



Parametrized topological complexity of spherical fibrations over spheres

Yuki Minowa¹

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Abstract

Parametrized topological complexity is a homotopy invariant that represents the degree of instability of motion planning problem that involves external constraints. We consider the parametrized topological complexity in the case of spherical fibrations over spheres. We explicitly compute a lower bound in terms of weak category and determine the parametrized topological complexity of some spherical fibrations.

Keywords Parametrized topological complexity · Spherical fibration · Sectional category · Weak category

Mathematics Subject Classification 55M30 · 55S40

1 Introduction

Farber [9] defined topological complexity as a quantity for topological study of the robot motion planning problem. The topological complexity of a space X represents the degree of instability of motion planning in X as a configuration space. Recently, Cohen, Farber and Weinberger [6] introduced a variant of topological complexity, called *parametrized topological complexity*, to formulate the robot motion planning problem that involves external constraints as parameters, such as a set of obstacles that block the motion of robots. The configuration space of this motion planning problem is described as a fibration $p: X \rightarrow B$, where X represents the configuration space and B represents the external constraints. Then parametrized topological complexity is defined by a topological invariant of the map $p: X \rightarrow B$, where we denote it by $\mathrm{TC}[X \rightarrow B]$. See Sect. 3 for the precise definition.

Initially, the study of parametrized topological complexity was motivated by the computation of $\mathrm{TC}[F(\mathbb{R}^k, m+n) \rightarrow F(\mathbb{R}^k, m)]$, where $F(Y, m)$ denotes the configuration space

$$\{(x_1, \dots, x_m) \in Y^m \mid x_i \neq x_j \text{ if } i \neq j\}.$$

Several authors have attempted to determine or estimate the parametrized topological complexity of other specific fibrations. In particular, Farber and Weinberger [13, 14] studied the parametrized topological complexity of sphere bundles coming from vector bundles; they

✉ Yuki Minowa
minowa.yuki.48z@st.kyoto-u.ac.jp

¹ Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan

gave an upper bound in terms of the 2-frame bundle and a lower bound in terms of characteristic classes. Crabb [7] reformulated this lower bound in terms of stable cohomotopy Euler class and obtained a similar result for projective bundles coming from vector bundles. Farber and Oprea [10] considered a fiber bundle $F \rightarrow X \rightarrow B$ associated to a principal G -bundle; they gave upper and lower bounds in terms of G -equivariant homotopy invariants.

On the other hand, many variants of parametrized topological complexity have also been investigated. Farber and Paul [11] introduced sequential parametrized topological complexity as an analog of sequential topological complexity [26]. They also gave upper and lower bounds in terms of 2-frame bundle and characteristic classes [12]. Daundkar [8] considered equivariant parametrized topological complexity for a fibration $X \rightarrow B$ that is also a G -map. Grant [17] defined the parametrized topological complexity of an epimorphism of discrete groups, which coincides with the parametrized topological complexity of the induced fibration between corresponding Eilenberg–MacLane spaces; he gave an upper and lower bound in terms of the cohomology dimension of groups. However, in this paper, we will not delve into these variants.

In this paper, we investigate the parametrized topological complexity of spherical fibrations over spheres. Let

$$S^n \rightarrow X \rightarrow S^{m+1}$$

be a homotopy fibration with $n \geq 1$. In general, as in [6], there is a simple lower bound

$$\mathrm{TC}[X \rightarrow S^{m+1}] \geq \mathrm{TC}(S^n) = \begin{cases} 1 & n \text{ is odd} \\ 2 & n \text{ is even.} \end{cases}$$

This lower bound coincides with the cup-length type cohomological lower bound [6], as well as with that given by Farber and Weinberger [13, 14] mentioned above, when the homotopy fibration is a sphere bundle coming from a vector bundle. On the other hand, if n is odd, then we can modify Farber and Weinberger's upper bound [14] such that

$$\mathrm{TC}[X \rightarrow S^{m+1}] \leq 2. \quad (1)$$

Hence we may ask:

Question 1.1 *Does there exist a homotopy fibration $S^n \rightarrow X \rightarrow S^{m+1}$ with n odd and $\mathrm{TC}[X \rightarrow S^{m+1}] = 2$?*

The first purpose of this paper is to provide a sufficient condition for the existence of the homotopy fibration in Question 1.1. To state the results, we set notation. As in [22], the total space X has a cell decomposition

$$X \simeq S^n \cup_{\beta} e^{m+1} \cup e^{m+n+1}.$$

We set $\chi(X) = \beta$. Let $\nabla: S^n \rightarrow S^n \vee S^n$ be the suspension comultiplication of a sphere. Then, for $\alpha \in \pi_{m+n}(S^n)$, we can define the crude Hopf invariant $\overline{H}_{\nabla}(\alpha) \in \pi_{m+n+1}(S^{2n})$. See [4, Definition 2.11] for further details. Let η denote the generator of the stable stem $\pi_1^S \cong \mathbb{Z}/2$.

Theorem 1.2 *Let $2n + 1 \leq m \leq 4n - 1$ with $n \geq 1$. Suppose that there are $\alpha \in \pi_{m+2n+1}(S^{2n+1})$ and $\beta \in \pi_m(S^{2n+1})$ such that $\overline{H}_{\nabla}(\alpha) \neq 0$, $\Sigma(\eta \circ \beta) = 0$ and $[\beta, 1_{S^{2n+1}}] = 0$. Then there exists a homotopy fibration $S^{2n+1} \rightarrow X \rightarrow S^{m+1}$ with $\chi(X) = \beta$ and*

$$\mathrm{TC}[X \rightarrow S^{m+1}] = 2.$$

We prove that Theorem 1.2 is applicable to infinitely many cases.

Corollary 1.3 *There exist infinitely many pairs (m, n) , e.g. $m - 2n - 2 = 0$ and n is odd, such that there exist homotopy fibrations $S^{2n+1} \rightarrow X \rightarrow S^{m+1}$ with*

$$\mathrm{TC}[X \rightarrow S^{m+1}] = 2.$$

To prove these statements, we begin by considering $\mathrm{TC}[X \rightarrow S^{m+1}]$ as the sectional category of the fiberwise diagonal map $\Delta: X \rightarrow X_{S^{m+1}}^2$. We then recall weak sectional category introduced by García-Calines and Vandembroucq [16], which gives a sharper lower bound than the aforementioned lower bounds, but is much harder to compute. As we will see later, the weak sectional category of Δ , denoted by $\mathrm{wsecat}(\Delta)$, coincides with the weak category of a certain 4-cell complex W . Therefore, we aim to provide a sufficient condition for the weak category of W to be equal to 2.

The next purpose of this paper is to determine the parametrized topological complexity for specific homotopy fibrations $S^n \rightarrow X \rightarrow S^{n+1}$. We first consider the cases where n is even and the homotopy fibration is the unit sphere bundle of a vector bundle and then consider the $n \equiv -1 \pmod{4}$ with $n \geq 11$ cases.

Theorem 1.4 1. *Let $S^{2n} \rightarrow X \rightarrow S^{2n+1}$ be the unit sphere bundle of a real vector bundle with $n \geq 1$. Then*

$$\mathrm{TC}[X \rightarrow S^{2n+1}] = 2.$$

2. *Let $S^{4n-1} \rightarrow X \rightarrow S^{4n}$ be a homotopy fibration with $\chi(X) \equiv 2 \pmod{4}$ and $n \geq 3$. Then*

$$\mathrm{TC}[X \rightarrow S^{4n}] = 2.$$

We now concentrate our attention on a specific family of sphere bundles; let $S^n \rightarrow T \rightarrow S^{n+1}$ be the unit tangent bundle. Then $\chi(T)$ coincides with the Euler characteristic of S^{n+1} , which is 2 for n odd and 0 for n even. Then, for n even or for $n \equiv -1 \pmod{4}$ with $n \geq 11$, we obtain $\mathrm{TC}[T \rightarrow S^{n+1}] = 2$ by Theorem 1.4. This equation holds even for $n = 3, 7$, which we prove separately in Sect. 7.2. Thus we obtain:

Theorem 1.5 *If n is even with $n \geq 2$ or $n \equiv -1 \pmod{4}$, then $\mathrm{TC}[T \rightarrow S^{n+1}] = 2$.*

The final purpose of this paper is to show that there are homotopy fibrations $S^n \rightarrow X \rightarrow S^{m+1}$ such that $\mathrm{TC}[X \rightarrow S^{m+1}]$ cannot be determined by the upper and lower bounds mentioned above. We consider the unit tangent bundle $S^n \rightarrow T \rightarrow S^{n+1}$ with $n \equiv 1 \pmod{4}$. We compute $\mathrm{wsecat}(\Delta)$, or equivalently, the weak category of W and obtain:

Proposition 1.6 *There are inequalities*

$$\mathrm{wsecat}(\Delta) = 1 \leq \mathrm{TC}[T \rightarrow S^{4n+2}] \leq 2.$$

For $n = 1, 2$, $\mathrm{TC}[T \rightarrow S^{4n+2}]$ is determined to be 1. However, for $n = 3$, we have not been able to determine $\mathrm{TC}[T \rightarrow S^{4n+2}]$.

This paper is organized as follows. Section 2 recalls notation and terminology for fiberwise spaces and provides a pushout construction for a spherical fibration. Section 3 recalls the definition of parametrized topological complexity and shows (1). Section 4 investigates the cell structure of the complex W . Section 5 provides a sufficient condition for $\mathrm{wsecat}(\Delta) = 2$ in relation to crude Hopf invariant and Theorem 1.2 and Corollary 1.3. Section 6 computes

the mod 2 cohomology of X and the total spaces of the related sphere bundles, then proves Theorem 1.4 (1). Section 7 computes $\text{wsecat}(\Delta)$ for specific homotopy fibrations, then proves Theorems 1.4 (2) and 1.5. Section 8 also computes $\text{wsecat}(\Delta)$ and proves Proposition 1.6. Finally, Sect. 9 lists some problems that could lead to further investigation.

2 Preliminaries

In this section, we recall notation and terminology involving fiberwise spaces, and then describe a fibration over a suspension as a certain pushout, which we will use frequently in the sequel.

We first recall notation and terminology for fiberwise spaces. A space X is called a fiberwise space over B if it is equipped with a distinguished map $X \rightarrow B$. Let $p: X \rightarrow B$ and $q: Y \rightarrow B$ be fiberwise spaces over B . A fiberwise map from X to Y is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ p \downarrow & & \downarrow q \\ B & \xlongequal{\quad} & B. \end{array}$$

We abbreviate the above fiberwise map as $f: X \rightarrow Y$. Let $f_0, f_1: X \rightarrow Y$ be fiberwise maps over B . A fiberwise homotopy from f to g is a fiberwise map $H: X \times [0, 1] \rightarrow Y$ such that $H(x, 0) = f_0(x)$ and $H(x, 1) = f_1(x)$ for $x \in X$; here we regard $X \times [0, 1]$ as a fiberwise space over B equipped with

$$X \times [0, 1] \xrightarrow{\text{pr}_1} X \rightarrow B,$$

where pr_1 denotes the first projection. If such a fiberwise homotopy exists, then we write $f \simeq_B g$. We say that two fiberwise spaces X and Y are fiberwise homotopy equivalent if there exist fiberwise maps $f: X \rightarrow Y$ and $g: Y \rightarrow X$ such that

$$g \circ f \simeq_B 1_X \quad \text{and} \quad f \circ g \simeq_B 1_Y.$$

For fiberwise spaces $p_1: X_1 \rightarrow B$ and $p_2: X_2 \rightarrow B$, the fiberwise product $X_1 \times_B X_2$ is defined by the pullback of the triad $X_1 \xrightarrow{p_1} B \xleftarrow{p_2} X_2$; that is,

$$X_1 \times_B X_2 = \{(x_1, x_2) \in X_1 \times X_2 \mid p_1(x_1) = p_2(x_2)\}.$$

Then $X_1 \times_B X_2$ is a fiberwise space over B equipped with

$$p_1 \times_B p_2: X_1 \times_B X_2 \rightarrow B, \quad (x_1, x_2) \mapsto p_1(x_1) = p_2(x_2). \quad (2)$$

Observe that, if p_1 and p_2 are fibrations with fibers F_1 and F_2 , then $p_1 \times_B p_2$ is a fibration with fiber $F_1 \times F_2$. If $p_1 = p_2 = p: X \rightarrow B$, then we write the fiberwise product $X_1 \times_B X_2$ by X_B^2 . We also define the fiberwise diagonal (map) $\Delta: X \rightarrow X_B^2, x \mapsto (x, x)$; notice that Δ is a fiberwise map over B .

Next, we describe a fibration over a suspension as a pushout by generalizing the result of James and Whitehead [22, Section 3]. Let C_+B and C_-B denote the upper and lower cones in the suspension ΣB , and let $i_+: B \rightarrow C_+B$ and $i_-: B \rightarrow C_-B$ denote the inclusions.

Lemma 2.1 For a fibration $p: X \rightarrow \Sigma B$ with fiber F , there exists a map $\mu: B \times F \rightarrow F$ such that the total space X is homotopy equivalent to the pushout of the cotriad

$$C_+B \times F \xleftarrow{(i_+ \circ \text{pr}_1, \mu)} B \times F \xrightarrow{i_- \times 1_F} C_-B \times F$$

where pr_j denotes the j -th projection.

Proof Since C_+B is contractible, the pullback $p^{-1}(C_+B) \rightarrow C_+B$ of p and $\text{pr}_1: C_+B \times F \rightarrow C_+B$ are fiberwise homotopy equivalent over C_+B ; there are two fiberwise maps $\phi_+: p^{-1}(C_+B) \rightarrow C_+B \times F$ and $\psi_+: C_+B \times F \rightarrow p^{-1}(C_+B)$ such that

$$\psi_+ \circ \phi_+ \simeq_{C_+B} 1_{p^{-1}(C_+B)} \quad \text{and} \quad \phi_+ \circ \psi_+ \simeq_{C_+B} 1_{C_+B \times F}.$$

One also obtain fiberwise homotopy equivalences over C_-B , $\phi_-: p^{-1}(C_-B) \rightarrow C_-B \times F$ and $\psi_-: C_-B \times F \rightarrow p^{-1}(C_-B)$ verbatim. Then there is a homotopy commutative diagram

$$\begin{array}{ccccc} C_+B \times F & \xleftarrow{\phi_+ \circ (p^*i_+) \circ \psi} & B \times F & \xrightarrow{i_- \times 1_F} & C_-B \times F \\ \psi_+ \downarrow & & \downarrow \psi & & \downarrow \psi_- \\ p^{-1}(C_+B) & \xleftarrow{p^*i_+} & p^{-1}(B) & \xrightarrow{p^*i_-} & p^{-1}(C_-B) \end{array}$$

where $\psi: B \times F \rightarrow F$ denotes the restriction of ψ_- . Note that, since $i_-: B \rightarrow C_-B$ is a closed cofibration, so are the maps $i_- \times 1_F$ and p^*i_- ; the latter follows from the Ström's pullback theorem [28, Theorem 12]. Then, for each row, the homotopy pushout has the same homotopy type as the pushout. Now we take the (homotopy) pushout of the rows to get a map $Z \rightarrow X$, where Z denotes the (homotopy) pushout of the top row. Then since all columns are homotopy equivalences, so is the map $Z \rightarrow X$. Thus, by setting $\mu = \text{pr}_2 \circ \phi_+ \circ (p^*i_+) \circ \psi$, the statement is proved. $\square$

Remark 1 In the sequel, we will mainly consider the fibrations $S^{2n+1} \rightarrow X \rightarrow \Sigma B$. Since ψ and ϕ_+ are fiberwise homotopy equivalence, the restriction

$$\mu|_{* \times S^{2n+1}} = \phi_+|_{* \times S^{2n+1}} \circ \psi|_{* \times S^{2n+1}}: S^{2n+1} \rightarrow S^{2n+1}$$

is a homotopy equivalence. Then by taking a suitable choice of homotopy equivalences, one may always assume that $\mu|_{* \times S^{2n+1}} = 1_{S^{2n+1}}$.

