数理解析研究所講究録
1159 巻 2000 年 87-91

Asymptotic expansion of singular solutions and characteristic polygon of linear partial differential equations with holomorphic coefficients

大内 忠 (上智大学)

By Sunao ŌUCHI * (Sophia Univ. Tokyo)

Abstract

Consider the equation $P(z,\partial)u(z) = f(z)$ in a neighbourhood of $z = 0$, where $u(z)$ admits singularities on the surface $K = \{z_0 = 0\}$ and $f(z)$ has the asymptotic expansion of Gevrey type with respect to z_0 as $z_0 \to 0$. We study the possibility of asymptotic expansion of $u(z)$. We define the characteristic polygon of $P(z,\partial)$ with respect to K and characteristic indices γ_i $(0 \le i \le p)$. We discuss the behaviour of $u(z)$ in a neighbourhood of K, by using these notions. The main result is a generalization of that in [2]. The details of this paper is in [4] and will be appeared elsewhere.

KEY WORDS: complex partial differential equations, solutions with asymptotic expansion

§1 Notations and Characteristic Polygon.

The coordinate of $\mathbb{C}^{d+1}$ is denoted by $z = (z_0, z') = (z_0, z_1, \cdots, z_d) \in \mathbb{C} \times \mathbb{C}^d$. $|z| = \max\{|z_i|;\ 0 \le i \le d\}$. Its dual variables are $\xi = (\xi_0, \xi') = (\xi_0, \xi_1, \cdots, \xi_d)$. The differentiation is denoted by $\partial_i = \partial/\partial z_i$, and $\partial = (\partial_0, \partial') = (\partial_0, \partial_1, \cdots, \partial_d)$. $\alpha = (\alpha_0, \alpha') = (\alpha_0, \alpha_1, \cdots, \alpha_d) \in \mathbb{N} \times \mathbb{N}^d$ is a multi-index and $|\alpha| = \alpha_0 + |\alpha'| = \sum_{i=0}^{n} \alpha_i$.

Let $P(z,\partial) = \sum_{|\alpha| \le m} a_\alpha(z)\partial^\alpha$ be a linear partial differental operator with holomophic coefficients in a neighbourhood Ω of $z = 0$ in $\mathbb{C}^{d+1}$ and $K = \{z_0 = 0\}$. Let us define the characteristic polygon Σ of $P(z,\partial)$ with respect to the surface K. Let j_α be the valuation of $a_\alpha(z)$ with respect to z_0. Hence if $a_\alpha(z) \not\equiv 0$, $a_\alpha(z) = z_0^{j_\alpha} b_\alpha(z)$ with $b_\alpha(0, z') \not\equiv 0$. Put $e_\alpha = j_\alpha - \alpha_0$. We denote by $\Pi(a, b)$ the set $\{(x, y) \in \mathbb{R}^2; x \le a, y \ge b\}$. The characteristic polygon of Σ is defined by $\Sigma =$ *the convex hull of* $\cup_\alpha \Pi(|\alpha|, e_\alpha)$. The boundary of Σ consists of a vertical half line $\Sigma(0)$, a horizontal half line $\Sigma(p)$ and $p-1$ segments $\Sigma(i)$ $(1 \le i \le p-1)$ with slope γ_i, $0 = \gamma_p < \gamma_{p-1} < \cdots < \gamma_1 < \gamma_0 = +\infty$.

Let $\{(k_i, e(i)) \in \mathbb{R}^2; 0 \le i \le p-1\}$ be vertices of Σ, where $0 \le k_{p-1} < \cdots < k_i < k_{i-1} < \cdots < k_0 = m$. So the endpoints of $\Sigma(i)$ $(1 \le i \le p-1)$ are

*Department of Mathematics, Sophia University, Chiyoda-ku, Tokyo 102-8554, Japan e-mail s_ohuti@hoffman.cc.sophia.ac.jp 1991 Mathematical Subject Classification(s): Primary 35A20; Secondary 35B05, 35C20.

$(k_{i-1}, e(i-1))$ and $(k_i, e(i))$.

