G, \mathcal{K})\|_{L_2/L_2} = \|\mathcal{F}_l(G, \mathcal{H}K_d\mathcal{S})\|_{L_2/L_2} < 1/\epsilon \quad (4.9)$$

the same controller K_d also robustly stabilizes the system shown in Fig. 4.2 for any uncertainty $\Delta(s)$ satisfying

$$\bar{\sigma}(\Delta(j\omega)) < \epsilon|\delta(j\omega)|. \quad (4.10)$$

Thus $\epsilon_{\max}$ can be obtained from $\gamma_{\min}$, which can be obtained by the γ -iteration.

In the next subsection, we transform this sampled-data H_∞ problem into a norm equivalent discrete H_∞ problem.

4.2.2 A norm equivalent discrete LPTV H_∞ problem

In the preceding subsection, we stated our problem and showed that it can be formulated as the sampled-data H_∞ problem. However, it is not easy to solve this H_∞ problem directly. Therefore, in this subsection we transform this problem into a norm equivalent discrete LPTV H_∞ problem based on the method in [21].

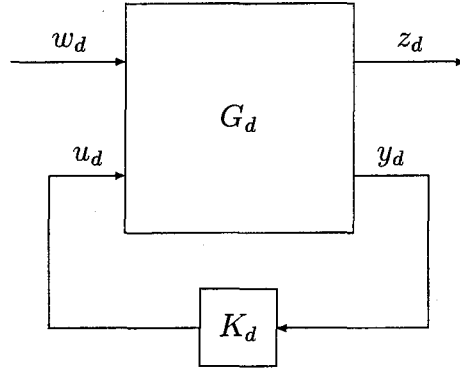


Figure 4.5: A norm equivalent discrete H_∞ problem

Let a state-space realization of $G(s)$ be given by

$$G(s) = \left[\begin{array}{c|cc} A_c & B_{c1} & B_{c2} \\ \hline C_{c1} & 0 & D_{c12} \\ C_{c2} & 0 & 0 \end{array} \right]. \quad (4.11)$$

where $C_{c1}e^{A_c t}B_{c1} = 0$ for all t . Assume that the sampling rate is N -periodic and the frame period is T . Furthermore, let us denote by T_i the interval between i -th sampling and $(i+1)$ -th sampling in a frame. Then the discrete-time generalized plant G_d in Fig. 4.5 satisfying

$$\|\mathcal{F}(G_d, K_d)\| = \|\mathcal{F}(G, \mathcal{H}K_d\mathcal{S})\|_{L_2/L_2} \quad (4.12)$$

can be described as follows:

$$G_d = \left[\begin{array}{c|c} A_d(\cdot) & B_d(\cdot) \\ \hline C_d(\cdot) & D_d(\cdot) \end{array} \right] = \left[\begin{array}{c|cc} A_d(\cdot) & B_{d1}(\cdot) & B_{d2}(\cdot) \\ \hline C_{d1}(\cdot) & 0 & D_{d12}(\cdot) \\ C_{d2}(\cdot) & 0 & 0 \end{array} \right] \quad (4.13)$$

where $A_d(k)$, $B_{d1}(k)$, $B_{d2}(k)$, $C_{d1}(k)$, $C_{d2}(k)$, and $D_{d12}(k)$ for $k = 0, 1, \dots, N-1$ are given by

$$A_d(k) = e^{A_c T_k} \quad (4.14)$$

$$B_{d1}(k) = W_k^{1/2}, \quad W_k = \int_0^{T_k} e^{A_c \tau} B_{c1} B_{c1}^T e^{A_c^T \tau} d\tau \quad (4.15)$$

$$B_{d2}(k) = \int_0^{T_k} e^{A_c \tau} B_{c2} d\tau \quad (4.16)$$

$$\left[\begin{array}{c} C_{d1}^T(k) \\ D_{d12}^T(k) \end{array} \right] \left[\begin{array}{cc} C_{d1}(k) & D_{d12}(k) \end{array} \right] = \int_0^{T_k} U_k^T(\sigma) U_k(\sigma) d\sigma \quad (4.17)$$

$$U_k(\sigma) = \left[\begin{array}{cc} C_{c1} e^{A_c \sigma} & C_{c1} \int_0^\sigma e^{A_c \tau} B_{c2} d\tau + D_{c12} \end{array} \right] \quad (4.18)$$

and,

$$\begin{aligned} A_d(k+N) &= A_d(k), & B_d(k+N) &= B_d(k) \\ C_d(k+N) &= C_d(k), & D_d(k+N) &= D_d(k), \quad \forall k. \end{aligned} \quad (4.19)$$

Now, let us assume that the minimal state space realization of $P(s)$ and $\delta(s)I$ are given by

$$P(s) = \left[\begin{array}{c|c} A_P & B_P \\ \hline C_P & 0 \end{array} \right], \quad \delta(s)I = \left[\begin{array}{c|c} A_\delta & B_\delta \\ \hline C_\delta & 0 \end{array} \right]. \quad (4.20)$$

Then, from (4.8), A_c , B_c , C_c , and D_c are given by

$$A_c = \left[\begin{array}{cc} A_\delta & 0 \\ 0 & A_P \end{array} \right] \quad (4.21)$$

$$B_c = \left[\begin{array}{c|c} B_{c1} & B_{c2} \end{array} \right] = \left[\begin{array}{c|c} B_\delta & 0 \\ 0 & B_P \end{array} \right] \quad (4.22)$$

$$C_c = \left[\begin{array}{c} C_{c1} \\ C_{c2} \end{array} \right] = \left[\begin{array}{cc} 0 & 0 \\ C_\delta & -C_P \end{array} \right] \quad (4.23)$$

$$D_c = \left[\begin{array}{cc} 0 & D_{c12} \\ 0 & 0 \end{array} \right] = \left[\begin{array}{cc} 0 & I \\ 0 & 0 \end{array} \right]. \quad (4.24)$$

Therefore, from (4.14)–(4.18), $A_d(k)$, $B_d(k)$, $C_d(k)$, and $D_d(k)$ for $k = 0, 1, \dots, N-1$ can be rewritten as

$$A_d(k) = \begin{bmatrix} e^{A_\delta T_k} & 0 \\ 0 & e^{A_P T_k} \end{bmatrix} \quad (4.25)$$

$$B_d(k) = \begin{bmatrix} B_{d1}(k) & B_{d2}(k) \end{bmatrix} = \begin{bmatrix} \widehat{W}_k^{1/2} & 0 \\ 0 & \widehat{V}_k \end{bmatrix} \quad (4.26)$$

$$C_d(k) = \begin{bmatrix} C_{d1}(k) \\ C_{d2}(k) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ C_\delta & -C_P \end{bmatrix} \quad (4.27)$$

$$D_d(k) = \begin{bmatrix} 0 & D_{d12}(k) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{T_k} I \\ 0 & 0 \end{bmatrix} \quad (4.28)$$

where $\widehat{W}_k$ and $\widehat{V}_k$ are defined by

$$\widehat{W}_k = \int_0^{T_k} e^{A_\delta \tau} B_\delta B_\delta^T e^{A_\delta^T \tau} d\tau \quad (4.29)$$

$$\widehat{V}_k = \int_0^{T_k} e^{A_P \tau} B_P B_P^T d\tau. \quad (4.30)$$

Since G_d is a discrete-time LPTV (N -periodic) system as shown in (4.25)–(4.28), our problem reduces to the discrete LPTV H_∞ problem, which we have already studied in Chapters 2 and 3. Therefore, $\gamma_{\min}$, i.e., $\epsilon_{\max}$ has some special relation with the timing of sampling. For example, consider the case that the sampling rate is 2-periodic. Let G_{d1} and G_{d2} denote the generalized plants of the norm equivalent discrete H_∞ problems for the cases of $t_1 = \tau$ and $t_1 = T - \tau$, respectively. Then, it can be shown from Lemma 2.2 that $\gamma_{\min}$ is identical for both the cases of $t_1 = \tau$ and $t_1 = T - \tau$, since G_{d1} and G_{d2} satisfy

$$G_{d2} = S_1(G_{d1}). \quad (4.31)$$

This implies that $\epsilon_{\max}$ is symmetric with regard to t_1 at $t_1 = T/2$.

We should note that the discrete LPTV H_∞ problem of G_d is not a standard problem in the sense that it does not satisfy the rank condition for $D_{d21}(k)$ (Assumption APO2). Therefore, we cannot apply the solution methods stated in Chapter 3 directly to our problem. The easiest way to avoid this difficulty is to adopt the method of [43, 31, 36], which was previously mentioned in Section 2.3. More specifically, we consider the modified plant G'_d by adding virtual exogenous inputs to G_d , which is described by

$$G'_d = \left[\begin{array}{c|cc|c} A_d(\cdot) & B_{d1}(\cdot) & 0 & B_{d2}(\cdot) \\ \hline 0 & 0 & 0 & D_{d12}(\cdot) \\ \hline C_{d2}(\cdot) & 0 & \nu I & 0 \end{array} \right] \quad (4.32)$$

where ν is some small positive number. Since this G'_d satisfies all the assumptions needed for the standard problem, the H_∞ problem of G'_d is a standard one. Moreover, any stabilizing controller for G'_d also stabilizes G_d , and, for such controller K_d ,

$$\|\mathcal{F}_l(G'_d, K_d)\| = (1 + f(\nu))\|\mathcal{F}_l(G_d, K_d)\| \quad (4.33)$$

holds, where $f(\cdot)$ is some positive function dependent on G_d and K_d satisfying $f(0) = 0$. Hence, by choosing arbitrarily small ν , $\gamma_{\min}$ for the original problem can be obtained.

4.3 Numerical Examples

In the preceding section, we showed that our problem can be reduced to the discrete LPTV H_∞ problem. In this section, we examine the relation between the timing of sampling and robust stability of the system in several settings. Throughout this section, the frame period T is set to 1(s), and the case that the sampling rate is 2-periodic is considered. In order to check the relation between the timing of sampling and $\epsilon_{\max}$, we calculate $\epsilon_{\max}$ for $t_1 = 0.001kT$ ($k = 1, 2, \dots, 999$). To calculate $\epsilon_{\max} = 1/\gamma_{\min}$, we apply the algorithm based on Lemma 3.4 (Method A) stated in 3.5.2. The interval and the tolerance for the γ -iteration are chosen as $[10^{-6}, 1000]$ and 10^{-6} , respectively. Since we calculate $\epsilon_{\max}$ for many parameter settings, we give up obtaining $\epsilon_{\max}$ to reduce calculation time when the initial interval is not appropriate, that is, $\gamma_{\min} > 1000$ or $\epsilon_{\max} < 0.001$.

Example 1

First, we consider the simplest case that $P(s)$ and $\delta(s)$ are given by

$$P(s) = \frac{p_0}{s + p_0}, \quad \delta(s) = \frac{d_0}{s + d_0}. \quad (4.34)$$

We change the parameters p_0 and d_0 , and for each setting of the parameters, we calculate t_1 that yields the best robust stability, i.e., t_1 that yields largest $\epsilon_{\max}$. The parameters p_0 and d_0 are changed as given in Table 4.1.