We also describe the fiberwise diagonal $\Delta: X \rightarrow X_{\Sigma B}^2$ for a fibration $F \rightarrow X \xrightarrow{p} \Sigma B$ as a map between pushouts. The pushout construction stated in Lemma 2.1 is natural with respect to Δ in the following sense. Let $p_{\Sigma B}^2: X_{\Sigma B}^2 \rightarrow \Sigma B$ denote the fiberwise map (2) for $p_1 = p_2 = p: X \rightarrow \Sigma B$. Then, by definition, the preimage $(p_{\Sigma B}^2)^{-1}(C_+B)$ is the pullback of the triad $p^{-1}(C_+B) \rightarrow C_+B \leftarrow p^{-1}(C_+B)$, where both maps are the pullback of p . Now we consider the commutative diagram

$$\begin{array}{ccccc} p^{-1}(C_+B) & \xrightarrow{p} & C_+B & \xleftarrow{p} & p^{-1}(C_+B) \\ \phi_+ \downarrow & & \parallel & & \downarrow \phi_+ \\ C_+B \times F & \xrightarrow{\text{pr}_1} & C_+B & \xleftarrow{\text{pr}_1} & C_+B \times F \\ \psi_+ \downarrow & & \parallel & & \downarrow \psi_+ \\ p^{-1}(C_+B) & \xrightarrow{p} & C_+B & \xleftarrow{p} & p^{-1}(C_+B) \end{array}$$

where ϕ_+ and ψ_+ are the fiberwise homotopy equivalences in the proof of Lemma 2.1. Then by taking the pullback of the rows, we get two maps

$$\Phi_+ : (p_{\Sigma B}^2)^{-1}(C_+B) \rightarrow C_+B \times F \times F, \quad \text{and} \quad \Psi_+ : C_+B \times F \times F \rightarrow (p_{\Sigma B}^2)^{-1}(C_+B)$$

that are fiberwise homotopy equivalences over C_+B . Similarly, one obtain two fiberwise homotopy equivalences over C_-B ,

$$\Phi_- : (p_{\Sigma B}^2)^{-1}(C_-B) \rightarrow C_-B \times F \times F, \quad \text{and} \quad \Psi_- : C_-B \times F \times F \rightarrow (p_{\Sigma B}^2)^{-1}(C_-B).$$

Let $\mu : B \times F \rightarrow F$ be a map as in the proof of Lemma 2.1, and let $\mu_B^2 : B \times F \times F \rightarrow F \times F$ denote the map between the pushouts of the rows in the commutative diagram

$$\begin{array}{ccccc} B \times F & \xrightarrow{\text{pr}_1} & B & \xleftarrow{\text{pr}_1} & B \times F \\ \mu \downarrow & & \downarrow & & \downarrow \mu \\ F & \longrightarrow & * & \longleftarrow & F. \end{array}$$

Arguing as in the proof of Lemma 2.1, we get a commutative diagram

$$\begin{array}{ccccc} C_+B \times F & \xleftarrow{(i_+ \circ \text{pr}_1, \mu)} & B \times F & \xrightarrow{i_- \times 1_F} & C_-B \times F \\ \downarrow 1 \times \Delta & & \downarrow 1 \times \Delta & & \downarrow 1 \times \Delta \\ C_+B \times F \times F & \xleftarrow{(i_+ \circ \text{pr}_1, \mu_B^2)} & B \times F \times F & \xrightarrow{i_- \times 1_{F \times F}} & C_-B \times F \times F \end{array} \quad (3)$$

where $\Delta : F \rightarrow F \times F$ denotes the diagonal. Moreover, this diagram is pointwise equivalent to the diagram

$$\begin{array}{ccccc} p^{-1}(C_+B) & \xleftarrow{p^*i_+} & p^{-1}(B) & \xrightarrow{p^*i_-} & p^{-1}(C_-B) \\ \downarrow & & \downarrow & & \downarrow \\ (p_{\Sigma B}^2)^{-1}(C_+B) & \xleftarrow{(p_{\Sigma B}^2)^*i_+} & (p_{\Sigma B}^2)^{-1}(B) & \xrightarrow{(p_{\Sigma B}^2)^*i_-} & (p_{\Sigma B}^2)^{-1}(C_-B) \end{array}$$

where all columns are the restriction of the fiberwise diagonal $\Delta : X \rightarrow X_{\Sigma B}^2$. Thus the map between the (homotopy) pushouts of the rows of (3) is equivalent to $\Delta : X \rightarrow X_{\Sigma B}^2$.

3 Parametrized topological complexity

Let $p : X \rightarrow B$ be a fiberwise space over B . We define the fiberwise path space $\mathcal{P}_B X$ by

$$\mathcal{P}_B X = \{\gamma : [0, 1] \rightarrow X \mid p \circ \gamma \text{ is a constant map}\}.$$

Observe that, if p is a fibration, then the evaluation map

$$\Pi : \mathcal{P}_B X \rightarrow X_B^2, \quad \gamma \mapsto (\gamma(0), \gamma(1)) \quad (4)$$

is a fibration. We now recall the definition of parametrized topological complexity.

Definition 1 [6, Definition 4.1] Let $p : X \rightarrow B$ be a fibration with path-connected fiber. The parametrized topological complexity of $p : X \rightarrow B$, denoted by $\text{TC}[p : X \rightarrow B]$, is the minimal integer k such that X_B^2 can be covered by $k + 1$ open sets, each of which admits

a local section of the evaluation fibration (4). If such an integer does not exist, then we set $\mathrm{TC}[p: X \rightarrow B] = \infty$.

For simplicity, we write $\mathrm{TC}[X \rightarrow B]$ whenever the indicated fibration is obvious.

Recall that the sectional category of a map $f: X \rightarrow Y$, denoted by $\mathrm{secat}(f)$, is defined as the minimal integer k such that Y can be covered by $k + 1$ open sets, each of which admits a local right homotopy inverse of f . If such an integer does not exist, then we set $\mathrm{secat}(f) = \infty$. Then by definition, we have

$$\mathrm{TC}[X \rightarrow B] = \mathrm{secat}(\Pi: \mathcal{P}_B X \rightarrow X_B^2).$$

Let $c: X \rightarrow \mathcal{P}_B X$ be the map sending $x \in X$ to the constant path at x . Then there is a commutative diagram of fiberwise maps over B

$$\begin{array}{ccc} X & \xrightarrow{c} & \mathcal{P}_B X \\ \Delta \downarrow & & \downarrow \Pi \\ X_B^2 & \xlongequal{\quad} & X_B^2 \end{array}$$

Notice that c is a fiberwise homotopy equivalence over B ; namely, Π is fiberwise homotopy equivalent to Δ . Thus, by the general property of sectional category, we get

$$\mathrm{TC}[X \rightarrow B] = \mathrm{secat}(\Delta: X \rightarrow X_B^2).$$

In the sequel, we will frequently consider the parametrized topological complexity of a map $p: X \rightarrow B$ that is not necessarily a fibration. In this case, we define

$$\mathrm{TC}[p: X \rightarrow B] = \mathrm{TC}[p': X' \rightarrow B],$$

where $p': X' \rightarrow B$ is a strict fibration that is equivalent to $p: X \rightarrow B$.

We recall two typical lower bounds for parametrized topological complexity. Let $\mathrm{TC}(Z)$ denote the topological complexity of a space Z . Namely, $\mathrm{TC}(Z) = \mathrm{TC}[Z \rightarrow *]$.

Proposition 3.1 [6, (4.4)] *Let $F \rightarrow X \rightarrow B$ be a homotopy fibration. Then*

$$\mathrm{TC}[X \rightarrow B] \geq \mathrm{TC}(F).$$

Example 1 Consider a homotopy fibration $S^n \rightarrow X \rightarrow B$. It is known that $\mathrm{TC}(S^n) = 1$ for n odd and $\mathrm{TC}(S^n) = 2$ for n even. Then by Proposition 3.1, we get

$$\mathrm{TC}[X \rightarrow B] \geq \begin{cases} 1 & n \text{ is odd} \\ 2 & n \text{ is even.} \end{cases} \quad (5)$$

For an augmented algebra A , let $\mathrm{nil}(I)$ denote the nilpotency of the augmented ideal I of A . Namely, $\mathrm{nil}(I)$ is the least integer k such that $I^{k+1} = 0$.

Proposition 3.2 [6, Proposition 7.3] *Let $X \rightarrow B$ be a homotopy fibration. Then there is an inequality*

$$\mathrm{TC}[X \rightarrow B] \geq \mathrm{nil}(\mathrm{Ker}\{\Delta^*: H^*(X_B^2) \rightarrow H^*(X)\}).$$

Example 2 Let $S^n \rightarrow X \rightarrow B$ be a homotopy fibration where $n \geq 1$ and B is simply-connected. We consider the Serre spectral sequence for this homotopy fibration. Then

$$E_2 \cong H^*(B) \otimes H^*(S^n).$$

Let u denote the generator of $H^n(S^n) \cong \mathbb{Z}$. Then $a = 1 \otimes (u \times 1 - 1 \times u)$ is a permanent cycle in the Serre spectral sequence for the homotopy fibration $S^n \times S^n \rightarrow X_B^2 \rightarrow B$. Clearly, $a \in \text{Ker}\{\Delta^*: H^*(X_B^2) \rightarrow H^*(X)\}$ is the generator of the augmented ideal. Note that $a^2 = 0$ for n odd, $a^2 \neq 0$ and $a^3 = 0$ for n even. Then by Proposition 3.2, we get the inequality (5).

Let us recall Farber and Weinberger's upper bound. Let $V \rightarrow B$ be a real vector bundle of rank $n + 1$, $X \rightarrow B$ be the unit sphere bundle, and $\tilde{X} \rightarrow B$ be the bundle of orthonormal 2-frames in V . Then there is a fiber bundle $S^{n-1} \rightarrow \tilde{X} \rightarrow X$.

Proposition 3.3 [14, Theorem 14] *Let $\tilde{X}$ and X be as above. Then there is an inequality*

$$\text{TC}[X \rightarrow B] \leq \text{secat}(\tilde{X} \rightarrow X) + 1.$$

We extend this upper bound to a homotopy fibration over a suspension with fiber S^{2n+1} .

Proposition 3.4 *Let $S^{2n+1} \rightarrow X \rightarrow \Sigma B$ be a homotopy fibration with B connected. Then*

$$\text{TC}[X \rightarrow \Sigma B] \leq 2.$$

Proof Recall the commutative diagram (3) for $F = S^{2n+1}$. Then the pushout of the second row, $X_{\Sigma B}^2$, has a disjoint partition

$$X_{\Sigma B}^2 = \Delta(X) \sqcup (C_+B \times A) \sqcup (\text{Int}(C_-B) \times A)$$

where $\Delta(X)$ denotes the image of the map between the pushouts of the rows of (3), A denotes the configuration space $\{(x, y) \in S^{2n+1} \times S^{2n+1} \mid x \neq y\}$, and $\text{Int}(C_-B)$ denotes the interior of C_-B . We aim to show that each disjoint subspaces admits a local right homotopy inverse of Δ , from which the statement follows by [15, Theorem 2.7]. Clearly, $\Delta: X \rightarrow X_{\Sigma B}^2$ has a section on $\Delta(X)$. We consider a commutative diagram

$$\begin{array}{ccccc} S^{2n+1} & \xrightarrow{x \mapsto (x, -x)} & A & \xrightarrow{\text{incl.}} & S^{2n+1} \times S^{2n+1} \\ \parallel & & \downarrow & & \downarrow \\ S^{2n+1} & \longrightarrow & C_+B \times A & \xrightarrow{\text{incl.}} & X_{\Sigma B}^2 \end{array}$$

where the right two columns are the fiber inclusions and the bottom left column is defined by composition. Since the top left column is a homotopy equivalence and C_+B is contractible, the bottom left column is a homotopy equivalence. Observe that, since the antipodal map $S^{2n+1} \rightarrow S^{2n+1}$ is homotopic to $1_{S^{2n+1}}$, the composite of the top rows is homotopic to the diagonal. Then $\Delta: X \rightarrow X_{\Sigma B}^2$ has a right homotopy inverse on $C_+B \times A$. We can see that $\Delta: X \rightarrow X_{\Sigma B}^2$ has a right homotopy inverse on $\text{Int}(C_-B) \times A$ verbatim. Thus local right homotopy inverses are constructed, finishing the proof. $\square$

4 The complex $W(\alpha)$

We begin by recalling the result of García-Calines and Vandembroucq [16]. Let $f: X \rightarrow Y$ be a map and let Z be the cofiber of f . The weak sectional category of f , denoted by $\text{wsecat}(f)$, is the minimal integer k such that the following composite is trivial:

$$Y \xrightarrow{\bar{\Delta}} Y^{\wedge(k+1)} \xrightarrow{g^{\wedge(k+1)}} Z^{\wedge(k+1)},$$

where $\bar{\Delta}$ denotes the reduced diagonal map, $Y^{\wedge(k+1)}$ denotes the smash product of $k+1$ copies of Y and $g: Y \rightarrow Z$ denotes the cofiber map of f . If such an integer does not exist, then we set $\text{wsecat}(f) = \infty$.

Recall that the weak category of a space Z , denoted by $\text{wcat}(Z)$, is defined as the minimal integer k such that $\bar{\Delta}: Z \rightarrow Z^{\wedge(k+1)}$ is trivial. If such an integer does not exist, then we set $\text{wcat}(Z) = \infty$. By definition, we clearly have $\text{wsecat}(f) \leq \text{wcat}(Z)$.

Proposition 4.1 [16, Theorem 21] *Let $f: X \rightarrow Y$ be a map and Z be the cofiber of f . If f admits a left homotopy inverse, then there are inequalities*

$$\text{nil}(\text{Ker}\{f^*: H^*(Y) \rightarrow H^*(X)\}) \leq \text{wcat}(Z) = \text{wsecat}(f) \leq \text{secat}(f).$$

Corollary 4.2 *For a homotopy fibration $p: X \rightarrow B$, there are inequalities*

$$\text{nil}(\text{Ker}\{\Delta^*: H^*(X_B^2) \rightarrow H^*(X)\}) \leq \text{wcat}(W) = \text{wsecat}(\Delta) \leq \text{TC}[X \rightarrow B]$$

where W is the cofiber of the fiberwise diagonal $\Delta: X \rightarrow X_B^2$.