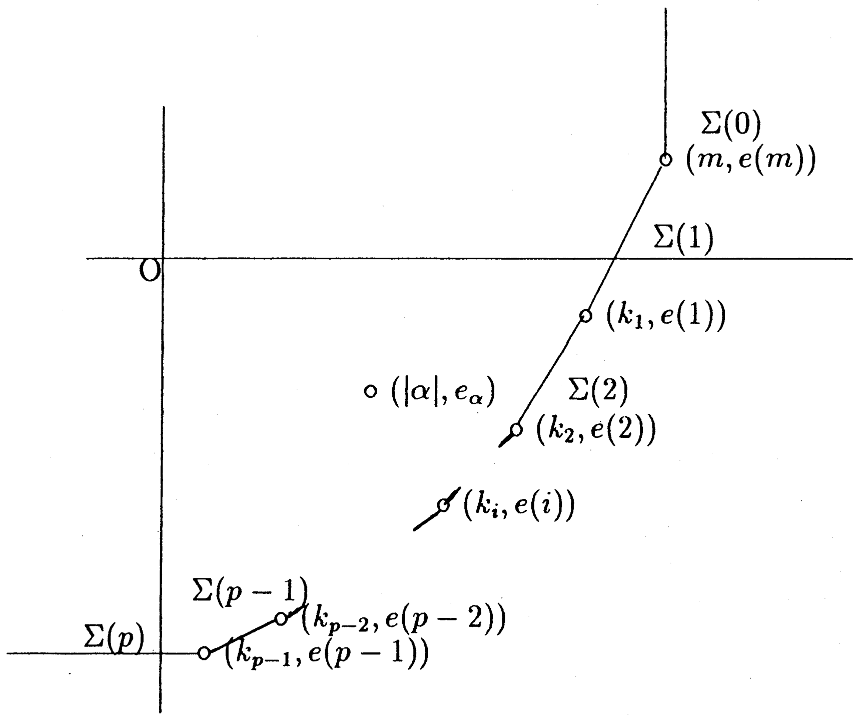

Figure 1: Characteristic polygon

Definition 1 *The slope γ_i of $\Sigma(i)$ is called the i-th characterisitic index of $P(z,\partial)$ with respect to $K = \{z_0 = 0\}$.*

Let us notice the vertices of the polygon Σ and define subsets $\Delta(i)$ and $\Delta_0(i)$ of multi-indices and operators $\mathfrak{P}_i(z,\partial)$ $(0 \le i \le p-1)$. Put

$$\begin{cases} \Delta(i) = \{\alpha \in \mathbb{N}^{d+1}; |\alpha| = k_i,\ j_\alpha - \alpha_0 = e(i)\}, \\ l_{k_i} = \max\{|\alpha'| : \alpha \in \Delta(i)\} \end{cases}$$

and

$$\begin{cases} \Delta_0(i) = \{\alpha \in \Delta(i); |\alpha'| = l_{k_i}\}, \\ \mathfrak{P}_i(z,\partial) = \sum_{\alpha \in \Delta_0(i)} z_0^{j_\alpha} b_\alpha(0,z')\partial_0^{\alpha_0}\partial^{\alpha'}. \end{cases} \tag{1.1}$$

$\mathfrak{P}_i(z,\partial)$ is a partial differential operator with total order k_i and order l_{k_i} with respect to ∂'. We have $e(i) = j_\alpha - \alpha_0 = j_\alpha - k_i + l_{k_i}$ for $\alpha \in \Delta_0(i)$. Hence we can wirte

$$(1.2) \qquad \mathfrak{P}_i(z,\partial) = z_0^{e(i)+k_i-l_{k_i}}\Big(\sum_{\alpha\in\Delta_0(i)} b_\alpha(0,z')\partial^{\alpha'}\Big)\partial_0^{k_i-l_{k_i}}\Big)$$

and define polynomial $\chi_{P,i}(z',\xi')$ in ξ' by

$$(1.3) \qquad \chi_{P,i}(z',\xi') = \sum_{\alpha\in\Delta_0(i)} b_\alpha(0,z')\xi^{\alpha'}.$$

§2 Function spaces.

Let $\Omega = \Omega_0 \times \Omega'$ be a polydisk with $\Omega_0 = \{z_0 \in C^1;\ |z_0| < R\}$ and $\Omega' = \{z' \in C^d; |z'| < R\}$. Put $\Omega_0(\theta) = \{z_0 \in \Omega_0 - \{0\};\ |\arg z_0| < \theta\}$ and $\Omega(\theta) = \Omega_0(\theta) \times \Omega'$. $\mathcal{O}(\Omega)$ $(\mathcal{O}(\Omega'),\ \mathcal{O}(\Omega(\theta)))$ is the set of all holomorphic functions on Ω (*resp.* Ω', $\Omega(\theta)$). We introduce subspaces of $\mathcal{O}(\Omega(\theta))$.

Definition 2 *$\mathcal{O}_{(\kappa)}(\Omega(\theta))$ $(0 < \kappa < +\infty)$ is the set of all $u(z) \in \mathcal{O}(\Omega(\theta))$ such that for any $\varepsilon > 0$ and any θ' with $0 < \theta' < \theta$*

$$(2.1) \qquad |u(z)| \le C\exp(\varepsilon|z_0|^{-\kappa}) \quad \textit{for } z \in \Omega(\theta')$$

holds for a constant $C = C(\varepsilon,\theta')$. We put $\mathcal{O}_{(+\infty)}(\Omega(\theta)) = \mathcal{O}(\Omega(\theta))$ for $\kappa = +\infty$.

Definition 3 *$Asy_{\{\kappa\}}(\Omega(\theta))$ $(0 < \kappa \le +\infty)$ is the set of all $u(z) \in \mathcal{O}(\Omega(\theta))$ such that for any θ' with $0 < \theta' < \theta$ and any $N \in \mathbb{N}$*

$$(2.2) \qquad |u(z) - \sum_{n=0}^{N} u_n(z')z_0^n| \le AB^N|z_0|^N\Gamma(\frac{N}{\kappa}+1) \quad z \in \Omega(\theta')$$

holds, where $u_n(z') \in \mathcal{O}(\Omega')$, $A = A(\theta')$ and $B = B(\theta')$.