The results are shown in Figs. 4.6, and 4.7. In Fig. 4.6, $\circ$ denotes a parameter setting for which the uniform sampling rate yields largest $\epsilon_{\max}$, and $+$ denotes a parameter setting for which $\epsilon_{\max} < 0.001$. Fig. 4.7, for which the parameters are selected as $p_0 = -5$ and $d_0 = 20$, demonstrates a typical relation between t_1 and $\epsilon_{\max}$. From Fig. 4.6, we can see that for all parameter settings except those for which $\epsilon_{\max}$ is not obtained, the uniform sampling rate always gives the best robust stability.

Example 2

Next, to see the effects of complex poles of $\delta(s)$, we consider the case that $P(s)$ and $\delta(s)$ are given by

$$P(s) = \frac{p_0}{s + p_0}, \quad \delta(s) = \frac{d_0}{s^2 + d_1s + d_0}. \quad (4.35)$$

The parameters p_0 , d_0 , d_1 are changed as given in Table 4.2.

In this case, as Example 1, the uniform sampling rate gives largest $\epsilon_{\max}$ whenever $\epsilon_{\max}$ is obtained. This fact implies that it does not matter whether $\delta(s)$ has complex poles or not, at least when $P(s)$ has the form given by (4.35).

Example 3

To see the effect of complex poles of the plant, we consider $P(s)$ and $\delta(s)$ given by

$$P(s) = \frac{p_0}{s^2 + p_1s + p_0}, \quad \delta(s) = \frac{d_0}{s + d_0}. \quad (4.36)$$

The parameters p_0 , p_1 , d_0 are changed as given in Table 4.3.

In Fig. 4.8, 4.9, $\circ$ and $+$ stand for the same cases with those in Fig. 4.6, and $\bullet$ stands for the cases that the uniform sampling rate *does not* yield largest $\epsilon_{\max}$. The solid curve is a boundary of parameters for which the plant $P(s)$ has complex poles. More specifically, the plant $P(s)$ has complex poles for p_0 above this curve. For p_0 and p_1 on the dotted curve, the discretized plant of $P(s)$ becomes unstabilizable and undetectable with the (uniform) sampling period $T/2 = 0.5$. That is, for p_0 , and p_1 on the dotted curve, $P(s)$ has complex poles whose imaginary parts are $\pm j2\pi$. Note that these figures only demonstrate the results for positive p_0 , while numerical calculation is also done for negative p_0 . However, for such p_0 (the poles of the plant become real) the uniform sampling rate always gives largest $\epsilon_{\max}$ whenever $\epsilon_{\max}$ is obtained, and so is omitted.

From Figs. 4.8 and 4.9, we can observe the following numerical results:

- For $P(s)$ which has complex poles, the uniform sampling rate does not always give largest $\epsilon_{\max}$.
- It does not affect this situation whether the pole of $\delta(s)$ is -10 ($d_0 = 10$) or -1 ($d_0 = 1$).
- periodically time-varying sampling rates give largest $\epsilon_{\max}$ for the plants $P(s)$ that has complex poles whose imaginary parts are around $\pm j2\pi$.

Fig. 4.10 is an illustrative example how the relation between t_1 and $\epsilon_{\max}$ changes as imaginary parts of complex poles of the plant approach to $\pm j2\pi$. In Fig. 4.10(a), the case that $P(s)$ has no complex poles, $\epsilon_{\max}$ monotonically becomes larger as t_1 approaches 0 or 1. In Fig. 4.10(b), the case that $P(s)$ has complex poles $5 \pm j\sqrt{15}$, $\epsilon_{\max}$ has local minimums around $t_1 = 0.1, 0.9$. These minimums seem to correspond to the (uniform) sampling period $\pi/\sqrt{15} \simeq 0.81$ with which the discretized plant of $P(s)$ becomes unstabilizable and undetectable, although unstabilizability and undetectability at this uniform sampling rate does not necessarily imply unstabilizability or undetectability of the discretized system with the corresponding periodically sampling rate. Local minimums also appear in Fig. 4.10(c), the case that $P(s)$ has complex poles $5 \pm j\sqrt{35}$. They correspond to the (uniform) sampling period $\pi/\sqrt{35} \simeq 0.53$ with which the discretized plant of $P(s)$ becomes unstabilizable and undetectable. The shape of the line in Fig. 4.10(c) is not so smooth as those in Fig. 4.10(a) or 4.10(b), and the peaks appear around $t_1 = 0.35$, not at $t_1 = 0.5$.

Example 4

Last, to see the influence of zeros of the plant, we consider $P(s)$ and $\delta(s)$ such as

$$P(s) = \frac{p_0}{p_2} \cdot \frac{(s + p_2)}{s^2 + p_1s + p_0}, \quad \delta(s) = \frac{d_0}{s + d_0}. \quad (4.37)$$

The parameters p_0, p_1, p_2, d_0 are changed as given in Table 4.4.

The result for $p_2 = -1$ is shown in Fig. 4.11. For $p_2 = -10, 1, 10$, the results do not differ very much from those of Example 3 (Fig. 4.8), and so are omitted. However, as Fig. 4.11 shows, when p_2 is -1 , that is, when the plant has an unstable zero near the imaginary axis, periodically time-varying sampling rates give largest $\epsilon_{\max}$ for almost every parameter that makes the poles of $P(s)$ complex.

A typical relation between t_1 and $\epsilon_{\max}$ is as shown in Fig. 4.12. In this figure, the parameters are the same with those in Fig. 4.10, except that $P(s)$ has an unstable zero at 1. By comparing Figs. 4.12 and 4.10, two peaks appear near $t_1 = 0, 1.0$ when $P(s)$ has an unstable zero and complex poles. Although we have no qualitative explanation for the relation between these peaks and unstable zeros of the plant, at least it is observed that if the plant does not have complex poles, the uniform sampling rate yields largest $\epsilon_{\max}$ whether $P(s)$ has unstable zeros or not.

Table 4.1: Parameter settings of p_0 and d_0 in Example 1

p_0	integer, $-15 \leq p_0 \leq -1$
d_0	integer, $1 \leq d_0 \leq 100$

Table 4.2: Parameter settings of p_0 , d_0 , d_1 in Example 2

p_0	integer, $-15 \leq p_0 \leq -1$
d_0	integer, $1 \leq d_0 \leq 50$
d_1	integer, $1 \leq d_1 \leq 20$

Table 4.3: Parameter settings of p_0 , p_1 , d_0 in Example 3

p_0	integer, $-100 \leq p_0 \leq -1, 1 \leq p_0 \leq 100$
p_1	integer, $-15 \leq p_1 \leq -1$
d_0	$d_0 = 1, 10$

Table 4.4: Parameter settings of p_0 , p_1 , p_2 , d_0 in Example 4

p_0	integer, $1 \leq p_0 \leq 100$
p_1	integer, $-15 \leq p_1 \leq -1$
p_2	$p_2 = -10, -1, 1, 10$
d_0	$d_0 = 1$

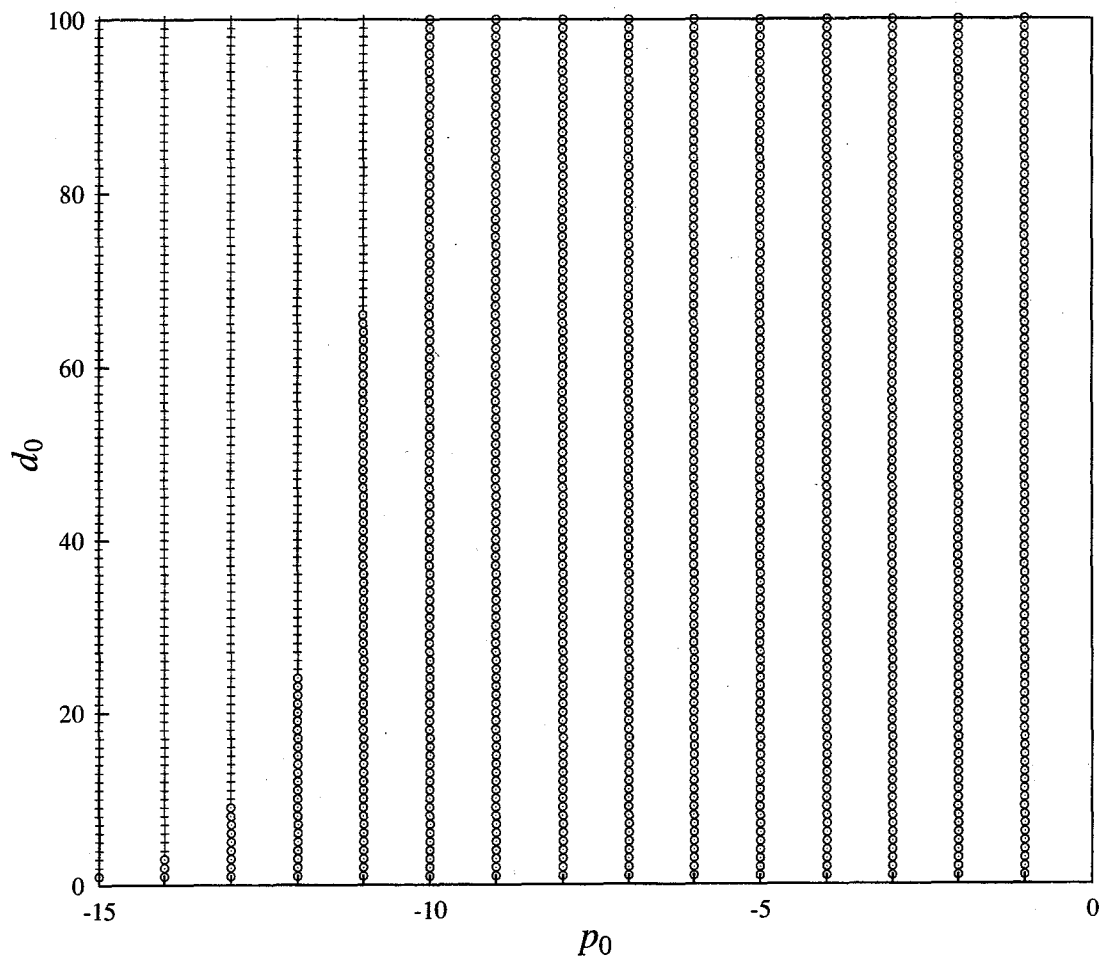


Figure 4.6: Results of Example 1: parameter settings for which the uniform sampling rate yields the best robust stability

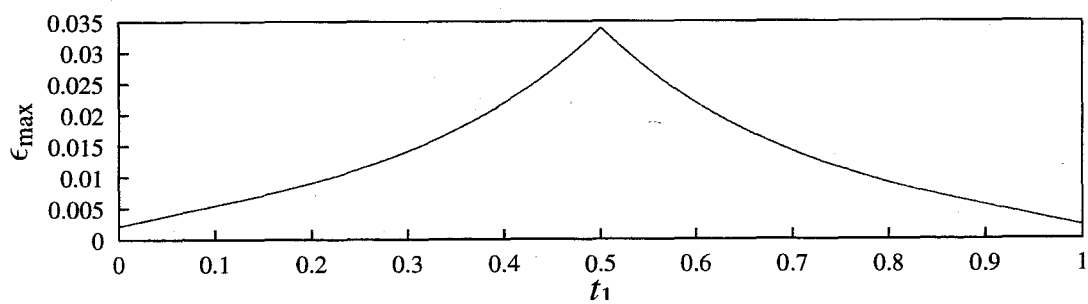


Figure 4.7: Results of Example 1: the relation between t_1 and $\epsilon_{\max}$ for $p_0 = -5$, $d_0 = 20$

