

On the expansion coefficients of Tau-functions of the KP and BKP hierarchies

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1 KP hierarchy

For the function $\tau(x)$ of $x = (x_1, x_2, \dots)$ the KP hierarchy [4] is the bilinear equation given by

$$\int \tau(x - y - [k^{-1}])\tau(x + y + [k^{-1}]) \exp\left(-2 \sum_{j=1}^{\infty} y_j k^j\right) dk = 0, \quad (1)$$

where $[k^{-1}] = (k^{-1}, k^{-1}/2, k^{-3}/3, \dots)$, $y = (y_1, y_2, \dots)$. The integral denotes taking the coefficient of k^{-1} in the Laurent expansion.

Any formal power series $\tau(x)$ can be expanded as

$$\tau(x) = \sum_{\lambda} \xi_{\lambda} s_{\lambda}(x), \quad (2)$$

where λ runs over all partitions.

A subset $M \subset \mathbb{Z}$ is called a Maya diagram of charge c if M satisfies the following conditions:

- (i) $\mathbb{Z}_{\geq 0} \cap M$ and $\mathbb{Z}_{< 0} \setminus M$ are finite,
- (ii) $\#(\mathbb{Z}_{\geq 0} \cap M) - \#(\mathbb{Z}_{< 0} \setminus M) = c$.

A partition is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ of nonnegative integers such that $|\lambda| = \sum_{i \geq 1} \lambda_i$ is finite. We identify a partition λ with its Young diagram, which is a left-justified array of $|\lambda|$ cells with λ_i cells in the i th row. Given a partition λ , we put

$$p(\lambda) = \#\{i : \lambda_i \geq i\}, \quad \alpha_i = \lambda_i - i, \quad \beta_i = \lambda'_i - i \quad (1 \leq i \leq p(\lambda)),$$

where λ'_i is the number of cells in the j th column of the Young diagram of λ . Then we write $\lambda = (\alpha_1, \dots, \alpha_{p(\lambda)} | \beta_1, \dots, \beta_{p(\lambda)})$ and call it the Frobenius notation of λ .

Example 1 If $\lambda = (3, 2, 1)$ the Frobenius notation of λ is $(2, 0 | 2, 0)$.

We can identify Maya diagrams M of charge 0 with partitions $\lambda = (\lambda_1, \lambda_2, \dots) = (\alpha_1, \dots, \alpha_r | \beta_1, \dots, \beta_r)$ by way of the following conditions:

- (i)

$$M = (\lambda_1 - 1, \lambda_2 - 2, \lambda_3 - 3, \dots),$$

(ii)

$$\mathbb{Z}_{\geq 0} \cap M = \{\alpha_1, \dots, \alpha_r\}, \quad \mathbb{Z}_{< 0} \setminus M = \{-\beta_1 - 1, \dots, -\beta_r - 1\}.$$

Example 2 If $\lambda = (3, 2, 1) = (2, 0|2, 0)$, Maya diagram M corresponded to λ becomes

$$M = (3 - 1, 2 - 2, 1 - 3, -4, -5, \dots) = (2, 0, -2, -4, -5, \dots).$$

Proposition 1 [17] The function $\tau(x)$ given as (2) is a solution of the KP hierarchy if and only if the coefficients $\{\xi[M]\}_M$ satisfy the Plücker relations

$$\sum_{i \geq 1} (-1)^i \xi[m_1, m_2, \dots, \widehat{m_i}, \dots] \xi[m_i, n_1, n_2, \dots] = 0, \quad (3)$$

where $M = (m_1, m_2, \dots)$ is Maya diagram of charge 1 and $N = (n_1, n_2, \dots)$ is Maya diagram of charge -1 . The $\widehat{m_i}$ means removing m_i from the sequence.