We say that $u(z) \in Asy_{\{\kappa\}}(\Omega(\theta))$ has asymptotic expansion with Gevrey exponent κ in $\Omega(\theta)$. $u(z) \in Asy_{\{+\infty\}}(\Omega(\theta))$ means that $u(z)$ is holomorphic at $z = 0$.

§3 Theorem

First we give a condition on $P(z,\partial)$ treated in this paper.

Condition-i $j_\alpha = 0$ *for all* $\alpha \in \Delta_0(i)$.

If $P(z,\partial)$ satisfies Condition-i, then $\mathfrak{P}_i(z,\partial) = (\sum_{\alpha\in\Delta_0(i)} b_\alpha(0,z')\partial^{\alpha'})\partial_0^{k_i - l_{k_i}}$. We have

Theorem 4 *Suppose that* $P(z,\partial)$ *satisfies Condition-i and* $\chi_{P,i}(0,\hat{\xi}') \neq 0$, $\hat{\xi}' = (1,0,\cdots,0)$. *Let* $u(z) \in \mathcal{O}_{(\gamma_i)}(\Omega(\theta))$ *be a solution of*

$$P(z,\partial)u(z) = f(z) \in Asy_{\{\gamma_i\}}(\Omega(\theta)) \tag{3.1}$$

satisfying

$$\partial_1^h u(z_0,0,z'') \in Asy_{\{\gamma_i\}}(\Omega(\theta)\cap\{z_1=0\}) \quad for \quad 0 \le h \le l_{k_i} - 1. \tag{3.2}$$

Then there is a polydisk W *centered at* $z=0$ *such that* $u(z) \in Asy_{\{\gamma_i\}}(W(\theta))$.

We studied in [1] and [2] similar problems for the case $i = p-1$ and $l_{k_{p-1}} = 0$. We gave in [2] a simple proof of the same result as Theorem 4 for this case. We show Theorem 4 by modifying the discussion in [2]. When Condition-i does not hold, solutions become less regular and we studied in [3] the behaviours of solutions under the condition that $i = p-1$ and $l_{k_{p-1}} = 0$ but Condition-(p-1) does not necessarily hold.

§4 Example

We give an example. Let us consider

$$P(z,\partial) = \partial_1^5 + \partial_1^3\partial_0 + \partial_0^2, \quad z = (z_0,z_1) \in \mathbb{C}^2. \tag{4.1}$$

We have

$$\begin{cases} \gamma_0 = +\infty, \quad \gamma_1 = 1, \quad \gamma_2 = 1/2, \quad \gamma_3 = 0, \\ \chi_{P,0}(z',\xi_1) = \xi_1^5, \quad \chi_{P,1}(z',\xi_1) = \xi_1^3, \quad \chi_{P,2}(z',\xi_1) = I. \end{cases}$$

Obviously $P(z,\partial)$ satisfies *Condition-i* and $\chi_{P,i}(z',1) \neq 0$ for $i = 0,1,2$. So it follows from Theorem 4 that there is a polydisk W centered at $z=0$ such that

$$
\begin{aligned}
i=0 &: u(z) \in \mathcal{O}_{(+\infty)}(\Omega(\theta)),\ \partial_1^h u(z_0,0) \in Asy_{\{+\infty\}}(\Omega_0(\theta))\ (0 \le h \le 4),\\
&\quad f(z) \in Asy_{\{+\infty\}}(\Omega(\theta)) \Rightarrow u(z) \in Asy_{\{+\infty\}}(W(\theta)),\\
i=1 &: u(z) \in \mathcal{O}_{(1)}(\Omega(\theta)),\ \partial_1^h u(z_0,0) \in Asy_{\{1\}}(\Omega_0(\theta))\ (0 \le h \le 2),\\
&\quad f(z) \in Asy_{\{1\}}(\Omega(\theta)) \Rightarrow u(z) \in Asy_{\{1\}}(W(\theta)),\\
i=2 &: u(z) \in \mathcal{O}_{(1/2)}(\Omega(\theta)),\ f(z) \in Asy_{\{1/2\}}(\Omega(\theta)) \Rightarrow u(z) \in Asy_{\{1/2\}}(W(\theta)).
\end{aligned}
$$

References

[1] Ōuchi,S., An integral representation of singular solutions and removable singularities to linear partial differential equations, *Publ. RIMS Kyoto Univ.,* **26** (1990), 735-783.

[2] Ōuchi, S., Singular solutions with asymptotic expansion of linear partial differential equations in the complex domain, *Publ. RIMS Kyoto Univ.,* **34** (1998), 291-311.

[3] Ōuchi, S., Growth property and slowly increasing behaviour of singular solutions of linear partial differential equations in the complex domain (*preprint*).

[4] Ōuchi, S., Asymptotic expansion of singular solutions and the characteristic polygon of linear partial differential equations in the complex domain (*preprint*).