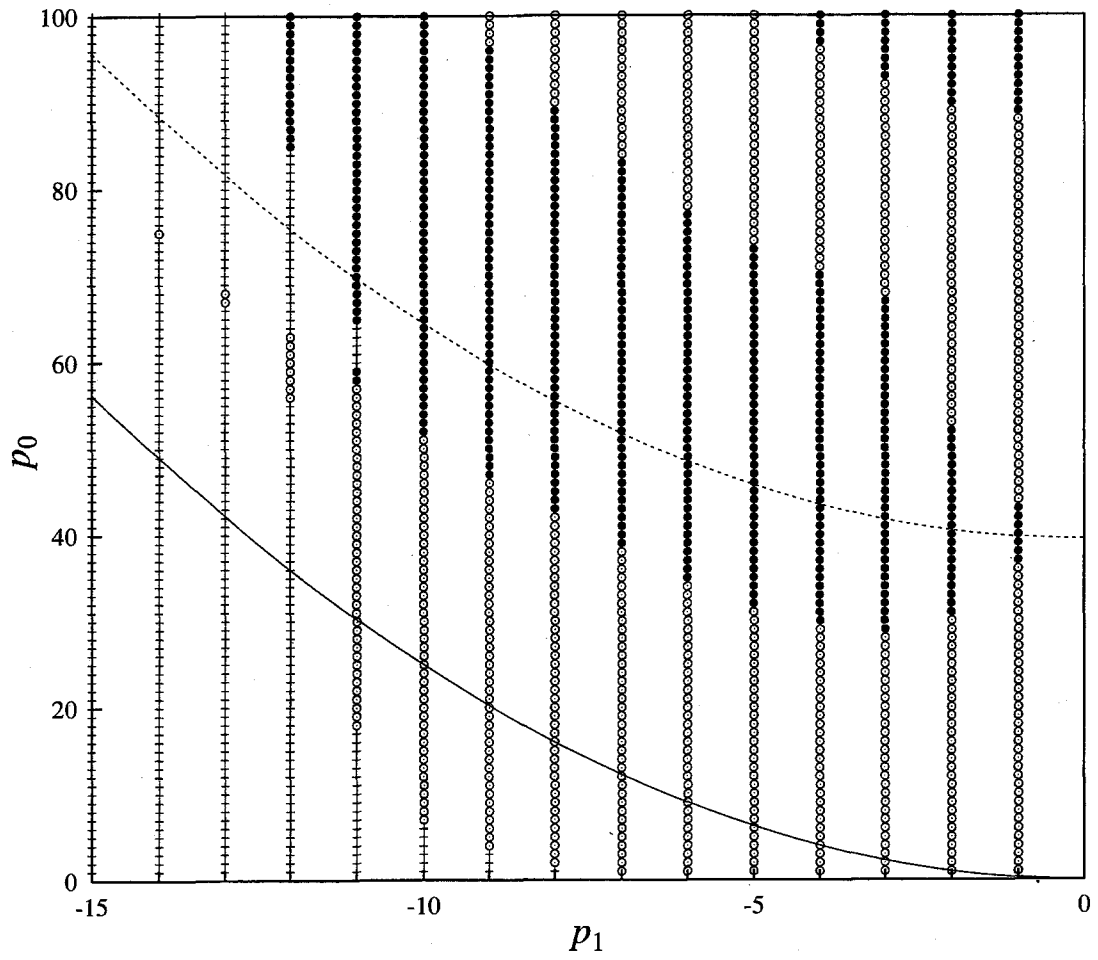


Figure 4.8: Results of Example 3: parameter settings for which the uniform sampling rate yields the best robust stability ($d_0 = 1$)

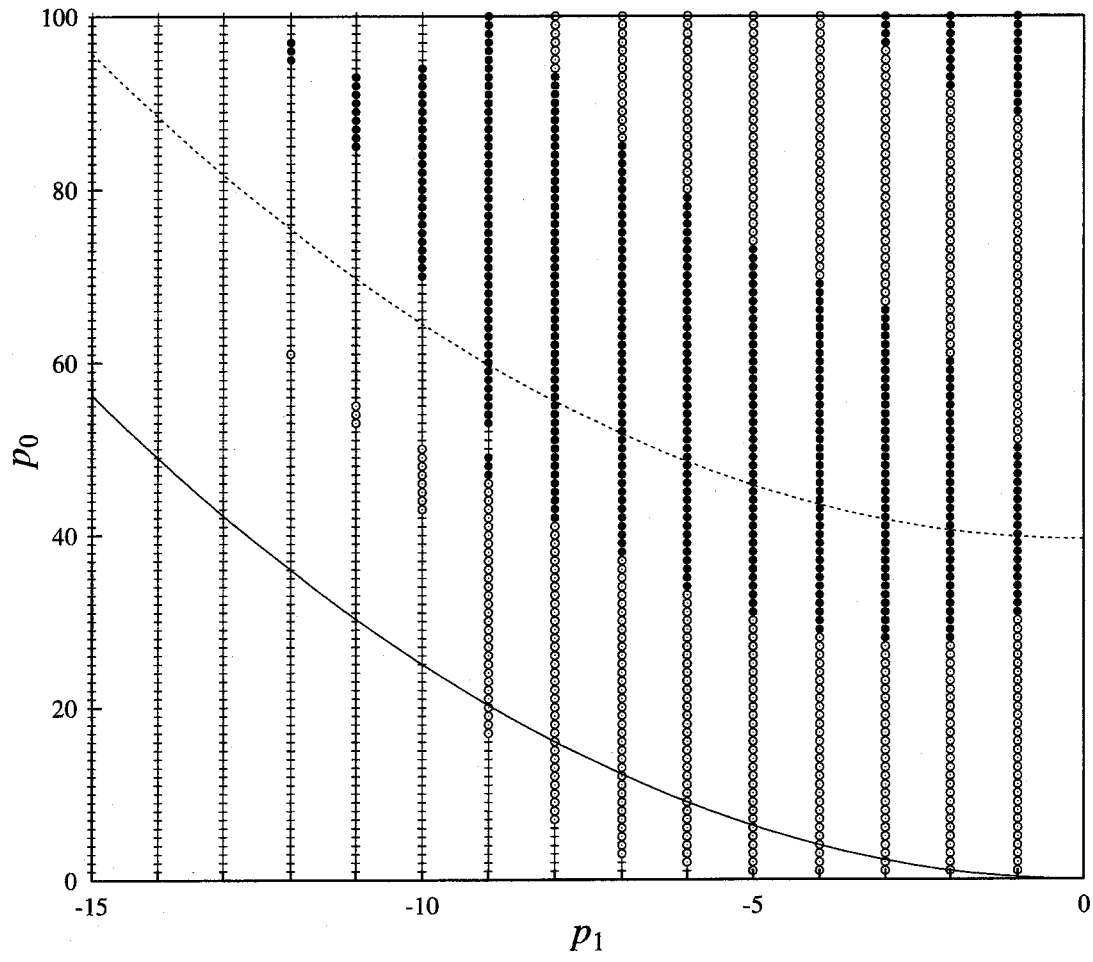
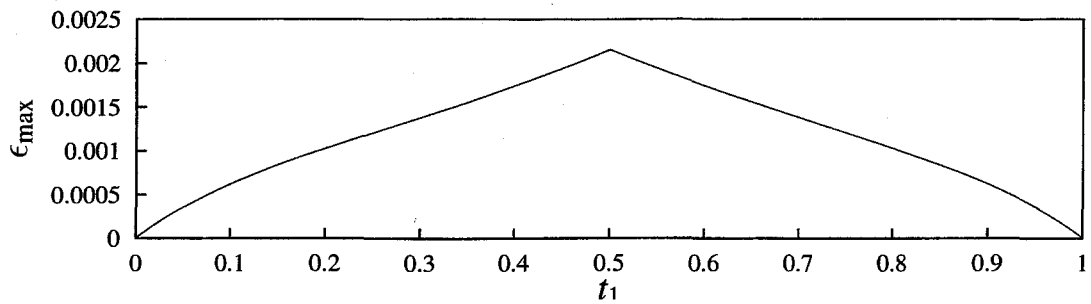
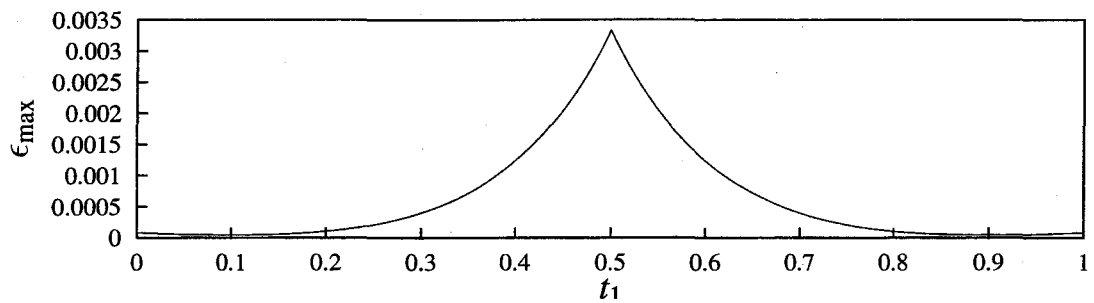


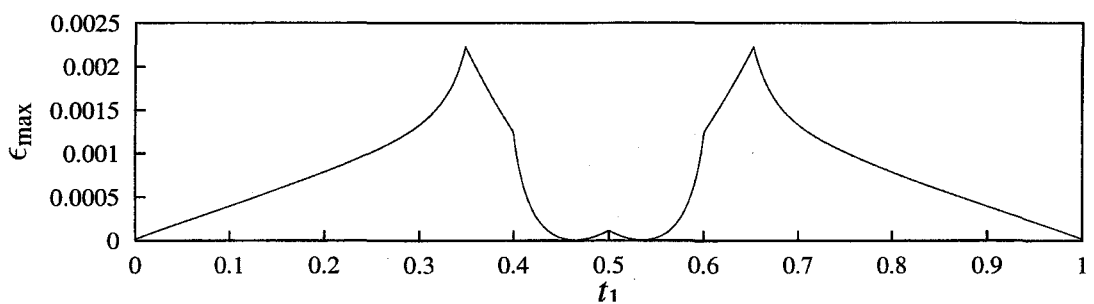
Figure 4.9: Results of Example 3: parameter settings for which the uniform sampling rate yields the best robust stability ($d_0 = 10$)



(a) $p_0 = 20$ (the poles of $P(s)$ are $5 \pm \sqrt{5}$)



(b) $p_0 = 40$ (the poles of $P(s)$ are $5 \pm j\sqrt{15}$)



(c) $p_0 = 60$ (the poles of $P(s)$ are $5 \pm j\sqrt{35}$)

Figure 4.10: Results of Example 3: the relation between t_1 and $\epsilon_{\max}$ ($p_1 = -10$, $d_0 = 1$)