Proposition 2 The function $\tau(x)$ is a solution of the KP hierarchy if and only if the coefficients ξ_λ satisfy the following Plücker relations:

$$\begin{aligned} \sum_{i=1}^{p+1} (-1)^i \xi \left(\begin{matrix} m_1, \dots, \widehat{m_i}, \dots, m_{p+1} \\ m'_1, \dots, m'_p \end{matrix} \right) \xi \left(\begin{matrix} m_i, n_1, \dots, n_q \\ n'_1, \dots, n'_{q+1} \end{matrix} \right) \\ = \sum_{j=1}^{q+1} (-1)^{p+j} \xi \left(\begin{matrix} m_1, \dots, m_{p+1} \\ m'_1, \dots, m'_p, n'_j \end{matrix} \right) \xi \left(\begin{matrix} n_1, \dots, n_q \\ n'_1, \dots, \widehat{n'_j}, \dots, n'_{q+1} \end{matrix} \right), \quad (4) \end{aligned}$$

for any sequences $m_1, \dots, m_{p+1}, m'_1, \dots, m'_p, n_1, \dots, n_q, n'_1, \dots, n'_{q+1}$ of nonnegative integers.

Corollary 1 The function $\tau(x)$ is a solution of the KP hierarchy if and only if the coefficients ξ_λ satisfy the following Plücker relations:

$$\begin{aligned} \xi \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_r \end{matrix} \right) \xi \left(\begin{matrix} c_1, \dots, c_s \\ d_1, \dots, d_s \end{matrix} \right) \\ = \sum_{k=1}^r (-1)^{r-k} \xi \left(\begin{matrix} a_1, \dots, \widehat{a_k}, \dots, a_r \\ b_1, \dots, b_{r-1} \end{matrix} \right) \xi \left(\begin{matrix} a_k, c_1, \dots, c_s \\ b_r, d_1, \dots, d_s \end{matrix} \right) \\ + \sum_{l=1}^s (-1)^{l-1} \xi \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_{r-1}, d_l \end{matrix} \right) \xi \left(\begin{matrix} c_1, \dots, c_s \\ b_r, d_1, \dots, \widehat{d_l}, \dots, d_s \end{matrix} \right), \quad (5) \end{aligned}$$

and

$$\begin{aligned} \xi \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_r \end{matrix} \right) \xi \left(\begin{matrix} c_1, \dots, c_s \\ d_1, \dots, d_s \end{matrix} \right) \\ = \sum_{k=1}^r (-1)^{r-k} \xi \left(\begin{matrix} a_1, \dots, a_{r-1} \\ b_1, \dots, \widehat{b_k}, \dots, b_r \end{matrix} \right) \xi \left(\begin{matrix} a_r, c_1, \dots, c_s \\ b_k, d_1, \dots, d_s \end{matrix} \right) \\ + \sum_{l=1}^s (-1)^{l-1} \xi \left(\begin{matrix} a_1, \dots, a_{r-1}, c_l \\ b_1, \dots, b_r \end{matrix} \right) \xi \left(\begin{matrix} a_r, c_1, \dots, \widehat{c_l}, \dots, c_s \\ d_1, \dots, d_s \end{matrix} \right), \quad (6) \end{aligned}$$

for any sequence of nonnegative integers $(a_1, \dots, a_r)$, $(b_1, \dots, b_r)$, $(c_1, \dots, c_s)$ and $(d_1, \dots, d_s)$.