Proof As the map Δ has an obvious left inverse, the statement follows from Proposition 4.1. $\square$

Let $S^n \rightarrow X \rightarrow S^{m+1}$ be a homotopy fibration with $m \geq n$. Then by Lemma 2.1, there is a map $\mu: S^m \times S^n \rightarrow S^n$ such that $\mu|_{* \times S^n} = 1_{S^n}$ and X is homotopy equivalent to the homotopy pushout of the cotriad

$$S^n \xleftarrow{\mu} S^m \times S^n \xrightarrow{\text{pr}_2} S^n \quad (6)$$

where pr_j denotes the j -th projection. Arguing as in [22, Section 3], we obtain the cell decomposition of X .

Lemma 4.3 *There is a homotopy equivalence*

$$X \simeq S^n \cup_{\chi(X)} e^{m+1} \cup e^{m+n+1}$$

such that $X/S^n \simeq S^{m+1} \vee S^{m+n+1}$ and the composite

$$X \rightarrow X/S^n \simeq S^{m+1} \vee S^{m+n+1} \xrightarrow{q_1} S^{m+1}$$

is identified with the projection $X \rightarrow S^{m+1}$, where q_1 denotes the quotient map.

Here we note that $\chi(X) = \mu|_{S^m \times *}$. Then as in [2, Theorem 4.4.8], for any map $\hat{\mu}: S^m \times S^n \rightarrow S^n$ with $\hat{\mu}|_{S^m \vee S^n} = \chi(X) + 1_{S^n}$, there is $\alpha \in \pi_{m+n}(S^n)$ such that $\hat{\mu}$ equals the composite

$$\mu(\alpha): S^m \times S^n \rightarrow (S^m \times S^n) \vee S^{m+n} \xrightarrow{\mu+\alpha} S^n$$

where the first map is the homotopy coaction associated to the homotopy cofibration $S^{m+n-1} \rightarrow S^m \vee S^n \rightarrow S^m \times S^n$. Let $X(\alpha)$ denote the homotopy pushout of the cotriad $S^n \xleftarrow{\mu(\alpha)} S^m \times S^n \xrightarrow{\text{pr}_2} S^n$.

Proposition 4.4 *There exists a homotopy fibration $S^n \rightarrow X(\alpha) \rightarrow S^{m+1}$ with $\chi(X(\alpha)) = \chi(X)$.*

Proof We recall Lemma 2.1. Consider a commutative diagram

$$\begin{array}{ccccc} C_+(S^m) \times S^n & \xleftarrow{(i_+ \circ \text{pr}_1, \mu(\alpha))} & S^m \times S^n & \xrightarrow{i_- \times 1_{S^n}} & C_-(S^m) \times S^n \\ \text{pr}_1 \downarrow & & \text{pr}_1 \downarrow & & \downarrow \text{pr}_1 \\ C_+(S^m) & \xleftarrow{i_+} & S^m & \xrightarrow{i_-} & C_-(S^m). \end{array}$$

Recall that $\mu(\alpha)|_{* \times S^n} = 1_{S^n}$. Then by [25, Lemma 2.9], we obtain a quasifibration $X(\alpha) \rightarrow S^{m+1}$ with fiber S^n by taking the (homotopy) pushouts of the rows. Thus the first statement is proved. The second statement immediately follows from Lemma 4.3. $\square$

Moreover, for any homotopy fibration $S^n \rightarrow \widehat{X} \rightarrow S^{m+1}$ with $\chi(\widehat{X}) = \chi(X)$, there exists an element $\alpha \in \pi_{m+n}(S^n)$ such that $\widehat{X}$ has the same homotopy type as $X(\alpha)$. Remark that $X = X(0)$.

We define the complex $W(\alpha)$ as the cofiber of the fiberwise diagonal

$$\Delta: X(\alpha) \rightarrow X(\alpha)_{S^{m+1}}^2.$$

In the sequel, we concentrate our attention on the homotopy fibrations

$$S^{2n+1} \rightarrow X(\alpha) \rightarrow S^{m+1} \quad \text{with} \quad 1 \leq n \quad \text{and} \quad 2n+1 \leq m \leq 4n-1. \quad (7)$$

We now consider the cell decomposition of $W(\alpha)$. We denote the universal Whitehead product $A \star B \rightarrow \Sigma A \vee \Sigma B$ by w and the k -skeleton of a CW complex A by A_k .

Proposition 4.5 *There is a homotopy equivalence*

$$W(\alpha)_{4n+2} \simeq S^{2n+1} \cup_{-[1_{S^{2n+1}}, 1_{S^{2n+1}}]} e^{4n+2}.$$

Proof Recall that $m+1 < 4n+2 < m+2n+2$ by (7). By (3) and Lemma 4.3,

$$(X(\alpha)_{S^{m+1}}^2)_{4n+2} \simeq (S^{2n+1} \times S^{2n+1}) \cup_{\Delta \circ \chi(X)} e^{m+1}$$

such that the diagram

$$\begin{array}{ccc} X(\alpha)_{4n+2} & \xrightarrow{\Delta} & (X(\alpha)_{S^{m+1}}^2)_{4n+2} \\ \parallel & & \parallel \\ S^{2n+1} \cup_{\chi(X)} e^{m+1} & \xrightarrow{\tilde{\Delta}} & (S^{2n+1} \times S^{2n+1}) \cup e^{m+1} \\ \downarrow & & \downarrow \\ S^{m+1} & \xlongequal{\quad} & S^{m+1} \end{array} \quad (8)$$

commutes up to homotopy, where $\tilde{\Delta}$ is an extension of the diagonal $\Delta: S^{2n+1} \rightarrow S^{2n+1} \times S^{2n+1}$ and the bottom columns are the projections. Then $W(\alpha)_{4n+2}$ is the cofiber of

$\Delta: S^{2n+1} \rightarrow S^{2n+1} \times S^{2n+1}$, and so there is a homotopy commutative diagram

$$\begin{array}{ccccc}
 * & \xrightarrow{\quad} & S^{4n+1} & \xlongequal{\quad} & S^{4n+1} \\
 \downarrow & & \downarrow w & & \downarrow \\
 S^{2n+1} & \xrightarrow{\quad \nabla \quad} & S^{2n+1} & \vee & S^{2n+1} \xrightarrow{1+(-1)} S^{2n+1} \\
 \parallel & & \downarrow & & \downarrow \\
 S^{2n+1} & \xrightarrow{\quad \Delta \quad} & S^{2n+1} \times S^{2n+1} & \longrightarrow & W(\alpha)_{4n+2}
 \end{array}$$

such that all rows are homotopy cofibrations, where ∇ denotes the suspension comultiplication. Since the left and middle columns are homotopy cofibrations, so is the right column. Thus since $(1_{S^{2n+1}} + (-1)_{S^{2n+1}}) \circ w = [1_{S^{2n+1}}, -1_{S^{2n+1}}] = -[1_{S^{2n+1}}, 1_{S^{2n+1}}]$, the statement follows. $\square$

Corollary 4.6 *The complex $W(\alpha)_{4n+2}$ is a suspension, and there is a homotopy equivalence*

$$\Sigma(W(\alpha)_{4n+2}) \simeq S^{2n+2} \vee S^{4n+3}.$$

Proof Consider the EHP sequence

$$\pi_{4n}(S^{2n}) \xrightarrow{E} \pi_{4n+1}(S^{2n+1}) \xrightarrow{H} \pi_{4n+1}(S^{4n+1}) \xrightarrow{P} \pi_{4n-1}(S^{2n}).$$

Then $\pi_{4n+1}(S^{2n+1})$ is a finite abelian group, and so the map H is trivial. Then the map E is surjective, implying that the Whitehead product $[1_{S^{2n+1}}, 1_{S^{2n+1}}]$ is a suspension. Thus the first statement follows from Proposition 4.5. The second statement follows from Proposition 4.5 and the fact that the suspension of $[1_{S^{2n+1}}, 1_{S^{2n+1}}]$ is trivial. $\square$

We describe $W(\alpha)$ as the homotopy pushout of a certain cotriad. Let $A \ltimes B$ be the half-smash product $A \times B / (A \times *)$, and let $\text{pr}_2: A \ltimes B \rightarrow B$ denotes the second projection. Observe that, if A is contractible, then pr_2 is a homotopy equivalence.

Lemma 4.7 *There is a map $\tilde{\mu}(\alpha): S^m \ltimes W(\alpha)_{4n+2} \rightarrow W(\alpha)_{4n+2}$ such that the complex $W(\alpha)$ is homotopy equivalent to the homotopy pushout of the cotriad*

$$W(\alpha)_{4n+2} \xleftarrow{\tilde{\mu}(\alpha)} S^m \ltimes W(\alpha)_{4n+2} \xrightarrow{\text{pr}_2} W(\alpha)_{4n+2}.$$

Proof Recall the commutative diagram (3) for $B = S^m$ and $\mu = \mu(\alpha)$. Then the map between the pushouts of the rows is the fiberwise diagonal $\Delta: X(\alpha) \rightarrow X(\alpha)_{S^{m+1}}^2$. Notice that, as in the proof of Proposition 4.5, $W(\alpha)_{4n+2}$ is the cofiber of the diagonal $\Delta: S^{2n+1} \rightarrow S^{2n+1} \times S^{2n+1}$. Then by taking the cofibers of the columns, we get a cotriad

$$C_+ S^m \ltimes W(\alpha)_{4n+2} \leftarrow S^m \ltimes W(\alpha)_{4n+2} \xrightarrow{i_- \ltimes 1} C_- S^m \ltimes W(\alpha)_{4n+2}.$$

Since colimits commute with colimits, the homotopy pushout of the above cotriad is $W(\alpha)$. Observe that the composite

$$S^m \ltimes W(\alpha)_{4n+2} \xrightarrow{i_- \ltimes 1} C_- S^m \ltimes W(\alpha)_{4n+2} \xrightarrow{\text{pr}_2} W(\alpha)_{4n+2}$$

is the second projection pr_2 . Let $\tilde{\mu}(\alpha)$ be the composite

$$\tilde{\mu}(\alpha): S^m \ltimes W(\alpha)_{4n+2} \rightarrow C_+ S^m \ltimes W(\alpha)_{4n+2} \xrightarrow{\text{pr}_2} W(\alpha)_{4n+2} \quad (9)$$

where the first map is the left map in the above cotriad. Thus since $C_+ S^m$ and $C_- S^m$ are contractible, the proof is complete. $\square$

We now define the map $\tilde{\mu}(\alpha): S^m \ltimes W(\alpha)_{4n+2} \rightarrow W(\alpha)_{4n+2}$ by (9). Notice that, by Proposition 4.5, $W(\alpha)_{4n+2} = W(0)_{4n+2}$. Let $i: S^{2n+1} \rightarrow W(\alpha)_{4n+2} = W(0)_{4n+2}$ denote the bottom cell inclusion, and define

$$\Lambda(\alpha): S^m \ltimes S^{2n+1} \rightarrow (S^m \ltimes S^{2n+1}) \vee S^{m+2n+1} \xrightarrow{\tilde{\mu}(0) \circ (1 \times i) + i \circ \alpha} W(0)_{4n+2} = W(\alpha)_{4n+2},$$

where the first map is the homotopy coaction associated to the homotopy cofibration $S^{m+2n} \rightarrow S^{2n+1} \rightarrow S^m \ltimes S^{2n+1}$.

Lemma 4.8 $\tilde{\mu}(\alpha) \circ (1_{S^m} \ltimes i) = \Lambda(\alpha)$.

Proof Let $q_1: S^m \times S^{2n+1} \rightarrow S^m \ltimes S^{2n+1}$ be the projection. We first aim to show that

$$\tilde{\mu}(\alpha) \circ (1_{S^m} \ltimes i) \circ q_1 = \Lambda(\alpha) \circ q_1.$$

Let $i_1: S^n \rightarrow S^n \times S^n$ denote the first inclusion. By (3) and Lemma 4.7, there is a homotopy commutative diagram

$$\begin{array}{ccc} S^m \times S^{2n+1} & \xrightarrow{\mu(\alpha)} & S^{2n+1} \\ \downarrow 1 \times i_1 & & \downarrow i_1 \\ S^m \times (S^{2n+1} \times S^{2n+1}) & \xrightarrow{\mu(\alpha) \times \mu(\alpha)} & S^{2n+1} \times S^{2n+1} \\ \downarrow & & \downarrow \\ S^m \ltimes W(\alpha)_{4n+2} & \xrightarrow{\tilde{\mu}(\alpha)} & W(\alpha)_{4n+2} \end{array}$$

where the bottom square is given by taking the cofiber of the left square of (3). Note that the composites of the left and right columns are $(1_{S^m} \ltimes i) \circ q_1$ and i , respectively. Then we get

$$\tilde{\mu}(\alpha) \circ (1_{S^m} \ltimes i) \circ q_1 = i \circ \mu(\alpha).$$

In particular, for $\alpha = 0$, we get $\tilde{\mu}(0) \circ (1_{S^m} \ltimes i) \circ q_1 = i \circ \mu(0)$. Then by the naturality of the homotopy coaction, there is a homotopy commutative diagram

$$\begin{array}{ccccc} S^m \times S^{2n+1} & \xrightarrow{\text{coaction}} & (S^m \times S^{2n+1}) \vee S^{m+2n+1} & \xrightarrow{i \circ \mu(0) + i \circ \alpha} & W(\alpha)_{4n+2} \\ q_1 \downarrow & & q_1 \vee 1 \downarrow & & \parallel \\ S^m \ltimes S^{2n+1} & \xrightarrow{\text{coaction}} & (S^m \ltimes S^{2n+1}) \vee S^{m+2n+1} & \xrightarrow{\tilde{\mu}(0) \circ (1 \times i) + i \circ \alpha} & W(\alpha)_{4n+2}. \end{array}$$

Thus we obtain

$$\tilde{\mu}(\alpha) \circ (1_{S^m} \ltimes i) \circ q_1 = i \circ \mu(\alpha) = \Lambda(\alpha) \circ q_1.$$

It is well known that the loop space ΩS^{2n+2} has the homotopy type of a CW complex

$$S^{2n+1} \cup_{-[1_{S^{2n+1}}, 1_{S^{2n+1}}]} e^{4n+2} \cup e^{6n+3} \cup \dots$$

Then by Proposition 4.5, $W(\alpha)_{4n+2}$ can be regarded as the $(6n+2)$ -skeleton of ΩS^{n+2} . Since $m+2n+1 < 6n+2$ by (7), we get a commutative diagram of homotopy sets

$$\begin{array}{ccc} [S^m \times S^{2n+1}, W(\alpha)_{4n+2}] & \xrightarrow{q_1^*} & [S^m \times S^{2n+1}, W(\alpha)_{4n+2}] \\ \cong \downarrow & & \downarrow \cong \\ [S^m \times S^{2n+1}, \Omega S^{n+2}] & \xrightarrow{q_1^*} & [S^m \times S^{2n+1}, \Omega S^{n+2}] \end{array}$$

where the two columns are isomorphisms of pointed sets. Moreover, the bottom row can be considered as a group homomorphism. Note that the connecting map $\delta: S^m \times S^{2n+1} \rightarrow S^{m+1}$ of the cofibration

$$S^m \xrightarrow{i_1} S^m \times S^{2n+1} \xrightarrow{q_1} S^m \times S^{2n+1}$$

is null-homotopic; then the bottom row is an injective homomorphism. Thus the top row is an injective pointed map, completing the proof. $\square$