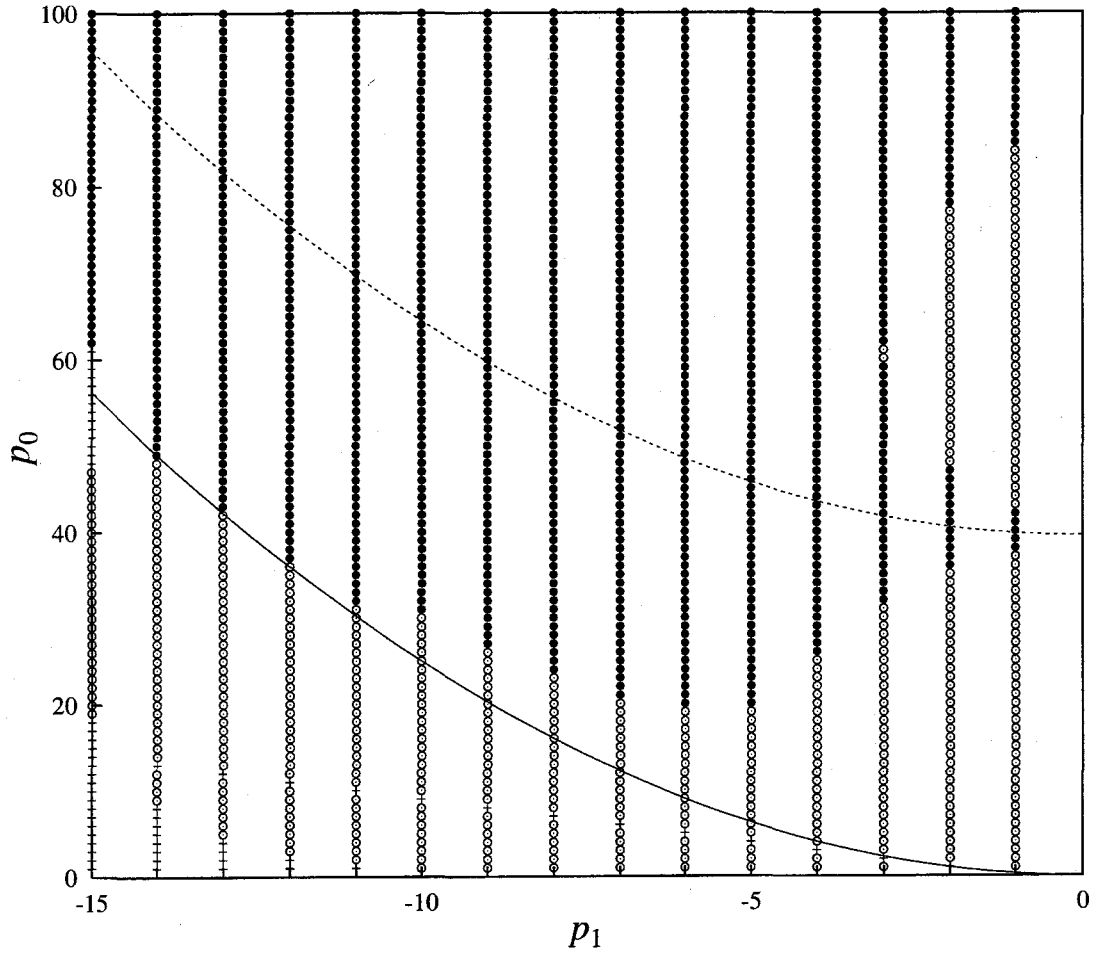
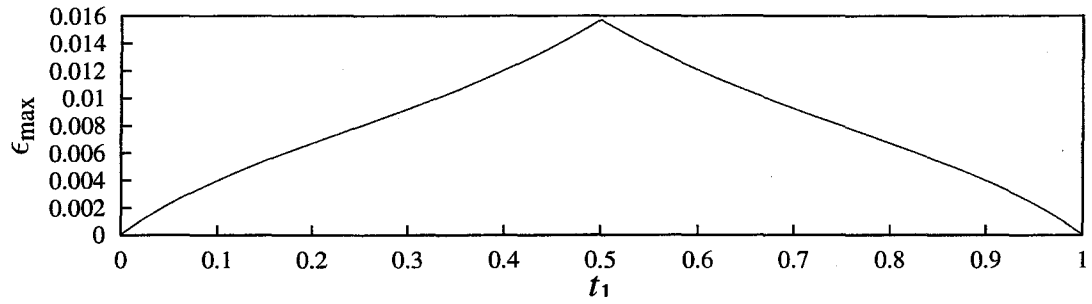
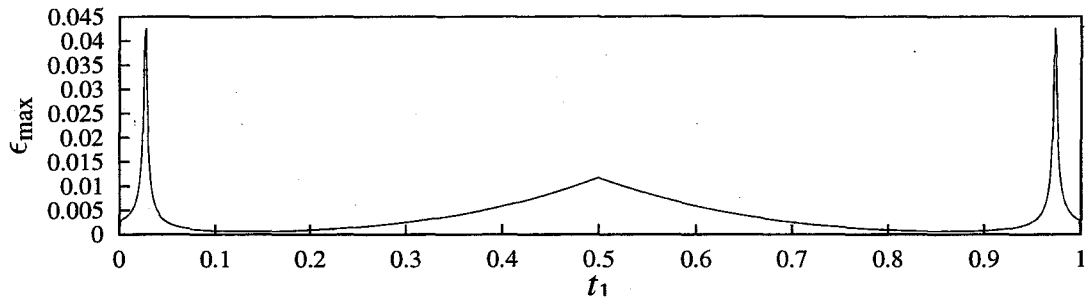


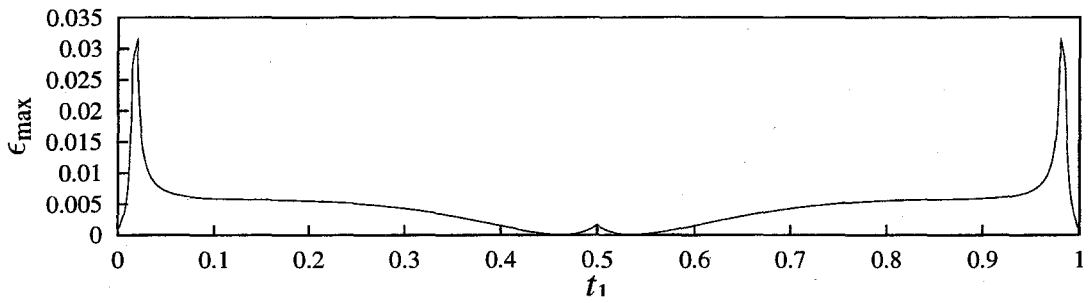
Figure 4.11: Results of Example 4: parameter settings for which the uniform sampling rate yields the best robust stability ($p_2 = -1$)



(a) $p_0 = 20$ (the poles of $P(s)$ are $5 \pm \sqrt{5}$)



(b) $p_0 = 40$ (the poles of $P(s)$ are $5 \pm j\sqrt{15}$)



(c) $p_0 = 60$ (the poles of $P(s)$ are $5 \pm j\sqrt{35}$)

Figure 4.12: Results of Example 4: the relation between t_1 and $\epsilon_{\max}$ ($p_1 = -10$, $p_2 = -1$, $d_0 = 1$)

4.4 Concluding remarks

In this chapter, we have investigated the relation between the timing of sampling and robust stability of the system by mainly focusing on whether uniform sampling gives the best robust stability. From numerical examples, we have observed the following results.

- If the continuous-time plant has no complex poles, the uniform sampling rate gives the best robust stability.
- If the continuous-time plant has complex poles, the uniform sampling rate does not always give the best robust stability. The situation varies whether the plant has unstable zeros near the imaginary axis. If the plant has such zeros, periodically time-varying sampling rates almost always give the best robust stability. If not, periodically time-varying sampling rates give the best robust stability when the complex poles of the plant are such that the discretized plant with the (uniform) sampling period around $T/2$ (2-periodic case) becomes unstabilizable and undetectable.

These results imply the possibilities to improve robust stability by using periodically time-varying sampling rates. However, there are no guidelines yet on the choice of the timing of sampling, although the numerical examples indicate that it may have some relationship with complex poles and unstable zeros of the plant. Theoretical analysis of this relationship is so far left as an open problem.

Chapter 5

The filtering problem of unreliable sampling systems

In this chapter we will give a minimum-variance estimate filter for unreliable sampling systems. We first give a system description of unreliable sampling systems, and then we derive a minimum-variance estimate filter for such systems. Next, we derive a stability condition of this filter and convert it into such a form that it is explicitly connected with the degree of the unreliability of sampling. Finally, we show through numerical examples the effectiveness of this filter in the context of control.

5.1 Unreliable sampling systems

In this section, we give a state expression of the systems under the setting of “unreliable sampling.” Unreliable sampling systems include multirate systems as special cases in the sense that any multirate systems can be described by using the state expression of unreliable sampling systems. We show this relationship through examples.

5.1.1 The expression for systems under unreliable sampling

As explained in Introduction, we use the term “unreliable sampling” to describe a situation of sampled-data systems in which the measurement of the outputs fails occasionally at some sampling instants. Fig. 5.1 is an illustrative example of this situation. Under this situation, systems are expressed in such a way that the dimension of the outputs varies with time. Here we give such an expression.

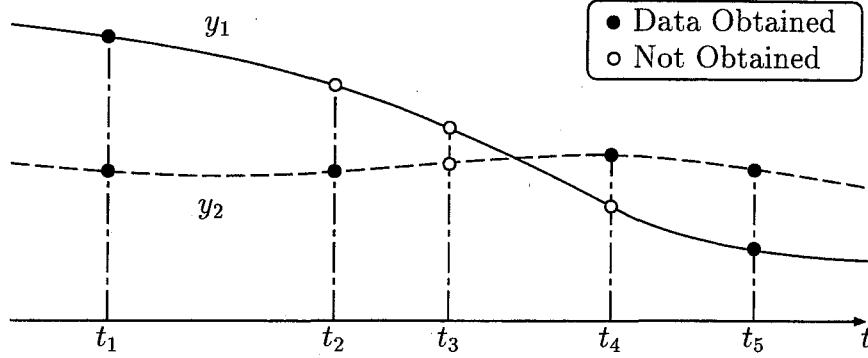


Figure 5.1: An example of unreliable sampling

Consider the plant

$$\begin{aligned} \frac{dx_c(t)}{dt} &= A_c x_c(t) + B_c u_c(t) + G_c w_c(t) \\ y_c(t) &= C_c x_c(t) \end{aligned} \quad (5.1)$$

where x_c is an n -dimensional state vector, u_c is an m -dimensional input vector, y_c is a p -dimensional output vector, and w_c is a q -dimensional Gaussian white-noise process with zero mean and covariance matrix

$$E\{w_c(t)w_c^T(s)\} = Q_c \delta(t-s) \quad (5.2)$$

where $\delta(\tau)$ is the Dirac delta function. The initial state $x_c(t_0)$ is also a Gaussian random vector with mean $\bar{x}_0$ and covariance matrix Σ_0 . The pair (C_c, A_c) is assumed to be observable.