2 Main theorem

Fix a partition $\mu = (\gamma_1, \dots, \gamma_s | \delta_1, \dots, \delta_s)$. We assume that $\tau(x)$ has the following expansion:

$$\tau(x) = s_\mu(x) + \sum_{\lambda \supset \mu} \xi_\lambda s_\lambda(x). \quad (7)$$

Theorem 1 [14] *The function $\tau(x)$ given by (7) is a solution of the KP hierarchy if and only if the expansion coefficients $\{\xi_\lambda\}_\lambda$ is the following formulae for a partition $\lambda = (\alpha_1, \dots, \alpha_r | \beta_1, \dots, \beta_r)$:*

$$\xi_\lambda = (-1)^s \det \begin{pmatrix} (z_{\alpha_i, \beta_j})_{1 \leq i, j \leq r} & (u_{\alpha_i}^{(j)})_{1 \leq i \leq r, 1 \leq j \leq s} \\ (v_{\beta_j}^{(i)})_{1 \leq i \leq s, 1 \leq j \leq r} & O \end{pmatrix}, \quad (8)$$

where $z_{\alpha, \beta}$, $u_\alpha^{(j)}$, $v_\beta^{(i)}$ satisfy

$$\begin{cases} z_{\alpha, \beta} = \xi \left(\begin{smallmatrix} \alpha, \gamma_1, \dots, \gamma_s \\ \beta, \delta_1, \dots, \delta_s \end{smallmatrix} \right), \\ u_\alpha^{(j)} = \xi \left(\begin{smallmatrix} \alpha, \gamma_1, \dots, \hat{\gamma}_j, \dots, \gamma_s \\ \delta_1, \dots, \delta_s \end{smallmatrix} \right), \\ v_\beta^{(i)} = \xi \left(\begin{smallmatrix} \gamma_1, \dots, \gamma_s \\ b, \delta_1, \dots, \hat{\delta}_i, \dots, \delta_s \end{smallmatrix} \right). \end{cases} \quad (9)$$

To derive the determinant formulae (8) we need the following lemma.

Lemma 1 *Fix a partition μ . Suppose that $\tau(x)$ given by (7) is a solution of the KP hierarchy. Then ξ_λ can be expressed as a polynomial in*

$$\begin{aligned} I_\mu = & \left\{ \xi \left(\begin{smallmatrix} a, \gamma_1, \dots, \gamma_s \\ b, \delta_1, \dots, \delta_s \end{smallmatrix} \right) : a, b \in \mathbb{Z}_{\geq 0} \right\} \\ & \cup \left\{ \xi \left(\begin{smallmatrix} a, \gamma_1, \dots, \hat{\gamma}_j, \dots, \gamma_s \\ \delta_1, \dots, \delta_s \end{smallmatrix} \right) : a \in \mathbb{Z}_{\geq 0}, 1 \leq j \leq s \right\} \\ & \cup \left\{ \xi \left(\begin{smallmatrix} \gamma_1, \dots, \gamma_s \\ b, \delta_1, \dots, \hat{\delta}_i, \dots, \delta_s \end{smallmatrix} \right) : b \in \mathbb{Z}_{\geq 0}, 1 \leq i \leq s \right\}. \end{aligned}$$

Example 3 *We consider the case of $\mu = (\gamma | \delta)$ and $\lambda = (\alpha_1, \alpha_2 | \beta_1, \beta_2)$. The set I_μ becomes*

$$I_\mu = \left\{ \xi \left(\begin{smallmatrix} \alpha_i, \gamma \\ \beta_j, \delta \end{smallmatrix} \right) : i, j = 1, 2 \right\} \cup \left\{ \xi \left(\begin{smallmatrix} \alpha_i \\ \delta \end{smallmatrix} \right) : i = 1, 2 \right\} \cup \left\{ \xi \left(\begin{smallmatrix} \gamma \\ \beta_j \end{smallmatrix} \right) : j = 1, 2 \right\}$$