We now give a cell structure of $W(\alpha)$, which has 4 cells. Recall that, for a suspension space A , there is a map $s: \Sigma^m A \rightarrow S^m \times A$ which induces a homotopy cofibration

$$\Sigma^m A \xrightarrow{s} S^m \times A \xrightarrow{\text{pr}_2} A \quad (10)$$

and is natural with respect to suspensions. Let $\bar{\mu}(\alpha): \Sigma^m(W(\alpha)_{4n+2}) \rightarrow W(\alpha)_{4n+2}$ denote the composite

$$\Sigma^m(W(\alpha)_{4n+2}) \xrightarrow{s} S^m \times W(\alpha)_{4n+2} \xrightarrow{\bar{\mu}(\alpha)} W(\alpha)_{4n+2}.$$

Proposition 4.9 *The complex $W(\alpha)$ is homotopy equivalent to the cofiber of $\bar{\mu}(\alpha)$.*

Proof By (10), there is a homotopy commutative diagram

$$\begin{array}{ccccc} * & \xleftarrow{\quad} & \Sigma^m(W(\alpha)_{4n+2}) & \xrightarrow{\bar{\mu}(\alpha)} & W(\alpha)_{4n+2} \\ \downarrow & & \downarrow s & & \parallel \\ W(\alpha)_{4n+2} & \xleftarrow{\text{pr}_2} & S^m \times W(\alpha)_{4n+2} & \xrightarrow{\bar{\mu}(\alpha)} & W(\alpha)_{4n+2} \\ \parallel & & \downarrow \text{pr}_2 & & \downarrow \\ W(\alpha)_{4n+2} & \xlongequal{\quad} & W(\alpha)_{4n+2} & \longrightarrow & * \end{array} \quad (11)$$

where all columns are homotopy cofibrations. Notice that colimits commute with colimits; then we take the homotopy pushouts of the rows to get a homotopy cofibration

$$Z \rightarrow W(\alpha) \rightarrow *$$

by Lemma 4.7, where Z denotes the cofiber of the map $\bar{\mu}(\alpha): \Sigma^m(W(\alpha)_{4n+2}) \rightarrow W(\alpha)_{4n+2}$. Thus we obtain $Z \simeq W(\alpha)$, as stated. $\square$

Corollary 4.10 *There are maps $\lambda(\alpha): S^{m+2n+1} \rightarrow W(\alpha)_{4n+2}$ and $\phi(\alpha): S^{m+4n+2} \rightarrow W(\alpha)_{4n+2}$ such that*

$$W(\alpha) \simeq W(\alpha)_{4n+2} \cup_{\lambda(\alpha)} e^{m+2n+2} \cup_{\phi(\alpha)} e^{m+4n+3}.$$

Proof Combine Corollary 4.6 and Proposition 4.9. $\square$

Corollary 4.11 $1 \leq \text{cat}(W(\alpha)) \leq 2$.

Proof By Corollary 4.6 and Proposition 4.9, the cone-length of $W(\alpha)$ is at most 2, implying $\text{cat}(W(\alpha)) \leq 2$. On the other hand, we have $\text{cat}(W(\alpha)) \geq 1$ as $W(\alpha)$ is not contractible. $\square$

We compare the attaching maps $\lambda(\alpha)$ and $\lambda(0)$ of $(m + 2n + 2)$ -cell. By Proposition 4.5, $W(\alpha)_{4n+2} = W(0)_{4n+2}$; thus, both $\lambda(\alpha)$ and $\lambda(0)$ are maps into $W(0)_{4n+2}$. Recall that $i: S^{2n+1} \rightarrow W(\alpha)_{4n+2} = W(0)_{4n+2}$ denotes the bottom cell inclusion.

Proposition 4.12 $\lambda(\alpha) = \lambda(0) + i \circ \alpha$.

Proof The map $s: \Sigma^m(W(\alpha)_{4n+2}) \rightarrow S^m \times W(\alpha)_{4n+2}$ restricts to a map $\bar{s}: S^{m+2n+1} \rightarrow S^m \times S^{2n+1}$, so there is a homotopy commutative diagram

$$\begin{array}{ccc} S^{m+2n+1} & \xrightarrow{\bar{s}} & S^m \times S^{2n+1} \\ \parallel & & \downarrow \Lambda(\alpha) \\ S^{m+2n+1} & \xrightarrow{\lambda(0)+i \circ \alpha} & W(\alpha)_{4n+2}. \end{array}$$

On the other hand, i is a suspension by Corollary 4.7. Thus by Lemma 4.8, we obtain

$$\lambda(\alpha) = \tilde{\mu}(\alpha) \circ s \circ \Sigma^m i = \tilde{\mu}(\alpha) \circ (1_{S^{2n+1}} \times i) \circ \bar{s} = \Lambda(\alpha) \circ \bar{s} = \lambda(0) + i \circ \alpha,$$

completing the proof. $\square$

5 Crude Hopf invariant

By Corollary 4.11, we can deduce that $\text{wcat}(W(\alpha))$ is 1 or 2. In this section, we provide a sufficient condition for $\text{wcat}(W(\alpha)) = 2$ in terms of the crude Hopf invariant of the attaching map of a cell of $W(\alpha)$ and prove Theorem 1.2 and Corollary 1.3.

Let us recall the definition of the crude Hopf invariant [4]. For $k > 1$, let $T^k(Z)$ denote the k -fat wedge of Z , that is, $T^k(Z)$ is the subspace of Z^k consisting of points $(x_1, \dots, x_k)$ such that at least one of x_j is the basepoint. Let F denote the homotopy fiber of the inclusion $T^k(Z) \rightarrow Z^k$ and let $\hat{F}$ denote the homotopy fiber of the projection $Z^k \rightarrow Z^{\wedge k}$. Since $T^k(Z) \rightarrow Z^k \rightarrow Z^{\wedge k}$ is a homotopy cofibration, the inclusion $T^k(Z) \rightarrow Z^k$ lifts to $\hat{F}$. Then there is a homotopy commutative diagram

$$\begin{array}{ccccccc} F & \xrightarrow{i} & T^k(Z) & \longrightarrow & Z^k & & \\ \downarrow & & \downarrow & & \parallel & & \\ \Omega Z^{\wedge k} & \longrightarrow & \hat{F} & \longrightarrow & Z^k & \longrightarrow & Z^{\wedge k} \end{array}$$

where two rows are homotopy fiber sequences. Arguing as in the proof of [4, Corollary 2.10], we get the following lemma.

Lemma 5.1 *There exists a natural left homotopy inverse $r: \Omega T^k(Z) \rightarrow \Omega F$ of Ωi .*

Recall that $\text{cat}(Z) \leq k - 1$ if and only if the diagonal map $Z \rightarrow Z^k$ lifts to a map $\gamma: Z \rightarrow T^k(Z)$. We call such a map γ a k -th structure map of Z .

Definition 2 [4, Definition 2.11] Let Z be a space with $\text{cat}(Z) = k - 1$ and a k -th structure map γ . The *crude Hopf γ -invariant*

$$\overline{H}_\gamma(\zeta) \in \pi_{p+1}(Z^{\wedge k})$$

of $\zeta \in \pi_p(Z)$ is defined by the adjoint of the composite

$$S^{p-1} \xrightarrow{\widehat{\zeta}} \Omega Z \xrightarrow{\Omega \gamma} \Omega T^k(Z) \xrightarrow{r} \Omega F \rightarrow \Omega^2 Z^{\wedge k}$$

where $\widehat{\zeta}$ denotes the adjoint of ζ .

Theorem 5.2 [4, Corollary 3.9] Let X be a space with $\text{cat}(Z) = k - 1$ and let $\zeta \in \pi_p(Z)$. If $\text{wcat}(Z \cup_\zeta e^{p+1}) = k$, then $\overline{H}_\gamma(\zeta) \neq 0$ for any k -th structure map γ .

We now apply this theorem to the attaching map $\lambda(\alpha): S^{m+2n+1} \rightarrow W(\alpha)_{4n+2}$ of the $(m + 2n + 2)$ -cell of $W(\alpha)$ (see Corollary 4.10). Let $\rho: W(\alpha)_{4n+2} \rightarrow S^{4n+2}$ denote the pinch map onto the top cell. Since $\text{cat}(W(\alpha)_{4n+2}) = 1$ by Corollary 4.6, there is a second structure map $W(\alpha)_{4n+2} \rightarrow W(\alpha)_{4n+2} \vee W(\alpha)_{4n+2}$.

Proposition 5.3 If $\overline{H}_\gamma(\lambda(\alpha)) \neq 0$ for any second structure map γ and $\eta \circ \rho \circ \lambda(\alpha) = 0$, then $\text{wcat}(W(\alpha)_{4n+2} \cup_{\lambda(\alpha)} e^{m+2n+2}) = 2$.

Proof By [5, Theorem 5.14], the reduced diagonal map

$$\overline{\Delta}: W(\alpha)_{m+2n+2} \rightarrow (W(\alpha)_{m+2n+2})^{\wedge 2}$$

is homotopic to the composite

$$W(\alpha)_{m+2n+2} \rightarrow S^{m+2n+2} \xrightarrow{\overline{H}_\gamma(\lambda(\alpha))} (W(\alpha)_{m+2n+2})^{\wedge 2}$$

for any second structure map γ , where the first map is the pinch map onto the top cell. Suppose that $\overline{\Delta}$ is null-homotopic. Then since there is the homotopy cofibration

$$W(\alpha)_{m+2n+2} \rightarrow S^{m+2n+2} \xrightarrow{\Sigma \lambda(\alpha)} \Sigma(W(\alpha)_{4n+2}),$$

$\overline{H}_\gamma(\lambda(\alpha))$ extends to a map $f: \Sigma(W(\alpha)_{4n+2}) \rightarrow (W(\alpha)_{m+2n+2})^{\wedge 2}$. Since the $(4n + 3)$ -skeleton of $(W(\alpha)_{m+2n+2})^{\wedge 2}$ is the bottom cell S^{4n+2} , the map f is homotopic to the composite

$$\Sigma(W(\alpha)_{4n+2}) \xrightarrow{\Sigma \rho} S^{4n+3} \xrightarrow{c \Sigma \eta} S^{4n+2} \xrightarrow{i} (W(\alpha)_{m+2n+2})^{\wedge 2}$$

for some $c \in \mathbb{Z}/2$, where i denotes the bottom cell inclusion. Then we get

$$\overline{H}_\gamma(\lambda(\alpha)) = f \circ \Sigma \lambda(\alpha) = i \circ \Sigma((c\eta) \circ \rho \circ \lambda(\alpha)) = 0,$$

which contradicts the assumption. Thus $\overline{\Delta}$ is not null-homotopic, implying

$$\text{wcat}(W(\alpha)_{m+2n+2}) \geq 2.$$

On the other hand, since $\text{cat}(W(\alpha)_{4n+2}) = 1$, we have $\text{wcat}(W(\alpha)_{m+2n+2}) \leq 2$, completing the proof. $\square$

Note that the crude Hopf invariant is determined by the choice of the k -th structure map.

Lemma 5.4 There are exactly two homotopy classes in the second structure maps

$$W(\alpha)_{4n+2} \rightarrow W(\alpha)_{4n+2} \vee W(\alpha)_{4n+2}.$$

Proof Let F denote the homotopy fiber of the inclusion $W(\alpha)_{4n+2} \vee W(\alpha)_{4n+2} \rightarrow W(\alpha)_{4n+2} \times W(\alpha)_{4n+2}$. Then since $W(\alpha)_{4n+2}$ is a suspension by Corollary 4.6, there is an exact sequence of groups

$$\begin{aligned} 1 \rightarrow [W(\alpha)_{4n+2}, F] &\rightarrow [W(\alpha)_{4n+2}, W(\alpha)_{4n+2} \vee W(\alpha)_{4n+2}] \\ &\rightarrow [W(\alpha)_{4n+2}, W(\alpha)_{4n+2} \times W(\alpha)_{4n+2}] \end{aligned}$$

by Lemma 5.1. Then the order of the set of the homotopy classes of second structure maps of $W(\alpha)_{4n+2}$ is equal to the order of $[W(\alpha)_{4n+2}, F]$. On the other hand, F has the homotopy type of $\Omega(W(\alpha)_{4n+2}) \star \Omega(W(\alpha)_{4n+2})$. Then by Proposition 4.5 and Corollary 4.6, the $(6n+1)$ -skeleton of F is $S^{4n+1} \vee S^{6n+1} \vee S^{6n+1}$, implying

$$[W(\alpha)_{4n+2}, F] \cong [W(\alpha)_{4n+2}, S^{4n+1}].$$

Moreover, the homotopy cofibration $S^{2n+1} \rightarrow W(\alpha)_{4n+2} \xrightarrow{\rho} S^{4n+2}$ induces the exact sequence

$$\pi_{2n+2}(S^{4n+1}) \rightarrow \pi_{4n+2}(S^{4n+1}) \rightarrow [W(\alpha)_{4n+2}, S^{4n+1}] \rightarrow \pi_{2n+1}(S^{4n+1}).$$

Thus since $\pi_{2n+2}(S^{4n+1}) = \pi_{2n+1}(S^{4n+1}) = 0$ and $\pi_{4n+2}(S^{4n+1}) \cong \mathbb{Z}/2$, the statement is proved. $\square$

Let $\nabla: W(\alpha)_{4n+2} \rightarrow W(\alpha)_{4n+2} \vee W(\alpha)_{4n+2}$ denote the suspension comultiplication, and let $\bar{\eta}$ be the composite

$$W(\alpha)_{4n+2} \rightarrow F \rightarrow W(\alpha)_{4n+2} \vee W(\alpha)_{4n+2}$$

where the first map is the generator of $[W(\alpha)_{4n+2}, F]$ and the second map is the fiber inclusion. Then by Lemma 5.4 and its proof, the set of all homotopy classes of second structure map of $W(\alpha)_{4n+2}$ is

$$\{\nabla, \nabla + \bar{\eta}\}.$$

From the proof of Lemma 5.4, we can see that $\bar{\eta}$ is the composite

$$W(\alpha)_{4n+2} \xrightarrow{\rho} S^{4n+2} \xrightarrow{\eta} S^{4n+1} \xrightarrow{w} S^{2n+1} \vee S^{2n+1} \rightarrow W(\alpha)_{4n+2} \vee W(\alpha)_{4n+2}. \quad (12)$$

For $k = 1, 2$, let $i_k: W(\alpha)_{4n+2} \rightarrow W(\alpha)_{4n+2} \vee W(\alpha)_{4n+2}$ denote the k -th inclusion.

Lemma 5.5 *Let X be a finite CW complex of dimension $\leq 6n - 1$. Then, for $\zeta \in [\Sigma X, W(\alpha)_{4n+2}]$, we have*

$$i_1 \circ \zeta + i_2 \circ \zeta = (i_1 + i_2) \circ \zeta + [i_1, i_2] \circ H(\zeta)$$

where $H(\zeta)$ is the second James–Hopf invariant of ζ .