Suppose that we are to sample the plant outputs at t_k :

$$0 = t_0 < t_1 < \dots < t_k < \dots \quad (5.3)$$

and to hold the plant inputs for the interval of $[t_k, t_{k+1})$, so that

$$u_c(t) = u_c(t_k), \quad t_k \leq t < t_{k+1} \quad k = 0, 1, 2, \dots \quad (5.4)$$

Let x_k, u_k stand for $x(t_k), u_c(t_k)$, respectively, and $\Phi(t_i, t_j)$ be the state transition matrix from t_j to t_i given by

$$\Phi(t_i, t_j) = e^{A_c(t_i - t_j)}. \quad (5.5)$$

Then the state transition between x_k and x_{k+1} is described by

$$x_{k+1} = A_k x_k + B_k u_k + w_k \quad (5.6)$$

where A_k , B_k and w_k are

$$A_k = \Phi(t_{k+1}, t_k) \quad (5.7)$$

$$B_k = \int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, t) B_c dt \quad (5.8)$$

$$w_k = \int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, t) G_c w_c(t) dt. \quad (5.9)$$

The above w_k is a Gaussian white-noise vector sequence with zero mean and covariance matrix

$$\begin{aligned} E\{w_k w_l^T\} &= Q_k \delta_{kl} \\ Q_k &= \int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, t) G_c Q_c G_c^T \Phi^T(t_{k+1}, t) dt \end{aligned} \quad (5.10)$$

where δ_{kl} is the Kronecker delta.

Suppose that the sampled outputs at the k th sampling instant, when all the measurement of the outputs is successful, are given by

$$y_k = y_c(t_k) + v_k = Cx_k + v_k. \quad (5.11)$$

Here, $C = C_c$, and v_k ($k = 0, 1, \dots$) is the measurement noise that is a Gaussian white-noise vector sequence uncorrelated with w_k ($k = 0, 1, \dots$) and $x_c(t_0)$, and has zero mean and covariance matrix

$$E\{v_k v_l^T\} = R_k \delta_{kl}, \quad R_k > 0. \quad (5.12)$$

Under unreliable sampling, however, the measurement of some outputs might fail. Suppose that only l_k ($0 \leq l_k \leq p$) entries of the p outputs are obtained successfully at the k th sampling instant, and let y_k^u be an l_k -dimensional vector consisting of the obtained measurement data (if $l_k = 0$, then we need not define y_k^u). Then, y_k^u can be related to y_k with some appropriate matrix E_k with full row rank by $y_k^u = E_k y_k$. From (5.11), it follows that

$$y_k^u = E_k y_k = E_k C x_k + E_k v_k. \quad (5.13)$$

Hence the system resulting from unreliable sampling of the plant (5.1) can be described by

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k + w_k \\ y_k^u &= E_k C x_k + E_k v_k. \end{aligned} \quad (5.14)$$

Remark 5.1 Our standing assumption in the following is that E_k is not necessarily known for future time instants t_k , but is known for the present and past time instants. This assumption reflects the situation in which we can immediately know the fact that the measurement data were lost.

5.1.2 The relationship between unreliable sampling systems and multirate systems

Although the state expression (5.14) is intended for unreliable sampling systems, we can treat multirate systems by this expression, too. From this point of view, it can be said that multirate systems are special cases of unreliable sampling systems. Hence the arguments in the following are also valid for multirate systems, namely, the minimum-variance linear estimate filter that we will derive in the next section can be implemented for multirate systems without any modifications.

As a matter of course, there is a difference between unreliable sampling systems and multirate systems. In multirate systems it is deterministic and periodic which outputs are measured at each sampling instant, i.e., E_k becomes deterministic and periodic. This difference will become clear through the following examples that indicate how multirate systems are described by (5.14).

Consider the continuous-time system (5.1) with three outputs ($p = 3$), and let the sampling periods of y_1 , y_2 , and y_3 be $1(s)$, $2(s)$, and $3(s)$, respectively. Furthermore, let us assume for simplicity that the system has no inputs ($u_c(t) = 0$). By setting $t_k = k$, this multirate system can be described by (5.14) with the term u_k eliminated. In this case, E_k becomes

$$E_k = \begin{cases} [e_1^T, e_2^T, e_3^T]^T = I & (k = 6i) \\ [e_1^T, e_2^T]^T & (k = 6i + 2, 6i + 4) \\ [e_1^T, e_3^T]^T & (k = 6i + 3) \\ e_1 & (k = 6i + 1, 6i + 5) \end{cases} \quad (5.15)$$

where i is any integer, and e_1 , e_2 , and e_3 are row unit vectors defined by

$$e_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}. \quad (5.16)$$

Although we set $u_c(t)$ to zero in the above example, we can treat the case $u_c(t) \neq 0$ similarly if the change of the inputs $u_c(t)$ and the measurement of the outputs $y_c(t)$ are synchronous: all the changes of the inputs $u_c(t)$ are done at sampling instants of the outputs $y_c(t)$. If they are asynchronous, however, we need to add the sampling instants corresponding to the change of the outputs in order to describe the system by (5.14). For example, if $u_c(t)$ is changed at $t = 1.5k$ ($k = 0, 1, 2, \dots$), we need to take into consideration the additional time instants $6i + 1.5$ and $6i + 4.5$. In this case, no outputs are obtained at $t_k = 6i + 1.5$ and $t_k = 6i + 4.5$, and E_k is not defined for such k .

5.2 A minimum-variance linear estimate filter for unreliable sampling systems

In this section, we derive a minimum-variance linear estimate filter for unreliable sampling systems as natural extension of an ordinary Kalman filtering algorithm [26, 28, 6] for systems in which all the measurement of the outputs can be obtained. Then, we show a condition to assure asymptotic stability of (the homogeneous part of) this filter by a similar argument taken for the ordinary Kalman filter in, for example, [9]. Last, we show this stability condition can be converted into a form that is connected with the unreliability of sampling.

5.2.1 The filtering algorithm under unreliable sampling

In the preceding section, we showed a basic state expression for systems under unreliable sampling. Here, we give a minimum-variance linear estimate filter for such systems.

First, we consider the ideal case that all the measurement of the outputs is successful at any sampling instants. Let us assume that u_k is a function of $(y_0, y_1, \dots, y_k)$. Then the minimum-variance linear estimate $\hat{x}_{i|j}$ of the state x_i based on the information $(y_0, y_1, \dots, y_j)$ can be obtained by the well-known Kalman filtering algorithm:

$$\hat{x}_{k+1|k} = A_k \hat{x}_{k|k} + B_k u_k \quad (5.17a)$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k (y_k - C \hat{x}_{k|k-1}) \quad (5.17b)$$

$$K_k = P_{k|k-1} C^T [C P_{k|k-1} C^T + R_k]^{-1} \quad (5.17c)$$

$$P_{k+1|k} = A_k P_{k|k} A_k^T + Q_k \quad (5.17d)$$

$$P_{k|k} = P_{k|k-1} - K_k C P_{k|k-1} \quad (5.17e)$$

where $P_{i|j}$ denotes the error covariance matrix of the estimate $\hat{x}_{i|j}$ and the initial condition is

$$\hat{x}_{0|-1} = \bar{x}_0, \quad P_{0|-1} = \Sigma_0. \quad (5.18)$$

In the case of unreliable sampling, the filter to obtain the minimum-variance linear estimate $\hat{x}_{i|j}^u$ based on the information $(y_0^u, y_1^u, \dots, y_j^u)$ can be described by slightly modifying the ordinary Kalman filtering algorithm. Let $P_{i|j}^u$ denote the error covariance matrix of the estimate $\hat{x}_{i|j}^u$ and assume that u_k in (5.14) is a function of $(y_0^u, y_1^u, \dots, y_k^u)$. Then, the filter is given by

$$\hat{x}_{k+1|k}^u = A_k \hat{x}_{k|k}^u + B_k u_k \quad (5.19a)$$

$$\hat{x}_{k|k}^u = \hat{x}_{k|k-1}^u + K'_k(y_k - C\hat{x}_{k|k-1}^u) \quad (5.19b)$$

$$K'_k := \begin{cases} K_k^u E_k & (l_k \neq 0) \\ 0 & (l_k = 0) \end{cases} \quad (5.19c)$$

$$K_k^u = P_{k|k-1}^u C^T E_k^T [E_k (C P_{k|k-1}^u C^T + R_k) E_k^T]^{-1} \quad (5.19d)$$

$$P_{k+1|k}^u = A_k P_{k|k}^u A_k^T + Q_k \quad (5.19e)$$

$$P_{k|k}^u = P_{k|k-1}^u - K'_k C P_{k|k-1}^u \quad (5.19f)$$

where the initial condition is

$$\hat{x}_{0|-1}^u = \bar{x}_0, \quad P_{0|-1}^u = \Sigma_0. \quad (5.20)$$

The difference between this filter and the ordinary Kalman filter is that when no outputs are measured successfully, this filter only performs prediction of the states because information does not increase at all, then.

Remark 5.2 For the sake of brevity in description, the filtering algorithm (5.19) is described by using y_k , which is not obtainable actually. Of course, this algorithm only requires successfully measured outputs y_k^u in practice: note that K'_k has the factor E_k .

In the following two subsections, we will discuss stability of this filter (5.19).

5.2.2 Stability of the filter

We consider stability of the homogeneous part of the filter (5.19), i.e.,

$$\hat{x}_{k+1|k}^u = A_k \hat{x}_{k|k}^u \quad (5.21a)$$

$$\hat{x}_{k|k}^u = (I - K'_k C) \hat{x}_{k|k-1}^u. \quad (5.21b)$$

Our concern is to derive a stability condition of the filter in connection with the degree of the unreliability of the sampling. To do this, we first derive a stability condition of our filter that has no explicit connection with the degree of the unreliability in this subsection. Then, in the next subsection, we will rewrite it into a form explicitly connected with the degree of unreliability.

The stability condition of the ordinary Kalman filter is already developed in, for example, [9]. Applying a similar method to our filter (5.19) we can obtain the following lemma. Here, the system (5.21) is called “uniformly asymptotically stable in the large” if the following three conditions are satisfied [25].

- (a) For any number $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that if the initial state $x_{i|i-1}$ at i (i : arbitrary nonnegative integer) satisfies

$$\|x_{i|i-1}\| \leq \delta(\epsilon) \quad (5.22)$$

then, for any $k \geq i$

$$\|x_{k|k-1}\| \leq \epsilon. \quad (5.23)$$

- (b) For any number $r > 0$ and $\nu > 0$ there exists a nonnegative integer j such that if the initial state $x_{i|i-1}$ at i (i : arbitrary nonnegative integer) satisfies

$$\|x_{i|i-1}\| \leq r \quad (5.24)$$

then, for any $k \geq i + j$

$$\|x_{k|k-1}\| \leq \nu. \quad (5.25)$$

- (c) For any number $\eta > 0$ there exists $\xi(\eta) > 0$ such that if the initial state $x_{i|i-1}$ at i (i : arbitrary nonnegative integer) satisfies

$$\|x_{i|i-1}\| \leq \eta \quad (5.26)$$

then, for any $k \geq i$

$$\|x_{k|k-1}\| \leq \xi(\eta). \quad (5.27)$$

Lemma 5.1 The homogeneous part (5.21) of the filter (5.19) is uniformly asymptotically stable in the large if the following two conditions are satisfied.

$$\text{CF1} \quad Q_k > 0, \quad \forall k \quad (5.28)$$

CF2 There exists a positive integer L such that

$$\bar{R}_{Lk} = \sum_{i=k-L}^{k-1} \Phi^T(t_i, t_k) C^T M_i C \Phi(t_i, t_k) > 0, \quad \forall k \geq L \quad (5.29)$$

$$M_i = \begin{cases} E_i^T (E_i R_k E_i^T)^{-1} E_i & (l_i \neq 0) \\ 0 & (l_i = 0) \end{cases} \quad (5.30)$$

Proof of Lemma 5.1. The outline of the proof is as follows.