Using (5) we have

$$\xi \left(\begin{smallmatrix} \alpha_1, \alpha_2 \\ \beta_1, \beta_2 \end{smallmatrix} \right) \xi \left(\begin{smallmatrix} \gamma \\ \delta \end{smallmatrix} \right) = -\xi \left(\begin{smallmatrix} \alpha_2 \\ \beta_1 \end{smallmatrix} \right) \xi \left(\begin{smallmatrix} \alpha_1, \gamma \\ \beta_2, \delta \end{smallmatrix} \right) + \xi \left(\begin{smallmatrix} \alpha_1 \\ \beta_1 \end{smallmatrix} \right) \xi \left(\begin{smallmatrix} \alpha_2, \gamma \\ \beta_2, \delta \end{smallmatrix} \right) + \xi \left(\begin{smallmatrix} \alpha_1, \alpha_2 \\ \beta_1, \delta \end{smallmatrix} \right) \xi \left(\begin{smallmatrix} \gamma \\ \beta_2 \end{smallmatrix} \right).$$

Similarly using (5) and (6) we have

$$\xi \begin{pmatrix} \alpha_i \\ \beta_j \end{pmatrix} = \xi \begin{pmatrix} \alpha_i \\ \delta \end{pmatrix} \xi \begin{pmatrix} \gamma \\ \beta_j \end{pmatrix}, \xi \begin{pmatrix} \alpha_1, \alpha_2 \\ \beta_1, \delta \end{pmatrix} = -\xi \begin{pmatrix} \alpha_1 \\ \delta \end{pmatrix} \xi \begin{pmatrix} \alpha_2, \gamma \\ \beta_1, \delta \end{pmatrix} + \xi \begin{pmatrix} \alpha_1, \gamma \\ \beta_1, \delta \end{pmatrix} \xi \begin{pmatrix} \alpha_2 \\ \delta \end{pmatrix}$$

Then we have

$$\xi \begin{pmatrix} \alpha_1, \alpha_2 \\ \beta_1, \beta_2 \end{pmatrix} = -\det \begin{pmatrix} \xi \begin{pmatrix} \alpha_1, \gamma \\ \beta_1, \delta \end{pmatrix} & \xi \begin{pmatrix} \alpha_1, \gamma \\ \beta_2, \delta \end{pmatrix} & \xi \begin{pmatrix} \alpha_1 \\ \delta \end{pmatrix} \\ \xi \begin{pmatrix} \alpha_2, \gamma \\ \beta_1, \delta \end{pmatrix} & \xi \begin{pmatrix} \alpha_2, \gamma \\ \beta_2, \delta \end{pmatrix} & \xi \begin{pmatrix} \alpha_2 \\ \delta \end{pmatrix} \\ \xi \begin{pmatrix} \gamma \\ \beta_1 \end{pmatrix} & \xi \begin{pmatrix} \gamma \\ \beta_2 \end{pmatrix} & O \end{pmatrix}.$$

3 BKP hierarchy

The BKP hierarchy [4] is a system of non-linear equations for $\tau(x)$ given by

$$\oint e^{-2\tilde{\xi}(y,k)} \tau(x-y-2[k^{-1}]_o) \tau(x+y+2[k^{-1}]_o) \frac{dk}{2\pi i k} = \tau(x-y) \tau(x+y),$$

where the integral means taking the coefficient of k^{-1} in the expansion of the integrand in the series of k .

A formal power series $\tau(x)$, $x = (x_1, x_3, \dots)$ can be expanded in terms of Schur's Q-function as

$$\tau(x) = \sum_{\mu} \xi_{\mu} Q_{\mu} \left(\frac{x}{2} \right), \quad (10)$$

where μ runs over all strict partitions.

For a skew summetric matrix $A = (a_{i,j})_{1 \leq i,j \leq 2m}$ Pfaffian $\text{Pf}(a_{i,j})$ [8] is defined by

$$\text{Pf}(a_{i,j}) = \sum \text{sgn}(i_1, \dots, i_{2m}) \cdot a_{i_1, i_2} a_{i_3, i_4} \cdots a_{i_{2m-1}, i_{2m}}, \quad (11)$$