Proof As in [3, II, Theorem 2.8], we have an expansion of $i_1 \circ \zeta + i_2 \circ \zeta$, and by degree reasons, terms higher than two vanish, although $W(\alpha)_{4n+2}$ is not a double suspension, in general. We then obtain equality in the statement. $\square$

Proposition 5.6 *If $\eta \circ \rho \circ \lambda(\alpha) = 0$, then $\overline{H}_{\nabla}(\lambda(\alpha)) = \overline{H}_{\nabla + \bar{\eta}}(\lambda(\alpha))$.*

Proof By the definition of crude Hopf invariants, it is sufficient to show that $\nabla \circ \lambda(\alpha) = (\nabla + \bar{\eta}) \circ \lambda(\alpha)$. By Lemma 5.5, we have

$$\nabla \circ \lambda(\alpha) + \bar{\eta} \circ \lambda(\alpha) = (\nabla + \bar{\eta}) \circ \lambda(\alpha) + [\nabla, \bar{\eta}] \circ H(\lambda(\alpha)).$$

By (12), $\bar{\eta} \circ \lambda(\alpha)$ is the composite

$$S^{m+2n+1} \xrightarrow{\lambda(\alpha)} W(\alpha)_{4n+2} \xrightarrow{\rho} S^{4n+2} \xrightarrow{\eta} S^{4n+1} \xrightarrow{w} S^{2n+1} \vee S^{2n+1} \rightarrow W(\alpha)_{4n+2} \vee W(\alpha)_{4n+2},$$

which is trivial by assumption. On the other hand, by (12), the Whitehead product $[\nabla, \bar{\eta}]$ is the composite

$$(\Sigma^{-1} W(\alpha)_{4n+2}) \star (\Sigma^{-1} W(\alpha)_{4n+2}) \xrightarrow{1 \star \Sigma^{-1} \rho} (\Sigma^{-1} W(\alpha)_{4n+2}) \star S^{4n+1} \xrightarrow{\omega} W(\alpha)_{4n+2} \vee W(\alpha)_{4n+2}$$

for some ω . By degree reasons, the composite

$$S^{m+2n+1} \xrightarrow{H(\lambda(\alpha))} (\Sigma^{-1} W(\alpha)_{4n+2}) \star (\Sigma^{-1} W(\alpha)_{4n+2}) \xrightarrow{1 \star \Sigma^{-1} \rho} (\Sigma^{-1} W(\alpha)_{4n+2}) \star S^{4n+1}$$

is trivial, implying $[\nabla, \bar{\eta}] \circ H(\lambda(\alpha)) = 0$. Thus the proof is complete. $\square$

Corollary 5.7 *If $\eta \circ \rho \circ \lambda(\alpha) = 0$ and $\overline{H}_{\nabla}(\lambda(\alpha)) \neq 0$, then $\text{wcat}(W(\alpha)) = 2$.*

Proof By Propositions 5.3 and 5.6, we get $\text{wcat}(W(\alpha)_{m+2n+2}) = 2$. Recall that there is a commutative diagram

$$\begin{array}{ccc} W(\alpha)_{m+2n+2} & \xrightarrow{\overline{\Delta}_1} & (W(\alpha)_{m+2n+2})^{\wedge 2} \\ \downarrow i & & \downarrow i^{\wedge 2} \\ W(\alpha) & \xrightarrow{\overline{\Delta}_2} & W(\alpha)^{\wedge 2} \end{array}$$

where $i: W(\alpha)_{m+2n+2} \rightarrow W(\alpha)$ denotes the inclusion and $\overline{\Delta}_1, \overline{\Delta}_2$ denote the reduced diagonals. Suppose that $\text{wcat}(W(\alpha)) = 1$. Then $\overline{\Delta}_2$ is trivial, and so $\overline{\Delta}_1$ can be lifted to $W(\alpha)_{m+2n+2} \rightarrow \tilde{F}$, where $\tilde{F}$ denotes the fiber of $i^{\wedge 2}$. By the Blakers–Massey theorem, $\tilde{F}$ is $(m + 6n + 2)$ -connected. Then the lift $W(\alpha)_{m+2n+2} \rightarrow \tilde{F}$ is trivial, which is a contradiction. Thus $\text{wcat}(W(\alpha)) \geq 2$, and by Corollary 4.11, the statement is proved. $\square$

We next aim to determine $\rho \circ \lambda(\alpha)$ for each homotopy fibration (7). Let $\text{Cyl}(f)$ denote the mapping cylinder of a map $f: X \rightarrow Y$ and let $\iota: A \star B \rightarrow \Sigma(A \times B)$ be the natural map $A \star B = (\text{Cyl}(\text{pr}_1)) \cup_{A \times B} (\text{Cyl}(\text{pr}_2)) \rightarrow (C_+(A \times B)) \cup_{A \times B} (C_-(A \times B)) = \Sigma(A \times B)$.

Then, for a map $\mu: A \times B \rightarrow C$, its Hopf construction $\mathcal{H}(\mu): A \star B \rightarrow \Sigma C$ is the composite

$$A \star B \xrightarrow{\iota} \Sigma(A \times B) \xrightarrow{\Sigma \mu} \Sigma C.$$

In particular, by setting μ as the projection $q: A \times B \rightarrow A \wedge B$, we obtain:

Lemma 5.8 *The composite*

$$A \star B \xrightarrow{\iota} \Sigma(A \times B) \xrightarrow{\Sigma q} \Sigma(A \wedge B)$$

is a homotopy equivalence.

Suppose that there is a cotriad

$$B \xleftarrow{\text{pr}_2} S^m \times B \xrightarrow{f} C.$$

We denote an extension of f by $\tilde{f}: S^m \times B \rightarrow C \cup e^{m+1}$. Let Z be the homotopy pushout of the above cotriad. Let $q_1: A \times B \rightarrow A \times B$ denote the projection.

Lemma 5.9 *The space ΣZ has the homotopy type of the cofiber of the Hopf construction*

$$\mathcal{H}(\tilde{f} \circ q_1): S^m \star B \rightarrow \Sigma(C \cup e^{m+1}).$$

Proof We consider a homotopy commutative diagram

$$\begin{array}{ccccc} S^m \star B & \xrightarrow{(\Sigma q_1) \circ \iota} & \Sigma(S^m \ltimes B) & \xrightarrow{\Sigma \tilde{f}} & \Sigma(C \cup e^{m+1}) \\ \downarrow & & \downarrow \Sigma \text{pr}_2 & & \downarrow \\ * & \xrightarrow{\quad} & \Sigma B & \xrightarrow{\quad} & \Sigma Z \end{array}$$

where the left and right squares are homotopy pushouts. Thus the total square is a homotopy pushout, completing the proof. $\square$

Observe that, if B is a suspension, then the map $\mathcal{H}(q_1) = \Sigma q_1 \circ \iota$ is homotopic to Σs , where s is the map as in (10).

We give a description of the map $\Sigma(\rho \circ \lambda(\alpha))$ in terms of Hopf construction. Let f denote the map $\mu(\alpha)_{S^m}^2 |_{S^m \times (S^{2n+1} \vee S^{2n+1})}$.

Lemma 5.10 *The map $\Sigma(\rho \circ \lambda(\alpha))$ is homotopic to the composite*

$$S^m \star S^{2n+1} \xrightarrow{1 \star i_1} S^m \star (S^{2n+1} \vee S^{2n+1}) \xrightarrow{\mathcal{H}(f)} \Sigma(S^{2n+1} \times S^{2n+1}) \xrightarrow{\Sigma q} \Sigma S^{2n+1} \wedge S^{2n+1} = S^{4n+3}.$$

Proof As in the proof of Proposition 3.4, there is a homotopy pushout

$$\begin{array}{ccc} S^m \times (S^{2n+1} \times S^{2n+1}) & \xrightarrow{\mu(\alpha)_{S^m}^2} & S^{2n+1} \times S^{2n+1} \\ \text{pr}_2 \downarrow & & \downarrow \\ S^{2n+1} \times S^{2n+1} & \xrightarrow{\quad} & (X(\alpha)_{S^{m+1}}^2). \end{array}$$

Then by Lemma 5.9, the skeleton $\Sigma((X(\alpha)_{S^{m+1}}^2)_{m+2n+2})$ has the homotopy type of the cofiber of the map

$$\mathcal{H}(\tilde{f} \circ q_1): S^m \star (S^{2n+1} \vee S^{2n+1}) \rightarrow \Sigma((S^{2n+1} \times S^{2n+1}) \cup e^{m+1}).$$

Clearly, we can get the similar description of $\Sigma(W(\alpha)_{m+2n+2})$ as the cofiber of $\Sigma \lambda(\alpha)$ such that there is a homotopy commutative diagram

$$\begin{array}{ccc} S^m \star (S^{2n+1} \vee S^{2n+1}) & \xrightarrow{1 \star (1+(-1))} & S^m \star S^{2n+1} \\ \downarrow \mathcal{H}(\tilde{f} \circ q_1) & & \downarrow \Sigma \lambda(\alpha) \\ \Sigma((S^{2n+1} \times S^{2n+1}) \cup e^{m+1}) & \xrightarrow{\Sigma j} & \Sigma(W(\alpha)_{4n+2}) \\ \downarrow & & \downarrow \\ \Sigma((X(\alpha)_{S^{m+1}}^2)_{m+2n+2}) & \xrightarrow{\quad} & \Sigma(W(\alpha)_{m+2n+2}), \end{array}$$

where j is the cofiber map of $\tilde{\Delta}$ in (8) and the two columns are homotopy cofibrations. On the other hand, there is a homotopy commutative diagram

$$\begin{array}{ccc} (S^{2n+1} \times S^{2n+1}) & \xrightarrow{q} & S^{4n+2} \\ \downarrow i & & \parallel \\ (S^{2n+1} \times S^{2n+1}) \cup e^{m+1} & \xrightarrow{\bar{\rho}} & S^{4n+2} \\ \downarrow j & & \parallel \\ W(\alpha)_{4n+2} & \xrightarrow{\rho} & S^{4n+2}, \end{array}$$

where i denotes the inclusion and $\bar{\rho}$ denotes the pinch map onto the top cell. Obviously we have $\mathcal{H}(\tilde{f} \circ q_1) = (\Sigma i) \circ \mathcal{H}(f)$, and then we obtain

$$\begin{aligned} (\Sigma \rho) \circ (\Sigma \lambda(\alpha)) &= (\Sigma \rho) \circ (\Sigma \lambda(\alpha)) \circ (1 \star (1 + (-1))) \circ (1 \star i_1) \\ &= (\Sigma \rho) \circ (\Sigma j) \circ \mathcal{H}(\tilde{f} \circ q_1) \circ (1 \star i_1) \\ &= (\Sigma \bar{\rho}) \circ \mathcal{H}(\tilde{f} \circ q_1) \circ (1 \star i_1) \\ &= (\Sigma \bar{\rho}) \circ (\Sigma i) \circ \mathcal{H}(f) \circ (1 \star i_1) \\ &= (\Sigma q) \circ \mathcal{H}(f) \circ (1 \star i_1). \end{aligned}$$

Thus the statement is proved. $\square$

Lemma 5.11 *The map $\Sigma(\rho \circ \lambda(\alpha))$ is homotopic to the composite*

$$S^m \star S^{2n+1} \xrightarrow{\mathcal{H}(\mu(\alpha), \chi(X) \circ \text{pr}_1)} \Sigma(S^{2n+1} \times S^{2n+1}) \xrightarrow{\Sigma q} \Sigma S^{2n+1} \wedge S^{2n+1} = S^{4n+3}.$$

Proof By the homotopy commutative diagram

$$\begin{array}{ccc} S^m \times S^{2n+1} & \xrightarrow{(\mu(\alpha), \chi(X) \circ \text{pr}_1)} & S^{2n+1} \times S^{2n+1} \\ \downarrow 1 \times i_1 & & \parallel \\ S^m \times (S^{2n+1} \vee S^{2n+1}) & \xrightarrow{\mu(\alpha)^2_{S^m}} & S^{2n+1} \times S^{2n+1}, \end{array}$$

we get $\mathcal{H}(f) \circ (1 \star i_1) = \mathcal{H}(\mu(\alpha), \chi(X) \circ \text{pr}_1)$. Thus by Lemma 5.10, the proof is complete. $\square$

We then give another description of the map $\rho \circ \lambda(\alpha)$ up to sign.

Theorem 5.12 *There is an equality*

$$\Sigma^{2n+1} \chi(X(\alpha)) = \pm \rho \circ \lambda(\alpha).$$

Proof Recall that $\chi(X(\alpha)) = \mu(\alpha)|_{S^m \times *}$. Then there is a homotopy commutative diagram

$$\begin{array}{ccccc} S^m \vee S^{2n+1} & \longrightarrow & S^m \times S^{2n+1} & \longrightarrow & S^m \wedge S^{2n+1} \\ \downarrow \nabla \circ \chi + i_1 & & \downarrow (\mu(\alpha), \chi \circ \text{pr}_1) & & \downarrow \theta \\ S^{2n+1} \vee S^{2n+1} & \longrightarrow & S^{2n+1} \times S^{2n+1} & \longrightarrow & S^{2n+1} \wedge S^{2n+1} \end{array} \quad (13)$$

where the two rows are homotopy cofibrations. Since $\Sigma\theta$ coincides with the composite in Lemma 5.11, up to sign,

$$\Sigma\theta = \pm\Sigma(\rho \circ \lambda(\alpha)).$$

As $m < 4n+2$, $\chi(X): S^m \rightarrow S^{2n+1}$ is a suspension by the Freudenthal suspension theorem, and so we have $\nabla \circ \chi(X(\alpha)) = i_1 \circ \chi(X(\alpha)) + i_2 \circ \chi(X(\alpha))$. Then by the additivity and the naturality of the Whitehead product, we get

$$\begin{aligned} [\nabla \circ \chi(X(\alpha)), i_1] &= [i_1 \circ \chi(X(\alpha)) + i_2 \circ \chi(X(\alpha)), i_1] \\ &= [i_1 \circ \chi(X(\alpha)), i_1] + [i_2 \circ \chi(X(\alpha)), i_1] \\ &= i_1 \circ [\chi(X(\alpha)), 1_{S^{2n+1}}] + [i_2 \circ \chi(X(\alpha)), i_1]. \end{aligned}$$

Then since $\chi(X(\alpha)) + 1_{S^{2n+1}}: S^m \vee S^{2n+1} \rightarrow S^{2n+1}$ extends to $\mu(\alpha): S^m \times S^{2n+1} \rightarrow S^{2n+1}$, we have $[\chi(X(\alpha)), 1_{S^{2n+1}}] = 0$, so

$$[\nabla \circ \chi(X(\alpha)), i_1] = [i_2 \circ \chi(X(\alpha)), i_1].$$

On the other hand, we have $[i_2 \circ \chi(X(\alpha)), i_1] = -[i_1 \circ \chi(X(\alpha)), i_2]$. Then the diagram (13) extends to

$$\begin{array}{ccccccc} S^{m-1} \star S^{2n} & \xrightarrow{w} & S^m \vee S^{2n+1} & \longrightarrow & S^m \times S^{2n+1} & \longrightarrow & S^m \wedge S^{2n+1} \\ \downarrow -\Sigma^{-1}\chi \star 1 & & \downarrow \nabla \circ \chi + i_1 & & \downarrow (\mu(\alpha), \chi \circ \text{pr}_1) & & \downarrow \theta \\ S^{2n} \star S^{2n} & \xrightarrow{w} & S^{2n+1} \vee S^{2n+1} & \longrightarrow & S^{2n+1} \times S^{2n+1} & \longrightarrow & S^{2n+1} \wedge S^{2n+1}. \end{array}$$

Hence we get

$$\pm\Sigma(\rho \circ \lambda(\alpha)) = \Sigma\theta = \pm\Sigma^{2n+2}\chi(X(\alpha)).$$

Thus since $\rho \circ \lambda(\alpha) \in \pi_{m+2n+1}(S^{4n+2})$ and $m+2n+1 \leq 2(4n+1)$, the statement is proved by the Freudenthal suspension theorem. $\square$

Corollary 5.13 *If $\Sigma(\eta \circ \chi(X(\alpha))) = 0$ and $\overline{H}_\nabla(\lambda(\alpha)) \neq 0$, then $\text{wcat}(W(\alpha)) = 2$.*

Proof Since $m \leq 4n-1$, $E^{2n}: \pi_{m+1}(S^{2n+1}) \rightarrow \pi_{m+2n+1}(S^{4n+1})$ is isomorphic by the Freudenthal suspension theorem. Then $\eta \circ \rho \circ \lambda(\alpha) = \pm\eta \circ \Sigma^{2n+1}(\chi(X(\alpha)))$ is trivial if and only if $\Sigma(\eta \circ \chi(X(\alpha)))$ is trivial. Thus by combining Corollary 5.7 and Theorem 5.12, the statement is proved. $\square$

We are now ready to prove Theorem 1.2. Let $i: S^{2n+1} \rightarrow W(\alpha)_{4n+2}$ denote the bottom cell inclusion.