According to Theorem 1 of [25], the system (5.21) is uniformly asymptotically stable in the large if there exists a Lyapunov function $V(x_k, k)$ such that

(a) $V(x_k, k)$ satisfies

$$0 < \gamma_1(\|x_k\|) \leq V(x_k, k) \leq \gamma_2(\|x_k\|), \quad \forall x_k \neq 0 \quad (5.31)$$

for some continuous, nondecreasing scalar functions $\gamma_1(\|x_k\|)$ and $\gamma_2(\|x_k\|)$ satisfying

$$\gamma_1(0) = \gamma_2(0) = 0, \quad \lim_{\rho \rightarrow \infty} \gamma_1(\rho) = \infty, \quad (5.32)$$

(b) $V(\hat{x}_{k|k-1}^u, k)$ satisfies for some finite positive integer L

$$V(\hat{x}_{k|k-1}^u, k) - V(\hat{x}_{k-L|k-L-1}^u, k-L) \leq -\gamma_3(\|\hat{x}_{k-L|k-L-1}^u\|) < 0, \quad \forall \hat{x}_{k-L|k-L-1}^u \neq 0 \quad (5.33)$$

where $\gamma_3(\|x_k\|)$ is a continuous scalar function satisfying

$$\gamma_3(0) = 0, \quad \lim_{\rho \rightarrow \infty} \gamma_3(\rho) = \infty \quad (5.34)$$

For the system (5.21), let us show that $V(x_k, k)$ defined by

$$V(x_k, k) = x_k^T (P_{k|k-1}^u)^{-1} x_k. \quad (5.35)$$

is one such Lyapunov function. To this end, it is enough to show that under Conditions CF1 and CF2,

$$0 < \alpha_1 I \leq (P_{k|k-1}^u)^{-1} \leq \alpha_2 I < \infty \quad (5.36)$$

and

$$V(\hat{x}_{k|k-1}^u, k) - V(\hat{x}_{k-L|k-L-1}^u, k-L) \leq -\alpha_3 \|\hat{x}_{k-L|k-L-1}^u\|^2 < 0 \quad (5.37)$$

hold for some positive numbers α_1 , α_2 , and α_3 .

In the following, without loss of generality we assume $P_{0|-1} > 0$: since Condition CF1 yields

$$P_{1|0} > 0 \quad (5.38)$$

even if $P_{0|-1}$ is only nonnegative, the arguments taken in the following remain valid if we regard t_1 as an initial time.

First, we show (5.36) under Conditions CF1 and CF2. From the results of [9], in the case of the ordinary Kalman filter (5.17), the error covariance matrix $P_{k|k-1}$ satisfies

$$0 < \beta_1 I \leq P_{k|k-1}^{-1} \leq \beta_2 I < \infty \quad (5.39)$$

for all k , if $Q_k > 0$, and

$$\tilde{R}_L = \sum_{i=k-L}^{k-1} \Phi^T(t_i, t_k) C^T R_k^{-1} C \Phi(t_i, t_k) > 0 \quad (5.40)$$

holds for some positive integer L . Since for all k

$$R_k^{-1} \geq M_k \quad (5.41)$$

regardless of E_k , Condition CF2 implies (5.40). Therefore, under Conditions CF1 and CF2, $P_{k|k-1}$ satisfies (5.39) for all k . If we take into account that $P_{k|k-1} \leq P_{k|k-1}^u$ since $P_{k|k-1}$ is the error covariance matrix of the minimum variance linear estimate based on the information $(y_0, y_1, \dots, y_{k-1})$, it can be shown that for all k

$$(P_{k|k-1}^u)^{-1} \leq \alpha_2 I < \infty \quad (5.42)$$

holds under Conditions CF1 and CF2. Hence we need only to show

$$0 < \alpha_1 I < (P_{k|k-1}^u)^{-1}. \quad (5.43)$$

To show this, we consider the system (5.14) with $u_k = 0$ ($\forall k$) and the estimate $\hat{x}_{k|k-1}^\Delta$ based on the information $(y_0^u, y_1^u, \dots, y_{k-1}^u)$, which is defined by

$$\hat{x}_{k|k-1}^\Delta = \bar{R}_{Lk}^{-1} \sum_{i=k-L}^{k-1} \Phi^T(t_i, t_k) C^T M_i y_i. \quad (5.44)$$

Note that $\hat{x}_{k|k-1}^\Delta$ only uses the information of y_i^u not of y_i (also refer to Remark 5.2). By applying the relation

$$y_i = C \Phi(t_i, t_k) x_k - C \sum_{j=i}^{k-1} \Phi(t_i, t_{j+1}) w_j + v_i \quad (k-L \leq i \leq k-1) \quad (5.45)$$

to (5.44), we can rewrite $\hat{x}_{k|k-1}^\Delta$ as

$$\begin{aligned} \hat{x}_{k|k-1}^\Delta &= x_k - \bar{R}_{Lk}^{-1} \sum_{i=k-L}^{k-1} \Phi^T(t_i, t_k) C^T M_i C \Phi(t_i, t_k) \sum_{j=i}^{k-1} \Phi(t_i, t_{j+1}) w_j \\ &\quad + \bar{R}_{Lk}^{-1} \sum_{i=k-L}^{k-1} \Phi^T(t_i, t_k) C^T M_i v_i. \end{aligned} \quad (5.46)$$

Therefore, the error covariance matrix $P_{k|k-1}^\Delta$ of $\hat{x}_{k|k-1}^\Delta$ satisfies

$$\begin{aligned}
P_{k|k-1}^\Delta &= \text{cov}\{\bar{R}_{Lk}^{-1} \sum_{i=k-L}^{k-1} \Phi^T(t_i, t_k) C^T M_i C \Phi(t_i, t_k) \sum_{j=i}^{k-1} \Phi(t_k, t_{j+1}) w_j\} \\
&\quad + \text{cov}\{\bar{R}_{Lk}^{-1} \sum_{i=k-L}^{k-1} \Phi^T(t_i, t_k) C^T M_i v_i\} \\
&\leq \text{cov}\{\bar{R}_{Lk}^{-1} \sum_{i=k-L}^{k-1} \Phi^T(t_i, t_k) C^T M_i C \Phi(t_i, t_k) \sum_{j=k-L}^{k-1} \Phi(t_k, t_{j+1}) w_j\} + \bar{R}_{Lk}^{-1} \\
&= \bar{Q}_{Lk} + \bar{R}_{Lk}^{-1}
\end{aligned} \tag{5.47}$$

where $\text{cov}\{x\}$ denotes the covariance matrix of x , and $\bar{Q}_{Lk}$ is defined by

$$\bar{Q}_{Lk} = \sum_{i=k-L}^{k-1} \Phi(t_k, t_{i+1}) Q_i \Phi^T(t_k, t_{i+1}). \tag{5.48}$$

Since $P_{k|k-1}^u$ is the minimum-variance estimate of x_k based on the information $(y_0^u, y_1^u, \dots, y_{k-1}^u)$, we have

$$P_{k|k-1}^u \leq P_{k|k-1}^\Delta. \tag{5.49}$$

Moreover, it is clear that

$$0 < \bar{Q}_{Lk} + \bar{R}_{Lk}^{-1} < \infty. \tag{5.50}$$

Therefore, there exists a positive integer α_1 such that

$$0 < \alpha_1 I < (P_{k|k-1}^u)^{-1}. \tag{5.51}$$

Next, we show (5.37). The homogeneous part of the filter (5.21) can be rewritten as

$$\hat{x}_{k+1|k}^u = A_k \hat{x}_{k|k}^u \tag{5.52a}$$

$$\hat{x}_{k|k}^u = P_{k|k}^u (P_{k|k-1}^u)^{-1} \hat{x}_{k|k-1}^u. \tag{5.52b}$$

Furthermore,

$$(P_{k|k}^u)^{-1} = (P_{k|k-1}^u)^{-1} + C^T M_k C \tag{5.53}$$

holds. Applying (5.52b) and (5.53), (5.35) becomes

$$\begin{aligned}
V(\hat{x}_{k|k-1}^u, k) &= \hat{x}_{k|k-1}^{uT} (P_{k|k-1}^u)^{-1} \hat{x}_{k|k-1}^u \\
&= \hat{x}_{k|k}^{uT} (P_{k|k}^u)^{-1} \hat{x}_{k|k}^u + \hat{x}_{k|k}^{uT} \left((P_{k|k}^u)^{-1} - (P_{k|k-1}^u)^{-1} \right) \hat{x}_{k|k}^u \\
&\quad + (\hat{x}_{k|k}^u - \hat{x}_{k|k-1}^u)^T (P_{k|k-1}^u)^{-1} (\hat{x}_{k|k}^u - \hat{x}_{k|k-1}^u) \\
&= \hat{x}_{k|k}^{uT} (P_{k|k}^u)^{-1} \hat{x}_{k|k}^u + \hat{x}_{k|k}^{uT} C^T M_k C \hat{x}_{k|k}^u + \xi_k^T (P_{k|k-1}^u)^{-1} \xi_k
\end{aligned} \tag{5.54}$$

where

$$\xi_k = \hat{x}_{k|k}^u - \hat{x}_{k|k-1}^u. \quad (5.55)$$

Since

$$(P_{k+1|k}^u)^{-1} \leq A_k^{-T} (P_{k|k}^u)^{-1} A_k^{-1} \quad (5.56)$$

from (5.19e), it is verified that $V(\hat{x}_{k|k-1}^u, k)$ satisfies

$$\begin{aligned} V(\hat{x}_{k|k-1}^u, k) &\geq \hat{x}_{k|k}^{uT} A_k^T (P_{k+1|k}^u)^{-1} A_k \hat{x}_{k|k}^u + \hat{x}_{k|k}^{uT} C^T M_k C \hat{x}_{k|k}^u + \xi_k^T (P_{k|k-1}^u)^{-1} \xi_k \\ &= V(\hat{x}_{k+1|k}^u, k+1) + \hat{x}_{k+1|k}^{uT} A_k^{-T} C^T M_k C A_k^{-1} \hat{x}_{k+1|k}^u + \xi_k^T (P_{k|k-1}^u)^{-1} \xi_k. \end{aligned} \quad (5.57)$$

Thus $V(\hat{x}_{k|k-1}^u, k) - V(\hat{x}_{k-L|k-L-1}^u, k-L)$ satisfies

$$\begin{aligned} V(\hat{x}_{k|k-1}^u, k) - V(\hat{x}_{k-L|k-L-1}^u, k-L) \\ \leq - \sum_{i=k-L}^{k-1} \left(\hat{x}_{i+1|i}^{uT} A_i^{-T} C^T M_i C A_i^{-1} \hat{x}_{i+1|i}^u + \xi_i^T (P_{i|i-1}^u)^{-1} \xi_i \right). \end{aligned} \quad (5.58)$$

By calculating the minimum value of the summation in the right-hand side of (5.58) as in [9], it can be shown that

$$- \sum_{i=k-L}^{k-1} \left(\hat{x}_{i+1|i}^{uT} A_i^{-T} C^T M_i C A_i^{-1} \hat{x}_{i+1|i}^u + \xi_i^T (P_{i|i-1}^u)^{-1} \xi_i \right) \leq -\alpha_3 \|\hat{x}_{k-L|k-L-1}^u\|^2 \quad (5.59)$$

for some positive number α_3 . From (5.58) and (5.59), (5.33) is satisfied. Q.E.D.