where the sum is over all permutations of $(1, \dots, 2m)$ such that

$$i_1 < i_3 < \cdots < i_{2m-1}, \quad i_1 < i_2, \dots, i_{2m-1} < i_{2m},$$

and $\text{sgn}(i_1, \dots, i_{2m})$ is the signature of the permutation $(i_1, \dots, i_{2m})$. In order to describe $\text{Pf}(a_{i,j})$ more conveniently we use some set of symbols X_i , $1 \leq i \leq 2m$. Set $(X_i, X_j) = a_{ij}$ and define $\text{Pf}((X_i, X_j))$ as

$$\text{Pf}((X_i, X_j)) = (X_1, \dots, X_{2m}).$$

The Pfaffian can be expanded as

$$(X_1, \dots, X_{2m}) = \sum_{j=2}^{2m} (-1)^j (X_1, X_j) (X_2, \dots, \hat{X}_j, \dots, X_{2m})$$

For a strict partition $\lambda = (\lambda_1, \dots, \lambda_M)$ we assume that $\tau(x)$ is expanded as

$$\tau(x) = Q_\lambda\left(\frac{x}{2}\right) + \sum_{|\mu| > |\lambda|} \xi_\mu Q_\mu\left(\frac{x}{2}\right), \quad (12)$$

where $\mu = (\mu_1, \dots, \mu_k)$ is a strict partition.

Theorem 2 [19] Suppose that $\tau(x)$ has the expansion (12). Then $\tau(x)$ is a solution of the BKP hierarchy if and only if the coefficients ξ_μ , $\mu = (\mu_1, \dots, \mu_k)$, $l(\mu) = k$ are given by the following formulae.

$$(i) \ M = 2L - 1,$$

$$\xi_\mu = \begin{cases} (\Lambda^{(1)}, \dots, \Lambda^{(2L-1)}, \mu_1, \dots, \mu_{2l-1}), & \text{if } k = 2l - 1, \\ (\Lambda, \Lambda^{(1)}, \dots, \Lambda^{(2L-1)}, \mu_1, \dots, \mu_{2l}), & \text{if } k = 2l. \end{cases} \quad (13)$$

$$(ii) \ M = 2L,$$

$$\xi_\mu = \begin{cases} (\Lambda, \Lambda^{(1)}, \dots, \Lambda^{(2L)}, \mu_1, \dots, \mu_{2l-1}), & \text{if } k = 2l - 1, \\ (\Lambda^{(1)}, \dots, \Lambda^{(2L)}, \mu_1, \dots, \mu_{2l}), & \text{if } k = 2l, \end{cases} \quad (14)$$

where the elements of the Pfaffian are

$$\begin{aligned} (\Lambda^{(i)}, n) &= \xi_{(\lambda_1, \dots, \hat{\lambda}_i, \dots, \lambda_L, n)}, \\ (\Lambda, n) &= \xi_{(\lambda_1, \dots, \lambda_L, n)}, \\ (n_i, n_j) &= \xi_{(\lambda_1, \dots, \lambda_L, n_i, n_j)}, \\ (\Lambda, \Lambda^{(i)}) &= (\Lambda^{(i)}, \Lambda^{(j)}) = 0. \end{aligned}$$

Example 4 We consider the case of $\lambda = (\lambda_1)$ and $\mu = (\mu_1, \mu_2, \mu_3)$. Then

$$\begin{aligned} \xi_\mu &= (\Lambda^{(1)}, \mu_1, \mu_2, \mu_3) \\ &= (\Lambda^{(1)}, \mu_1)(\mu_2, \mu_3) - (\Lambda^{(1)}, \mu_2)(\mu_1, \mu_3) + (\Lambda^{(1)}, \mu_3)(\mu_1, \mu_2) \\ &= \xi_{(\mu_1)}\xi_{(\lambda_1, \mu_2, \mu_3)} - \xi_{(\mu_2)}\xi_{(\lambda_1, \mu_1, \mu_3)} + \xi_{(\mu_3)}\xi_{(\lambda_1, \mu_1, \mu_2)} \end{aligned}$$

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