Proof of Theorem 1.2 By Lemma 4.3, there is a map $\mu: S^m \times S^{2n+1} \rightarrow S^{2n+1}$ such that $\mu|_{S^m \times *} = \beta$ and X is homotopy equivalent to the homotopy pushout of the cotriad

$$S^{2n+1} \xleftarrow{\mu} S^m \times S^{2n+1} \xrightarrow{\text{pr}_2} S^{2n+1}.$$

By the assumption, $\Sigma(\eta \circ \beta) = 0$. If $\overline{H}_\nabla(\lambda(0))$ is nonzero, then $\text{wcat}(W(0)) = 2$ by Corollary 5.13. Hence by Proposition 3.4 and Corollary 4.2, we obtain $\text{TC}[X \rightarrow S^{m+1}] = 2$.

Suppose that $\overline{H}_\nabla(\lambda(0))$ is zero. By assumption, $\overline{H}_\nabla(\alpha) \neq 0 \in \pi_{m+2n+2}(S^{4n+2})$ and, by Corollary 4.6,

$$\begin{aligned} (W(\alpha)_{4n+2})^{\wedge 2} &\simeq (W(\alpha)_{4n+2} \wedge S^{2n+1}) \cup_{1_{W(\alpha)_{4n+2}} \wedge [1, -1]} (W(\alpha)_{4n+2} \wedge S^{4n+2}) \\ &\simeq (S^{4n+2} \vee S^{6n+3}) \vee (S^{6n+3} \vee S^{8n+4}). \end{aligned}$$

Then by the Blakers–Massey theorem, the $(6n + 2)$ -skeleton of the homotopy fiber of the map $i^{\wedge 2}: (S^{2n+1})^{\wedge 2} \rightarrow (W(\alpha)_{4n+2})^{\wedge 2}$ is $S^{6n+2} \vee S^{6n+2}$. Since $m + 2n + 2 \leq 6n + 1$, $i_*^{\wedge 2}: \pi_{m+2n+2}((S^{2n+1})^{\wedge 2}) \rightarrow \pi_{m+2n+2}((W(\alpha)_{4n+2})^{\wedge 2})$ is an isomorphism. Then since i is a suspension by Corollary 4.6, we obtain

$$\overline{H}_{\nabla}(\lambda(\alpha)) = \overline{H}_{\nabla}(\lambda(0) + i_*\alpha) = \overline{H}_{\nabla}(\lambda(0)) + \overline{H}_{\nabla}(i_*\alpha) = i_*^{\wedge 2}\overline{H}_{\nabla}(\alpha) \neq 0$$

by Proposition 4.12. Thus $\text{wcat}(W(\alpha)) = 2$ by Corollary 5.13; hence, $\text{TC}[X(\alpha) \rightarrow S^{m+1}] = 2$ by Proposition 3.4 and Corollary 4.2. $\square$

Next, we prove Corollary 1.3. Let us recall the relation between the second James–Hopf invariant and the crude Hopf invariant.

Theorem 5.14 [5, Theorem 5.12.] *Let $H: \pi_p(\Sigma B) \rightarrow \pi_p(\Sigma(B^{\wedge 2}))$ denote the second James–Hopf invariant and let $\overline{H}_{\nabla}: \pi_p(\Sigma B) \rightarrow \pi_{p+1}((\Sigma B)^{\wedge 2})$ denote the crude Hopf ∇ -invariant. Then $\Sigma H = -\overline{H}_{\nabla}$.*

Lemma 5.15 *Let $k = 1, 2, 5$ or $8i - 1$ with $i \geq 1$. Then, for each k , there are infinitely many n with $2n - 3 \geq k$ such that $\overline{H}_{\nabla}(\alpha) \neq 0$ for some $\alpha \in \pi_{k+4n+2}(S^{2n+1})$.*

Proof Note that $\overline{H}_{\nabla}(\alpha) \in \pi_{k+4n+3}(S^{4n+2})$ and $k + 4n + 2 \leq 6n - 1 < 8n$. Then by Theorem 5.14 and the Freudenthal suspension theorem, $E: \pi_{k+4n+2}(S^{4n+1}) \rightarrow \pi_{k+4n+3}(S^{4n+2})$ is isomorphic, and thus, $\overline{H}_{\nabla}(\alpha) \neq 0$ if and only if $H(\alpha) \neq 0$. Since $k + 4n + 2 \leq 6n - 1$, the EHP sequence

$$\pi_{k+4n+2}(S^{2n+1}) \xrightarrow{H} \pi_{k+4n+2}(S^{4n+1}) \xrightarrow{P} \pi_{k+4n}(S^{2n}) \xrightarrow{E} \pi_{k+4n+1}(S^{2n+1})$$

is exact. We claim that, for each k above, there are infinitely many n such that $[\alpha', 1_{S^{2n}}] = 0$ for some $\alpha' \neq 0 \in \pi_{k+2n+1}(S^{2n})$. Then by the EHP sequence, there exists $\alpha \in \pi_{k+4n+2}(S^{2n+1})$ such that $H(\alpha) = E^{2n+1}\alpha'$. Since $k + 2n + 1 \leq 4n - 2$,

$$E^{2n+1}: \pi_{k+2n+1}(S^{2n}) \rightarrow \pi_{k+4n+2}(S^{4n+1})$$

is an isomorphism by the Freudenthal suspension theorem and we obtain $E^{2n+1}\alpha' \neq 0$. Thus the statement follows.

Let ν denote the generator of the 2-component of the stable stem $\pi_3^S \cong \mathbb{Z}/12$. Then by [33, Chapter V], η^2 , η^3 and ν^2 have order 2 in the stable stem π_2^S , π_3^S and π_6^S , respectively. For $i \geq 1$, let $\eta\sigma_i \in \pi_{8i}^S$ denote the element of order 2 in [24, Chapter I]. Then we have:

- $[\eta^2, 1_{S^{2n}}] = 0$ if and only if $n \equiv 1 \pmod{2}$ by [18, Corollary 2].
- $[\eta^3, 1_{S^{2n}}] = 0$ if and only if $n \equiv 1 \pmod{2}$ by [32, 2.10].
- $[\nu^2, 1_{S^{2n}}] = 0$ if and only if $n \equiv 0 \pmod{4}$ by [23, Theorem 1.3] and [32, 2.11].
- $[\eta\sigma_i, 1_{S^{2n}}] = 0$ if $3 \leq \nu_2(8i + 2n + 2) \leq 4i$ by [24, Theorem D].

Here $\nu_2(l)$ denotes the 2-exponent of l . Therefore, our claim is proved. $\square$

Proof of Corollary 1.3 By Lemma 5.15, there are infinitely many pairs (m, n) of integers with

$$1 \leq n, \quad 2n + 1 \leq m \leq 4n - 1 \quad \text{and} \quad m - (2n + 1) = 1, 2, 5 \quad \text{or} \quad 8i - 1,$$

such that $\overline{H}_{\nabla}(\alpha) \neq 0$ for some $\alpha \in \pi_{m+2n+1}(S^{2n+1})$.

Let $S^{2n+1} \rightarrow X \rightarrow S^{m+1}$ be the trivial bundle. Then we have $\chi(X) = 0$ and $\text{TC}[X \rightarrow S^{m+1}] = 1$. By Corollary 5.13, it follows that $\text{wcat}(W(\alpha)) = 2$ as $\eta \circ \chi(X(\alpha)) = 0$. Thus by Proposition 3.4 and Corollary 4.2, we obtain $\text{TC}[X(\alpha) \rightarrow S^{m+1}] = 2$, completing the proof. $\square$

Remark 2 For the $m - (2n + 1) = 8i - 1$ cases above, there are infinitely many homotopy fibrations $S^{2n+1} \rightarrow X \rightarrow S^{m+1}$ such that

$$\chi(X) \neq 0 \quad \text{and} \quad \text{TC}[X \rightarrow S^{m+1}] = 2.$$

Namely by [1], there always exists an odd component $\beta \in \pi_{8i-1}^S$ for each i . Let $2l + 1$ be the order of β . Then since $[1_{S^{2n+1}}, 1_{S^{2n+1}}]$ have at most order 2, we get

$$\begin{aligned} [\beta, 1_{S^{2n+1}}] &= [1_{S^{2n+1}}, 1_{S^{2n+1}}] \circ \beta \\ &= [1_{S^{2n+1}}, 1_{S^{2n+1}}] \circ \beta \circ (2l + 1 - 2l) \\ &= [1_{S^{2n+1}}, 1_{S^{2n+1}}] \circ \beta \circ (2l + 1) - [1_{S^{2n+1}}, 1_{S^{2n+1}}] \circ \beta \circ 2l \\ &= 0 - [1_{S^{2n+1}}, 1_{S^{2n+1}}] \circ (2l)_{S^{4n+1}} \circ \beta = 0. \end{aligned}$$

Similarly, we get $\eta \circ \beta = 0$. Thus by combining Theorem 1.2 and Corollaries 5.13 and 5.15, we can prove the existence of such homotopy fibrations as stated.

6 Proof of Theorem 1.4 (1)

Throughout this section, let $V \rightarrow S^{2n+1}$ be a vector bundle of rank $2n + 1$ with $n \geq 1$. Let $X \rightarrow S^{2n+1}$ be the unit sphere bundle of V and let $\tilde{X} \rightarrow S^{2n+1}$ be the orthonormal 2-frame bundle of V . Then there is a fiber bundle $S^{2n-1} \rightarrow \tilde{X} \xrightarrow{p} X$.

For a map $p: X \rightarrow Y$, let $X \star_Y X$ denote the fiberwise join of two copies of X over Y , that is, the homotopy pushout of the cotriad $X \xleftarrow{\text{pr}_1} X_Y^2 \xrightarrow{\text{pr}_2} X$. The fiberwise join $X \star_Y X$ is a fiberwise space over Y equipped with a map between the homotopy pushouts of the rows of the commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{\text{pr}_1} & X_Y^2 & \xrightarrow{\text{pr}_2} & X \\ p \downarrow & & p \downarrow & & p \downarrow \\ Y & \xlongequal{\quad} & Y & \xlongequal{\quad} & Y. \end{array}$$

Notice that, if p is a fibration with fiber F , then the map $X \star_Y X \rightarrow Y$ is a homotopy fibration with fiber $F \star F$, the join of two copies of F .

Proposition 6.1 [29, Theorem 3] *For a fibration $X \rightarrow Y$, $\text{secat}(X \rightarrow Y) \leq 1$ if and only if $X \star_Y X \rightarrow Y$ has a right homotopy inverse.*

We aim to show $\text{secat}(p: \tilde{X} \rightarrow X) \leq 1$. By Proposition 6.1, this is equivalent to find a right homotopy inverse of the map $p \star_X p: \tilde{X} \star_X \tilde{X} \rightarrow X$. For the rest of this section, we only consider cohomology with mod 2 coefficients unless otherwise specified.

Lemma 6.2 *The mod 2 cohomology of X is given by*

$$H^*(X) = \Lambda(x_{2n}, x_{2n+1}), \quad |x_i| = i.$$

Proof It is sufficient to show that the mod 2 cohomology Serre spectral sequence for the fiber bundle $S^{2n} \rightarrow X \rightarrow S^{2n+1}$ collapses at the E_2 -term, which is equivalent to the Euler class $e(X)$ being trivial. Since $2e(X) = 0$ in $H^{2n+1}(S^{2n+1}; \mathbb{Z}) \cong \mathbb{Z}$, so $e(X) = 0$. $\square$

Lemma 6.3 *The mod 2 cohomology of $\tilde{X}$ is given by*

$$H^*(\tilde{X}) = \Lambda(x_{2n-1}, x_{2n}, x_{2n+1}), \quad |x_i| = i$$

such that $p^*(x_i) = x_i$ for $i = 2n, 2n + 1$.

Proof Let $V_2(\mathbb{R}^{2n+1})$ be the space of orthonormal 2-frames in $\mathbb{R}^{2n+1}$. Then there is a commutative diagram

$$\begin{array}{ccccc} S^{2n-1} & \xlongequal{\quad} & S^{2n-1} & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ V_2(\mathbb{R}^{2n+1}) & \longrightarrow & \tilde{X} & \longrightarrow & S^{2n+1} \\ \downarrow & & \downarrow p & & \parallel \\ S^{2n} & \longrightarrow & X & \longrightarrow & S^{2n+1} \end{array}$$

where all columns and rows are fibrations. We note that the left column is the unit tangent bundle of S^{2n} . Then the mod 2 cohomology Serre spectral sequence for the left column collapses at the E_2 -term. Thus by Lemma 6.2 and naturality of spectral sequences, the mod 2 cohomology Serre spectral sequence for the middle column also collapses at the E_2 -term. Therefore by Lemma 6.2, the statement follows. $\square$

Lemma 6.4 *The mod 2 cohomology of $\tilde{X} \star_X \tilde{X}$ is given by*

$$H^*(\tilde{X} \star_X \tilde{X}) = \Lambda(x_{2n}, x_{2n+1}, x_{4n-1}), \quad |x_i| = i$$

such that $(p \star_X p)^*(x_i) = x_i$ for $i = 2n, 2n + 1$.