5.2.3 The stability condition in connection with the degree of the unreliability

In the preceding subsection, we showed the condition to assure uniform asymptotic stability of our filter for unreliable sampling systems. However, the relationship between stability of the filter and the unreliability of the sampling mechanism is still unclear: we cannot know from Lemma 5.1 how often the measurement of the outputs may fail to assure stability of the filter. Hence, in this subsection, we will give a more explicit condition that assures the three conditions of Lemma 5.1 in connection with the degree of the unreliability of the sampling mechanism.

The following assumption plays the key role.

Assumption AU1 There exist an observability index vector $(\mu_1, \dots, \mu_p)$ of (A_c, C_c)

and a positive integer L such that the measurement data of the i th output can be obtained at least μ_i times in any L successive sampling instants. Here an observability index vector $(\mu_1, \dots, \mu_p)$ of (A_c, C_c) is defined as a set of nonnegative integers satisfying

$$\mu_1 + \dots + \mu_p = n \quad (5.60)$$

and

$$\text{rank}[c_{c1}^T, \dots, (c_{c1}A_c^{\mu_1-1})^T, \dots, c_{cp}^T, \dots, (c_{cp}A_c^{\mu_p-1})^T]^T = n \quad (5.61)$$

Our claim is that the filter becomes uniformly asymptotically stable under this assumption and some additional conditions. The details are given in the following theorem.

Theorem 5.1 Assume that the pair $(A_c, G_c Q_c^{1/2})$ is controllable and assume that the sampling rate is N -periodic. Then, if the unreliability of sampling is such that Assumption AU1 is satisfied, the homogeneous part (5.21) of the filter is uniformly asymptotically stable in the large for almost every set of t_k ($k = 0, 1, \dots$).

In this theorem, “almost every set of t_k ” means that even if stability of the filter is not assured for a certain set of t_k , the filter becomes stable by slightly changing t_k . More specifically, for a given set of t_k^0 ($k = 0, 1, \dots$), there exists at most finite number of α in any interval of finite length such that the filter is still not assured to be uniformly asymptotically stable in the large even if t_k is replaced by αt_k^0 .

Before we prove this theorem, we present the following lemma needed for the proof.

Lemma 5.2 Assume that t_i ($i = 0, 1, \dots$) satisfies for some $T > 0$

$$t_k - t_{k-N} = T, \quad \forall k = N, N+1, \dots \quad (5.62)$$

Then, $\tilde{C}_{Nk}$, defined by

$$\tilde{C}_{Nk} := \begin{bmatrix} E_{k-N}C \\ E_{k-N+1}Ce^{A_c(t_{k-N+1}-t_{k-N})} \\ \vdots \\ E_{k-1}Ce^{A_c(t_{k-1}-t_{k-N})} \end{bmatrix} \quad (5.63)$$

has full column rank for all $k \geq N$ and for almost every set of t_i regardless of E_i , if Assumption AU1 is satisfied.

Proof of Lemma 5.2. Let us assume that Assumption AU1 is satisfied, and denote by $N_{ik} \geq \mu_i$ the number of the successful measurement of y_i in N successive sampling instants $t_{k-N}, t_{k-N+1}, \dots, t_{k-1}$. Then, by permuting the rows of $\tilde{C}_{Nk}$, it can be transformed into

$$C_{Nk}^* = \begin{bmatrix} c_{c1} \exp(A_c l_{10} T) \\ \vdots \\ c_{c1} \exp(A_c l_{1, N_1-1} T) \\ \vdots \\ c_{cp} \exp(A_c l_{p0} T) \\ \vdots \\ c_{cp} \exp(A_c l_{p, N_p-1} T) \end{bmatrix} \quad (5.64)$$

where l_{ij} ($i = 1, 2, \dots, p; j = 0, 1, \dots, N_{ik} - 1$) are real numbers satisfying

$$0 \leq l_{i0} < l_{i1} < \dots < 1, \quad i = 1, 2, \dots, p. \quad (5.65)$$

From [51, 19], C_{Nk}^* has full column rank for almost every T in the sense that there exists at most finite number of T in every interval of finite length such that C_{Nk}^* does not have full column rank. Therefore, $\tilde{C}_{Nk}$ has full column rank for almost every set of t_k if $E_{k-N}, E_{k-N+1}, \dots, E_{k-1}$ are fixed. However, since the number of the combinations of the possible values of $E_{k-N}, E_{k-N+1}, \dots, E_{k-1}$ is finite (at most 2^{pN}), $\tilde{C}_{Nk}$ has full column rank for almost every set of t_k regardless of $E_{k-N}, E_{k-N+1}, \dots, E_{k-1}$. Moreover, since $\tilde{C}_{Nk+N} = \tilde{C}_{Nk}$ if $E_{k+i} = E_{k-N+i}$ ($i = 0, 1, \dots, N-1$), $\tilde{C}_{N,k+N}$ has full column rank regardless of $E_k, E_{k+1}, \dots, E_{k+N-1}$, if $\tilde{C}_{Nk}$ has full column rank regardless of $E_{k-N}, E_{k-N+1}, \dots, E_{k-1}$. Therefore, $\tilde{C}_{Nk}, \tilde{C}_{N,k+N}, \tilde{C}_{N,k+2N}, \dots$ have full column rank for almost every set of t_k regardless of E_k . The same argument is also true for $\tilde{C}_{Nk+i}, \tilde{C}_{N,k+N+i}, \tilde{C}_{N,k+2N+i}, \dots$ ($i = 1, 2, \dots, N-1$), and, consequently, $\tilde{C}_{Nk}$ has full column rank for all $k \geq N$ and for almost every set of t_i , regardless of E_i . Q.E.D.

We are now in a position to prove Theorem 5.1.

Proof of Theorem 5.1. It is clear from (5.10) that the Condition CF1 is satisfied if and only if the pair $(A_c, G_c Q_c^{1/2})$ is controllable. Next, let us consider Condition CF2. Without loss of generality, we can assume that $L = N$; if $L < N$, it is clear that Assumption AU1 is also true for $L = N$. If, on the other hand, $L > N$, there exists some positive integer i such that $L < iN$, and, we can choose L and N as iN . We further assume, for simplicity, that at least one output can be measured successfully at any sampling instants, i.e., $l_i > 0$ ($i = 0, 1, \dots$). Then, $\bar{R}_{Nk}(= \bar{R}_{Lk})$ in (5.29) can be written by $\tilde{C}_{Nk}$ as the form

$$\bar{R}_{Nk} = e^{A_c^T(t_k - N - t_k)} \tilde{C}_{Nk}^T \tilde{M}_{Nk} \tilde{C}_{Nk} e^{A_c(t_k - N - t_k)} \quad (5.66)$$

where $\tilde{M}_{Nk}$ is

$$\widetilde{M}_{Nk} = \text{block diag}(M_{k-N}, M_{k-N+1}, \dots, M_{k-1}) > 0. \quad (5.67)$$

Note that in the case that $l_i = 0$ for some $k - N \leq i \leq k - 1$, (5.66) is still true if we define $\widetilde{M}_{Nk}$ by

$$\widetilde{M}_{Nk} = \text{block diag}(M_{k-N}, \dots, M_{i-1}, M_{i+1}, \dots, M_{k-1}). \quad (5.68)$$

Since $\widetilde{M}_{Nk} > 0$, (5.29) is true if and only if $\widetilde{C}_{Nk}$ has full column rank for all $k \geq N$ and regardless of E_i ($i = 0, 1, 2, \dots$). Hence the proof becomes complete by applying Lemma 5.2. Q.E.D.

5.3 Numerical Examples

Here we construct a regulator with the filter proposed in 5.1.2 and show effectiveness of the filter through numerical simulations.

Consider the continuous-time plant (5.1) with the triplet (C_c, A_c, B_c) being controllable and observable. We are to sample the outputs with the sampling period T ($t_k = kT$) and construct a regulator by the feedback from the estimated states ($u_k = -F\hat{x}_k$), where the feedback gain F is determined so as to minimize

$$J = \sum_{k=0}^{\infty} (x_k^T x_k + u_k^T u_k). \quad (5.69)$$

We assume that sampling of the outputs is unreliable and that Assumption AU1 is satisfied for some L . Thus we use the filter (5.19) as the state estimator ($\hat{x}_k = \hat{x}_{k/k}^u$).

For comparison, we will construct two other types of control systems, namely, we compare the following three types of control systems.

- (a) A regulator incorporating the filter given in the preceding subsection.
- (b) A regulator incorporating the ordinary Kalman filter, with the lost measurement data replaced by the corresponding elements of $C\hat{x}_{k/k-1}^u$. That is, the corresponding elements of the innovation $y_k - C\hat{x}_{k/k-1}^u$ are set to 0.
- (c) A regulator incorporating the ordinary Kalman filter with full access to the output information (this is the ideal case, not the unreliable sampling case).

Clearly, the control system (c) is stable. Also, the control system (a) is stable because the plant is controllable and the filter given in the preceding subsection is stable from Theorem 5.1. However the control system (b) is not assured to be stable.

We will construct these regulators in two different settings.