Proof Note that the fiber bundle $S^{2n-1} \rightarrow \tilde{X} \rightarrow X$ is the unit sphere bundle of some vector bundle $V' \rightarrow X$; thus, the fiber bundle $S^{4n-1} = S^{2n-1} \star_X S^{2n-1} \rightarrow \tilde{X} \star_X \tilde{X} \xrightarrow{p \star_X p} X$ is the unit sphere bundle of the vector bundle $V' \oplus V' \rightarrow X$. In particular, $e(\tilde{X} \star_X \tilde{X}) = e(\tilde{X})^2$. The proof of Lemma 6.3 implies that $e(\tilde{X})$ is divisible by 2. Then $e(\tilde{X} \star_X \tilde{X})$ is also divisible by 2, hence the mod 2 cohomology Serre spectral sequence collapses at the E_2 -term. Thus the statement follows from Lemma 6.3. $\square$

Consider the homotopy commutative diagram

$$\begin{array}{ccccc} \tilde{X} & \longleftarrow & * & \longrightarrow & \tilde{X} \\ \parallel & & \downarrow & & \parallel \\ \tilde{X} & \xleftarrow{\text{pr}_1} & \tilde{X}_X^2 & \xrightarrow{p_2} & \tilde{X} \\ \downarrow & & \parallel & & \downarrow \\ * & \longleftarrow & \tilde{X}_X^2 & \longrightarrow & * \end{array}$$

such that all columns are cofibrations, where pr_i denotes the i -th projection. Notice that colimits commute with colimits; then by taking the homotopy pushouts of rows, we get a cofibration

$$\tilde{X} \vee \tilde{X} \xrightarrow{f} \tilde{X} \star_X \tilde{X} \xrightarrow{g} \Sigma(\tilde{X}_X^2).$$

Lemma 6.5 *The map $g^*: H^*(\Sigma(\tilde{X}_X^2)) \rightarrow H^*(\tilde{X} \star_X \tilde{X})$ is surjective for $* = 4n - 1$ and trivial for $* = 4n + 1$.*

Proof Consider the long exact sequence

$$\cdots \rightarrow H^*(\Sigma(\tilde{X}_X^2)) \xrightarrow{g^*} H^*(\tilde{X} \star_X \tilde{X}) \xrightarrow{f^*} H^*(\tilde{X} \vee \tilde{X}) \rightarrow \cdots.$$

By Lemma 6.3, pr_1 and pr_2 induce an isomorphism

$$H^{4n-1}(\tilde{X}) \oplus H^{4n-1}(\tilde{X}) \cong H^{4n-1}(\tilde{X}_X^2).$$

Then f^* is trivial for $* = 4n - 1$ and thus the $* = 4n - 1$ case is proved. By the definition of the map f , there is a homotopy commutative diagram

$$\begin{array}{ccc} \tilde{X} \vee \tilde{X} & \xrightarrow{f} & \tilde{X} \star_X \tilde{X} \\ \downarrow p \vee p & & \downarrow p \star p \\ X \vee X & \longrightarrow & X \end{array}$$

where the bottom arrow is the folding map. Then by Lemmas 6.2 and 6.3, the map f is injective in H^{4n+1} . Thus by the above long exact sequence, the $* = 4n + 1$ case is proved as well. $\square$

Proposition 6.6 *If $n \geq 2$, then $\text{secat}(p: \tilde{X} \rightarrow X) \leq 1$.*

Proof By Proposition 6.1, it is sufficient to show that $p \star_X p: \tilde{X} \star_X \tilde{X} \rightarrow X$ has a right homotopy inverse. Clearly, the only obstruction $\mathfrak{o}$ for the existence of such a right homotopy inverse lies in

$$H^{4n+1}(X; \pi_{4n}(S^{4n-1})) \cong H^{4n+1}(X).$$

Suppose $\mathfrak{o} \neq 0$. Then by Lemma 6.2, $\mathfrak{o} = x_{2n}x_{2n+1}$, and by Lemma 6.4, $(p \star_X p)^*(\mathfrak{o}) = x_{2n}x_{2n+1} \neq 0$. On the other hand, since $n \geq 2$, there is $y \in H^{4n-1}(\tilde{X} \star_X \tilde{X})$ such that $\text{Sq}^2 y = (p \star_X p)^*(\mathfrak{o})$ by [29, Chapter III, §4], and by Lemma 6.5, there is $z \in H^{4n-1}(\Sigma(\tilde{X}_X^2))$ such that $g^*(z) = y$. Then by Lemma 6.5, we get

$$0 \neq (p \star_X p)^*(\mathfrak{o}) = \text{Sq}^2 y = \text{Sq}^2 g^*(z) = g^*(\text{Sq}^2 z) = 0$$

which is a contradiction. Thus $\mathfrak{o} = 0$, as desired. $\square$

Proof of Theorem 1.4 (1) For $n \geq 2$, the statement is proved by combining (5), Propositions 3.3 and 6.6. For $n = 1$, all vector bundles over S^3 of rank 3 are trivial as $\pi_3(BO(3)) = 0$. Thus by [6, Example 4.2], we obtain $\text{TC}[X \rightarrow S^3] = \text{TC}(S^2) = 2$. $\square$

7 Proofs of Theorem 1.4 (2) and Theorem 1.5

7.1 Proof of Theorem 1.4 (2)

Throughout this subsection, we consider the homotopy fibrations

$$S^{2n+1} \rightarrow X \rightarrow S^{2n+2} \quad \text{with } n \neq 1, 3.$$

Namely, we exclude the cases in which the fiber is S^3 or S^7 . For $3 \leq k < l$, let $f: S^l \rightarrow S^k$ be a map of order $b < \infty$ and let Z denote the cofiber of f . By the Blakers–Massey theorem, the $(k + l - 1)$ -skeleton of the homotopy fiber of the pinch map $\rho: Z \rightarrow S^{l+1}$ is homotopy equivalent to S^k , so there is an exact sequence

$$\pi_{l+1}(S^k) \rightarrow \pi_{l+1}(Z) \xrightarrow{\rho_*} \pi_{l+1}(S^{l+1}) \xrightarrow{\delta} \pi_l(S^k).$$

Clearly, the composite

$$\pi_l(S^l) \xrightarrow{E} \pi_{l+1}(S^{l+1}) \xrightarrow{\delta} \pi_l(S^k)$$

represents the induced map f_* , where E denotes the suspension map. Then we obtain:

Lemma 7.1 $\pi_{l+1}(S^{l+1})/\ker \delta \cong \mathbb{Z}/b$.

As in the proof of Corollary 4.6, the Whitehead product $-[1_{S^{2n+1}}, 1_{S^{2n+1}}] \in \pi_{4n+1}(S^{2n+1})$ is a suspension. Here we consider the order of its desuspension.

Lemma 7.2 *The set $\{\bar{w} \in \pi_{4n}(S^{2n}) \mid \Sigma \bar{w} = -[1_{S^{2n+1}}, 1_{S^{2n+1}}]\}$ has exactly two elements, and each of its element has order*

$$\begin{cases} 2 & (n \equiv 0 \pmod{2}) \\ 4 & (n \equiv 1 \pmod{2}). \end{cases}$$

Proof Let A denote the set in the statement. Consider the EHP sequence

$$\mathbb{Z}/2 \cong \pi_{4n+2}(S^{4n+1}) \xrightarrow{P} \pi_{4n}(S^{2n}) \xrightarrow{E} \pi_{4n+1}(S^{2n+1}) \xrightarrow{H} \pi_{4n+1}(S^{4n+1}) \cong \mathbb{Z}. \quad (14)$$

Then by [19, Corollary 4.18], $P(\eta) = [\eta, 1_{S^{2n}}] \neq 0$, and thus the map P is injective. Thus $\text{Ker } E = \text{Im } P \cong \mathbb{Z}/2$, which implies that $|A| = 2$. Take any $a \in A$. Then $A = \{a, a + P(\eta)\}$. If a has order 2^k for $k \geq 1$, then as $2P(\eta) = 0$, $a + P(\eta)$ also has order 2^k . Thus it is sufficient to find an element of A having order 2 for $n \equiv 0 \pmod{2}$ and 4 for $n \equiv 1 \pmod{2}$. If $n \equiv 0 \pmod{2}$, then by [32, 2.24], there is $a \in \pi_{4n}(S^{2n})$ of order 2 such that

$$Ea = [1_{S^{2n-1}}, 1_{S^{2n-1}}] = -[1_{S^{2n-1}}, 1_{S^{2n-1}}].$$

Then we have $a \in A$ and thus the first case is proved.

Consider the EHP sequence

$$\mathbb{Z} \cong \pi_{4n+3}(S^{4n+3}) \xrightarrow{P} \pi_{4n+1}(S^{2n+1}) \xrightarrow{E} \pi_{4n+2}(S^{2n+2}) \xrightarrow{H} \pi_{4n+2}(S^{4n+3}) = 0.$$

Then $\text{Ker } E = \text{Im } P = \mathbb{Z}/2\{[1_{S^{2n+1}}, 1_{S^{2n+1}}]\}$, where $[1_{S^{2n+1}}, 1_{S^{2n+1}}]$ has order 2 as $n \neq 1, 3$. By the Freudenthal suspension theorem, $\text{Ker } E \cong \text{Ker}\{E^2: \pi_{4n+1}(S^{2n+1}) \rightarrow \pi_{4n+3}(S^{2n+3})\}$. In the EHP sequence (14), the map H is trivial as $\pi_{4n+1}(S^{2n+1})$ is a torsion group; therefore, the map E is surjective. Then the natural map

$$\text{Ker}\{E^3: \pi_{4n}(S^{2n}) \rightarrow \pi_{4n+3}(S^{2n+3})\} \rightarrow \text{Ker}\{E^2: \pi_{4n+1}(S^{2n+1}) \rightarrow \pi_{4n+3}(S^{2n+3})\}$$

is surjective too. If $n \equiv 1 \pmod{2}$, then $\text{Ker}\{E^3: \pi_{4n}(S^{2n}) \rightarrow \pi_{4n+3}(S^{2n+3})\} \cong \mathbb{Z}/4$ by [32, §3]. Thus there is $a \in \pi_{4n}(S^{2n})$ of order 4 such that $Ea = [1_{S^{2n+1}}, 1_{S^{2n+1}}]$. Namely, A has an element of order 4. $\square$

We choose $\bar{w} \in \pi_{4n}(S^{2n})$ such that $\Sigma \bar{w} = -[1_{S^{2n+1}}, 1_{S^{2n+1}}]$. Let $\overline{W} = S^{2n} \cup_{\bar{w}} e^{4n+1}$. Then by Proposition 4.5, $\Sigma \overline{W} = W(0)_{4n+2}$.

Proposition 7.3 *There is an isomorphism*

$$\text{Coker}\{E: \pi_{4n+1}(\overline{W}) \rightarrow \pi_{4n+2}(W(0)_{4n+2})\} \cong \begin{cases} \mathbb{Z}/2 & (n \equiv 1 \pmod{2}) \\ 0 & (n \equiv 0 \pmod{2}). \end{cases}$$

Proof By the Blakers–Massey Theorem, there is a commutative diagram with exact rows

$$\begin{array}{ccccccc} \pi_{4n+1}(S^{2n}) & \xrightarrow{\bar{i}_*} & \pi_{4n+1}(\overline{W}) & \longrightarrow & \pi_{4n+1}(S^{4n+1}) & \xrightarrow{\bar{\delta}} & \pi_{4n}(S^{2n}) \\ \downarrow E & & \downarrow E & & \downarrow E & & \downarrow E \\ \pi_{4n+2}(S^{2n+1}) & \xrightarrow{i_*} & \pi_{4n+2}(W(0)_{4n+2}) & \longrightarrow & \pi_{4n+2}(S^{4n+2}) & \xrightarrow{\delta} & \pi_{4n+1}(S^{2n+1}) \end{array}$$

where i and $\bar{i}$ are the bottom cell inclusions. Then we get a commutative diagram of exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Im } \bar{i}_* & \longrightarrow & \pi_{4n+1}(\overline{W}) & \longrightarrow & \text{Ker } \bar{\delta} \longrightarrow 0 \\ \parallel & & \downarrow f & & \downarrow E & & \downarrow g \\ 0 & \longrightarrow & \text{Im } i_* & \longrightarrow & \pi_{4n+2}(W(0)_{4n+2}) & \longrightarrow & \text{Ker } \delta \longrightarrow 0. \end{array}$$

Hence by the snake lemma, we get an exact sequence

$$\text{Coker } f \rightarrow \text{Coker}\{E: \pi_{4n+1}(\overline{W}) \rightarrow \pi_{4n+2}(W(0)_{4n+2})\} \rightarrow \text{Coker } g \rightarrow 0.$$

Consider the EHP sequence

$$\pi_{4n+1}(S^{2n}) \xrightarrow{E} \pi_{4n+2}(S^{2n+1}) \xrightarrow{H} \pi_{4n+2}(S^{4n+1}) \xrightarrow{P} \pi_{4n}(S^{2n}).$$

As in the proof of Lemma 7.2, the map P is injective, and thus the map E is surjective. Then $\text{Coker } f = 0$. By Lemmas 7.1 and 7.2, we have

$$\text{Ker } \bar{\delta} \cong \begin{cases} 4\mathbb{Z} & (n \equiv 1 \pmod{2}) \\ 2\mathbb{Z} & (n \equiv 0 \pmod{2}) \end{cases} \quad \text{and} \quad \text{Ker } \delta \cong 2\mathbb{Z} \subset \mathbb{Z} \cong \pi_{4n+2}(S^{4n+2}).$$

Thus since $E: \pi_{4n+1}(S^{4n+1}) \rightarrow \pi_{4n+2}(S^{4n+2})$ is an isomorphism, $\text{Coker } g \cong \mathbb{Z}/2$ for $n \equiv 1 \pmod{2}$ and $\text{Coker } g = 0$ for $n \equiv 0 \pmod{2}$. $\square$

We now consider a homotopy fibration $S^{4n-1} \rightarrow X \rightarrow S^{4n}$ with $n \geq 3$.

Proof of Theorem 1.4 (2) Let $\chi(X) = 4l+2 \in \pi_{4n-1}(S^{4n-1}) \cong \mathbb{Z}$. Then by Theorem 5.12, $\rho \circ \lambda(0) = \pm \Sigma^{4n-1} \chi(X) = \pm(4l+2)$. Consider the maps $E: \pi_{8n-3}(\overline{W}) \rightarrow \pi_{8n-2}(W(0)_{8n-2})$ and $g: \text{Ker } \bar{\delta} \rightarrow \text{Ker } \delta$ in the proof of Proposition 7.3. By Proposition 7.3, $\rho \circ \lambda(0) = \pm(4l+2) \notin \text{Im } g$, implying

$$\lambda(0) \notin \text{Im}\{E: \pi_{8n-3}(\overline{W}) \rightarrow \pi_{8n-2}(W(0)_{8n-2})\}.$$

Then by the EHP sequence

$$\pi_{8n-3}(\overline{W}) \xrightarrow{E} \pi_{8n-2}(W(0)_{8n-2}) \xrightarrow{H} \pi_{8n-2}(\Sigma(\overline{W})^{\wedge 2}) \xrightarrow{P} \pi_{8n-4}(\overline{W}),$$

$H(\lambda(0)) \neq 0$. Since the $(8n-1)$ -skeleton of $\Sigma(\overline{W})^{\wedge 2}$ is S^{8n-3} , we have

$$\pi_{8n-2}(\Sigma(\overline{W})^{\wedge 2}) \cong \pi_{8n-2}(S^{8n-3}) \cong \mathbb{Z}/2,$$

and by the Freudenthal suspension theorem,

$$\pi_{8n-2}(\Sigma(\overline{W})^{\wedge 2}) \xrightarrow{E} \pi_{8n-1}((W(0)_{8n-2})^{\wedge 2})$$

is an isomorphism. Then by Theorem 5.14, we get

$$\overline{H}_{\nabla}(\lambda(0)) = -\Sigma H(\lambda(0)) \neq 0.$$

Thus as $\eta \circ \chi(X) = 0$, we obtain $\text{wcat}(W(0)) = 2$ by Corollary 5.13. Therefore by Proposition 3.4 and Corollary 4.2, $\text{TC}[X \rightarrow S^{4n}] = 2$. $\square$

As an application, we obtain:

Corollary 7.4 *For any homotopy fibration $S^{4n+1} \rightarrow X \rightarrow S^{4n+2}$ with $n \geq 1$,*

$$\text{wcat}(W(0)_{8n+3}) = 1.$$

Proof By Proposition 7.3, $E: \pi_{8n+1}(\overline{W}) \rightarrow \pi_{8n+2}(W(0)_{8n+2})$ is surjective; thus, $\lambda(0) \in \text{Im } E \subset \pi_{8n+2}(W(0)_{8n+2})$. Then $W(0)_{8n+3}$ is a suspension and thus the statement follows. $\square$

7.2 The $n = 3, 7$ cases

Throughout this subsection, we consider the homotopy fibrations

$$S^n \rightarrow X \rightarrow S^{n+1} \quad \text{with } n = 3, 7,$$

and regard S^3 and S^7 as the unit spheres of $\mathbb{H}$ and $\mathbb{O}$, respectively. Then S^n has a product. We recall from [31] an explicit description of $\pi_n(SO(n+1))$. We define maps $\rho, \sigma: S^n \rightarrow SO(n+1)$ by

$$\rho(x)(y) = xyx^{-1}, \quad \sigma(x)(y) = xy$$

for $x, y \in S^n$. Let $\mathbb{Z}\{a_1, \dots, a_k\}$ denote the free abelian group generated by $a_1, \dots, a_k$ and let $\mathbf{r}: SU(\frac{n+1}{2}) \rightarrow SO(n+1)$ denote the realification map.