Example 1

The plant is given by

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} x_{c1}(t) \\ x_{c2}(t) \end{bmatrix} &= \begin{bmatrix} 0.5 & -0.2 \\ 0.4 & -0.1 \end{bmatrix} \begin{bmatrix} x_{c1}(t) \\ x_{c2}(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_c(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_{c1}(t) \\ w_{c2}(t) \end{bmatrix} \\ y_c(t) &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_{c1}(t) \\ x_{c2}(t) \end{bmatrix}. \end{aligned} \quad (5.70)$$

The sampling period is $T = 1.0(\text{s})$, and the covariance matrices of $w_c(t)$ and v_k are $Q_c = 0.01I$, $R_k = 0.01$ ($\forall k$), respectively. Assumption AU1 is supposed to be satisfied for $L = 4$. This means that the measurement data are obtained successfully at least twice out of any four successive sampling instants since the observability index of the system (5.70) is $\mu_1 = 2$. The initial conditions of the plant and the filter are

$$x_c(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \bar{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \Sigma_0 = 2I. \quad (5.71)$$

Example 2

The plant is given by

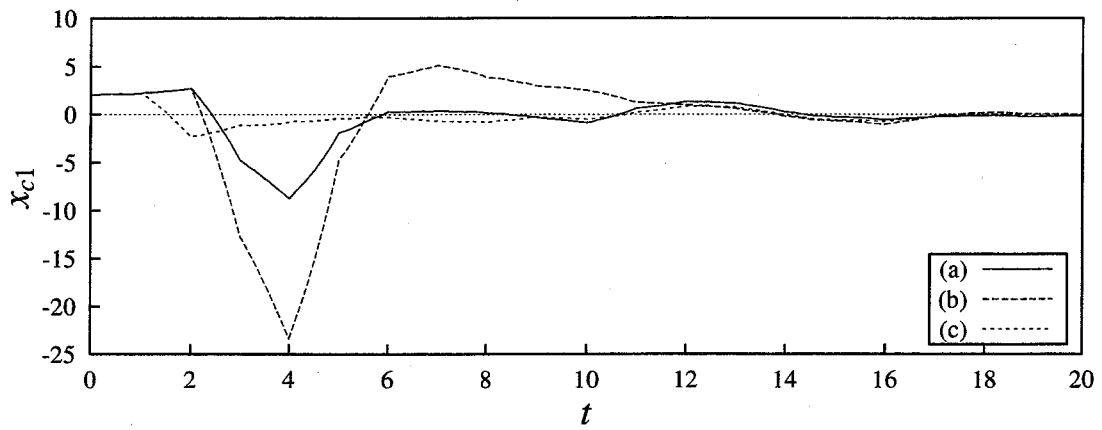
$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} x_{c1}(t) \\ x_{c2}(t) \\ x_{c3}(t) \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & -1.5 \end{bmatrix} \begin{bmatrix} x_{c1}(t) \\ x_{c2}(t) \\ x_{c3}(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_{c1}(t) \\ u_{c2}(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_{c1}(t) \\ w_{c2}(t) \\ w_{c3}(t) \end{bmatrix} \\ \begin{bmatrix} y_{c1}(t) \\ y_{c2}(t) \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_{c1}(t) \\ x_{c2}(t) \\ x_{c3}(t) \end{bmatrix}. \end{aligned} \quad (5.72)$$

The sampling period is $T = 1.0(\text{s})$, and the covariance matrices of $w_c(t)$ and v_k are $Q_c = 0.01I$, $R_k = 0.01I$ ($\forall k$), respectively. Assumption AU1 is supposed to be satisfied for $L = 4$ and for an observability index vectors $(\mu_1, \mu_2) = (1, 2)$ of the system (5.72). The initial conditions of the plant and the filter are

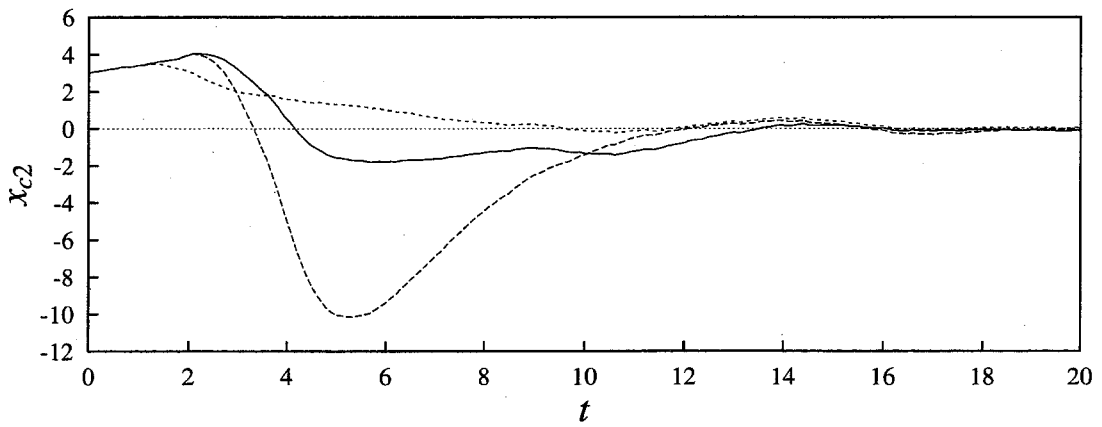
$$x_c(0) = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \quad \bar{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \Sigma_0 = 2I. \quad (5.73)$$

Fig. 5.2 and Fig. 5.3 show the responses of three types of control systems in both settings. In Fig. 5.2(c) and Fig. 5.3(d), • denotes the time instants at which the measurement of the outputs is successful.

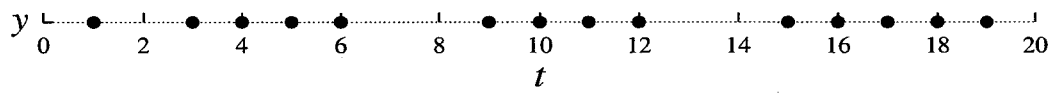
By the comparison of (a) and (b) in both settings, we can verify that our filter yields a better result than the ordinary Kalman filter that has a seemingly natural remedy to cope with the loss of the measurement data. This clearly shows the effectiveness of the use of this filter. The difference between (a) and (c) shows the deterioration caused by the loss of the measurement data due to the unreliable sampling.



(a) The Responses of x_{c1}

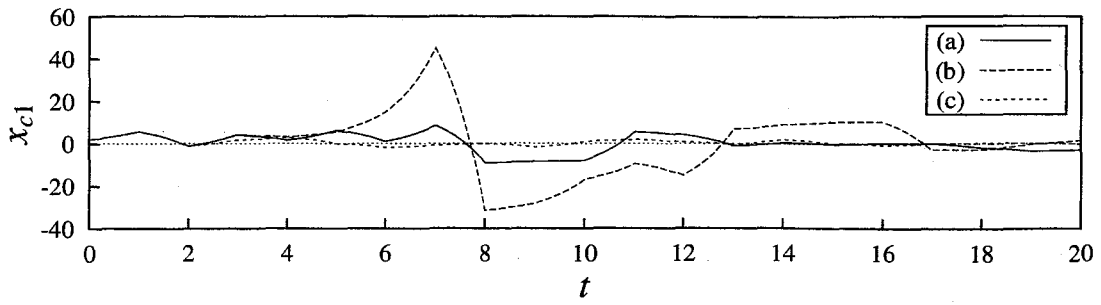


(b) The responses of x_{c2}

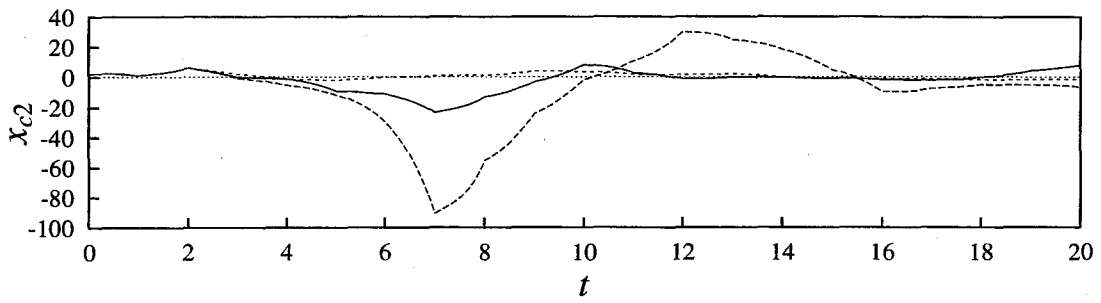


(c) The timing of successful measurement

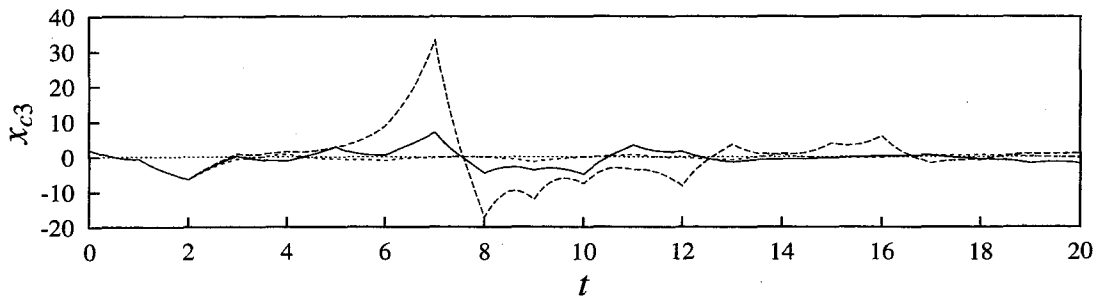
Figure 5.2: Simulation Results of Exmample 1



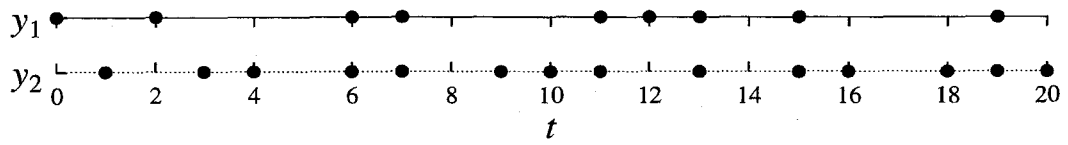
(a) The Responses of x_{c1}



(b) The responses of x_{c2}



(c) The responses of x_{c3}



(d) The timing of successful measurement

Figure 5.3: Simulation Results of Example 2

Chapter 6

Conclusion

In this thesis two sorts of topics concerning sampled-data control systems were treated.

The first topic, which was studied in Chapters 2 and 3, is concerned with the H_∞ problem of discrete linear periodically time-varying systems. In Chapter 2 we proposed a new method to calculate the optimal cost of the discrete LPTV H_∞ problem. Furthermore, applying our method, we showed some property of the discrete H_∞ problem of a special class of systems. In Chapter 3, we compared our method with other three methods for the discrete LPTV H_∞ problem in the state-space domain, and showed explicit relations among them. Especially for three methods that are categorized into the time-invariant approach, we investigated how the causality constraint is treated therein. We also gave numerical algorithms to calculate $\gamma_{\min}$ based on these three methods, and showed some numerical examples.

The second topic, which was treated in Chapters 4 and 5, is concerned with two types of nonuniform sampling problems. In Chapter 4 we treated a sampled-data robust stabilization problem with periodically time-varying sampling rates with the purpose of investigating whether uniform sampling yields the best robust stability. We converted this problem into the discrete LPTV H_∞ problem, and by applying the result of Chapter 3, we showed through numerical examples that whether uniform sampling yields the best robust stability has some relation to complex poles and unstable zeros of the plant. Although there is no qualitative explanation of these phenomena, these results lead to the possibilities of improving robust stability by using nonuniform sampling. In Chapter 5, we designed a minimum-variance linear estimate filter for unreliable sampling systems in which the measurement of the outputs fails occasionally. We derived a sufficient condition for asymptotic stability of this filter, and then, we showed that this stability condition can be rewritten in a form that is explicitly connected to unreliability of the measurement,

i.e., how often the measurement of the outputs should be successful. Through some numerical examples we verified the effectiveness of our proposed method.

In the above topics the following points are left for future research.

- (1) In the research of the discrete LPTV H_∞ problem, we only showed a method to obtain the optimal cost and no constructive method of (sub)optimal controllers was shown. It will be necessary to extend our method so that it can also be applied to controller synthesis.
- (2) In the research of a sampled-data robust stabilization problem with periodically time-varying sampling rates, we showed that whether uniform sampling yields the best robust stability has some relation to complex poles and unstable zeros of the plant. However, it is only a numerical result and the relation cannot be said to be clarified from a theoretical point of view. We will need to analyze this relation theoretically.

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