Lemma 7.5 1. $\pi_n(SO(n+1)) = \mathbb{Z}\{\rho, \sigma\}$.

2. $\sigma \in \text{Im}\{\mathbf{r}_*: \pi_n(SU(\frac{n+1}{2})) \rightarrow \pi_n(SO(n+1))\}$.

Let $T \rightarrow S^{n+1}$ denote the unit tangent bundle.

Lemma 7.6 *The map $-\rho + 2\sigma: S^n \rightarrow SO(n+1)$ determines T .*

Proof Let $V \rightarrow S^{n+1}$ be the real vector bundle of rank $n+1$ which is determined by the map $k\rho + l\sigma$. Then by [30, Theorems 4.1 and 4.4], we have

$$p_1(V) = \pm 2(2k + l)u \quad \text{for } n = 3, \quad \text{and} \quad p_2(V) = \pm 6(2k + l)u \quad \text{for } n = 7,$$

where $p_i(V) \in H^{4i}(S^{n+1}; \mathbb{Z})$ denotes the i -th Pontryagin class and u denotes the generator of $H^{n+1}(S^{n+1}; \mathbb{Z}) \cong \mathbb{Z}$.

Let $X \rightarrow S^{n+1}$ be the unit sphere bundle of V . Then X has the homotopy type of the homotopy pushout of the cotriad $S^n \xleftarrow{\text{Pr}_2} S^n \times S^n \xrightarrow{\mu} S^n$, where μ is defined by

$$\mu(x, y) = ((k\rho + l\sigma)(x))(y).$$

Since $\mu(x, 1) = x^l$, we get $\chi(X) = \mu|_{S^n \times *} = l$, which is the Euler class of $V \rightarrow S^{n+1}$. Then, for the tangent bundle $TS^{n+1} \rightarrow S^{n+1}$, we get $l = 2$ as the Euler class $e(TS^{n+1})$ is 2. Furthermore, the total Pontryagin class $p(TS^{n+1})$ is equal to 1; therefore, we get $k = -1$. Thus $TS^{n+1} \rightarrow S^{n+1}$ is determined by $-\rho + 2\sigma$, as stated. $\square$

We define

$$\mu: S^n \times S^n \rightarrow S^n, \quad (x, y) \mapsto xyx = ((-\rho + 2\sigma)(x))(y).$$

Then by Lemma 7.6, T has the homotopy type of the homotopy pushout of the cotriad $S^n \xleftarrow{\text{pr}_2} S^n \times S^n \xrightarrow{\mu} S^n$. We also define

$$\hat{\mu}: S^n \times S^n \rightarrow S^n \quad (x, y) \mapsto x^2y = (2\sigma(x))(y).$$

Then as $\mu|_{S^n \vee S^n} = \hat{\mu}|_{S^n \vee S^n}$, there exists $\alpha \in \pi_{2n}(S^n)$ such that $\hat{\mu} = \mu(\alpha)$. Thus by Lemma 7.5, the homotopy fibration $S^n \rightarrow T(\alpha) \rightarrow S^{n+1}$ defined by $\hat{\mu} = \mu(\alpha)$ is equivalent to the unit sphere bundle of a complex vector bundle $V \rightarrow S^{n+1}$ of rank $\frac{n+1}{2}$.

Recall that $\pi_{2n}(S^n)$ is isomorphic to $\mathbb{Z}/12$ for $n = 3$ and $\mathbb{Z}/120$ for $n = 7$.

Proposition 7.7 *The element $\alpha \in \pi_{2n}(S^n)$ is a generator.*

Proof Let $\omega = \mu(0) \cdot \mu(\alpha)^{-1}: S^n \times S^n \rightarrow S^n$. Then ω is given by a map

$$S^n \times S^n \rightarrow S^n, \quad (x, y) \mapsto (x, y) \mapsto (xyx)(x^2y)^{-1} = x(yxy^{-1}x^{-1})x^{-1}.$$

Let $\tau: S^n \wedge S^n \rightarrow S^n \wedge S^n$ denote the map $(x, y) \mapsto (xy, x)$ and $\gamma: S^n \wedge S^n \rightarrow S^n$ denote the commutator map. Then, for the pinch map $q: S^n \times S^n \rightarrow S^n \wedge S^n$, it is straightforward to check

$$\omega = \gamma \circ \tau \circ q.$$

On the other hand, we have $\omega = \alpha \circ q$ by definition. As S^n is an H-space, the map $q^*: [S^n \wedge S^n, S^n] \rightarrow [S^n \times S^n, S^n]$ is injective by [34, Lemma 1.3.5]; thus, we get $\alpha = \gamma \circ \tau$. Thus since τ is a homotopy equivalence, α is a generator of $\pi_{2n}(S^n)$ if and only if γ is so. Now by [20, §9] and [27, Theorem II], γ generates $\pi_{2n}(S^n)$, so α does too. $\square$

Proof of Theorem 1.5 for $n = 3, 7$ By Propositions 4.12 and 7.7, we have $\lambda(\alpha) = \lambda(0) + i_*\alpha$, where α is a generator of $\pi_{2n}(S^n)$ and i is the bottom cell inclusion $S^n \rightarrow W(\alpha)_{2n}$. Then by [33, Chapter V], we get

$$H(\alpha) = \eta \neq 0,$$

Therefore we have $\overline{H}_\nabla(\alpha) \neq 0$ by Theorem 5.14. Arguing as in the proof of Theorem 1.2, the map $i_*^{\wedge 2}: \pi_{2n+1}(S^{2n}) \rightarrow \pi_{2n+1}((W(\alpha)_{2n})^{\wedge 2})$ is an isomorphism, and so we get

$$\overline{H}_\nabla(\lambda(\alpha)) - \overline{H}_\nabla(\lambda(0)) = \overline{H}_\nabla(\lambda(\alpha) - \lambda(0)) = \overline{H}_\nabla(i_*\alpha) = i_*^{\wedge 2}\overline{H}_\nabla(\alpha) \neq 0.$$

Recall that there is a complex vector bundle $V \rightarrow S^{n+1}$ such that $T(\alpha)$ is equivalent to its unit sphere bundle. Then $\text{TC}[T(\alpha) \rightarrow S^{n+1}] = 1$ by [14, Corollary 17], implying that $\overline{H}_\nabla(\lambda(\alpha)) = 0$, and so we get $\overline{H}_\nabla(\lambda(0)) \neq 0$. We note that $\Sigma(\eta \circ \chi(T)) = \Sigma(\eta \circ 2) = 0$; then $\text{wcat}(W(0)) = 2$ by Corollary 5.13. Thus by Proposition 3.4 and Corollary 4.2, $\text{TC}[T \rightarrow S^{n+1}] = 2$ as stated. $\square$

Corollary 7.8 *Let k be an integer that is prime to 12 if $n = 3$, prime to 120 if $n = 7$, and let l even. If the bundle $S^n \rightarrow X \rightarrow S^{n+1}$ is determined by $k\rho + l\sigma$, then $\text{TC}[X \rightarrow S^{n+1}] = 2$.*

Proof Let $S^n \rightarrow \widehat{X} \rightarrow S^{n+1}$ be the sphere bundle that is determined by $l\sigma$. Then there is an element $\alpha \in \pi_{2n}(S^n)$ such that $\widehat{X} = X(\alpha)$. Arguing as in the proof of Proposition 7.7, it follows that α is a generator. Thus since $\chi(X) = l$ is even, the statement is proved in the same way as in the proof of Theorem 1.5 for $n = 3, 7$. $\square$

8 The unit tangent bundle

Throughout this subsection, we consider the unit tangent bundle

$$S^{4n+1} \rightarrow T \rightarrow S^{4n+2}.$$

Let $\tilde{T} \rightarrow S^{4n+2}$ be the orthonormal 2-frame bundle of the tangent bundle of S^{4n+2} . Then there is a fiber bundle $S^{4n} \rightarrow \tilde{T} \xrightarrow{p} T$, which is equivalent to the fiber bundle

$$S^{4n} \rightarrow V_3(\mathbb{R}^{4n+3}) \xrightarrow{p} V_2(\mathbb{R}^{4n+3}),$$

where $V_k(\mathbb{R}^{4n+3})$ is the space of the orthonormal k -frames in $\mathbb{R}^{4n+3}$ and p is the projection of the first 2 vectors. Assuming Proposition 1.6, we get an inequality

$$\text{wcat}(W(0)) = 1 \leq \text{TC}[T \rightarrow S^{4n+2}] \leq \text{secat}(p) + 1$$

by Proposition 3.3. On the other hand, by [21, Theorem 1.15], $\text{secat}(p) = 0$ if and only if $n = 0, 1$. Thus we obtain $\text{TC}[T \rightarrow S^{4n+2}] = 1$ for $n = 0, 1$. Observe that, for $n \geq 2$, $\text{TC}[T \rightarrow S^{4n+2}]$ cannot be determined by the upper and lower bounds we have mentioned.

We now aim to prove Proposition 1.6. First, we provide another description of $W(0)$ as the cofiber of a map.

Lemma 8.1 *The complex $W(0)$ has the homotopy type of the cofiber of $p: \tilde{T} \rightarrow T$.*

Proof Let $T_+ \rightarrow S^{4n+2}$ and $T_- \rightarrow S^{4n+2}$ be the subbundle of $T_{S^{4n+2}}^2 \rightarrow S^{4n+2}$ defined by

$$T_+ = \{(x, y) \in T_{S^{4n+2}}^2 \mid \langle x, y \rangle \geq 0\} \quad \text{and} \quad T_- = \{(x, y) \in T_{S^{4n+2}}^2 \mid \langle x, y \rangle \leq 0\}$$

where $\langle x, y \rangle$ denotes the fiberwise inner product. Then since $\tilde{T} = T_+ \cap T_-$, $T_{S^{4n+2}}^2$ is the pushout of the cotriad

$$T_+ \hookleftarrow \tilde{T} \hookrightarrow T_-.$$

Since $\tilde{T}$ is obviously a neighborhood deformation retract of T_- , the inclusion $\tilde{T} \rightarrow T_-$ is a cofibration. Then $T_{S^{4n+2}}^2$ is also the homotopy pushout of the cotriad above. Now we have a commutative diagram

$$\begin{array}{ccccc} T_+ & \xrightarrow{\text{pr}_1} & T & \xleftarrow{\text{pr}_1} & T_- \\ \downarrow & & \downarrow & & \downarrow \\ S^{4n+2} & \xlongequal{\quad} & S^{4n+2} & \xlongequal{\quad} & S^{4n+2} \end{array}$$

where the first projection pr_1 is a homotopy equivalence. Then since the composite

$$T_+ \xrightarrow{\text{pr}_1} T \xrightarrow{\Delta} T_{S^{4n+2}}^2$$

is homotopic to the inclusion $T_+ \rightarrow T_{S^{4n+2}}^2$, we get a homotopy pushout diagram

$$\begin{array}{ccc} \tilde{T} & \longrightarrow & T \\ \downarrow p & & \downarrow \Delta \\ T & \longrightarrow & T_{S^{4n+2}}^2. \end{array}$$

Notice that colimits commute with colimits; thus by taking the cofiber of the vertical maps, the statement is proved. $\square$

We consider the cell decomposition of $W(0)$ in relation to the cell structures of T and $\tilde{T}$. We recall the cell decomposition of the real Stiefel manifolds by James [21].

Lemma 8.2 *Let $Q_{k+l,l} = \mathbb{R}P^{k+l}/\mathbb{R}P^k$.*

1. *There is a cell decomposition $T = V_2(\mathbb{R}^{4n+3}) = Q_{4n+2,2} \cup e^{8n+3}$.*
2. *There is a cell decomposition*

$$\tilde{T} = V_3(\mathbb{R}^{4n+3}) = (Q_{4n+2,3} \cup e^{8n+1} \cup e^{8n+2}) \cup_{\psi} e^{8n+3} \cup e^{12n+3}.$$

3. *The map $p: \tilde{T} \rightarrow T$ is a cellular map.*

We also recall that there are cell decompositions

$$Q_{4n+2,2} = S^{4n+1} \cup_2 e^{4n+2} \quad \text{and} \quad Q_{4n+2,3} = S^{4n} \vee (S^{4n+1} \cup_2 e^{4n+2}).$$

From the construction, it follows that $\tilde{T}_{8n+2}$ is the pullback of the triad

$$Q_{4n+2,2} \hookrightarrow T \xleftarrow{p} \tilde{T}.$$

Since $Q_{4n+2,2}$ is a suspension and $Q_{4n+2,3}$ is the cofiber of the trivial map $S^{4n} \cup_2 e^{4n+1} \rightarrow S^{4n}$, there is a map $\mu: (S^{4n} \cup_2 e^{4n+1}) \times S^{4n} \rightarrow S^{4n}$ with

$$\mu|_{(S^{4n} \cup e^{4n+1}) \vee S^{4n}} = 0 + 1_{S^{4n}}$$

such that $\tilde{T}_{8n+2}$ has the homotopy type of the homotopy pushout of the cotriad

$$S^{4n} \xleftarrow{\text{pr}_2} (S^{4n} \cup_2 e^{4n+1}) \times S^{4n} \xrightarrow{\mu} S^{4n}.$$

Let W' be the cofiber of $p: \tilde{T}_{8n+2} \rightarrow Q_{4n+2,2}$. Clearly, $\dim W' = 8n + 3$ and W' is $4n$ -connected.

Lemma 8.3 *The connecting map $\delta: W' \rightarrow \Sigma(\tilde{T}_{8n+2})$ of the homotopy cofibration*

$$\tilde{T}_{8n+2} \xrightarrow{p} Q_{4n+2,2} \rightarrow W'$$

admits a left homotopy inverse.

Proof Consider the homotopy commutative diagram

$$\begin{array}{ccccc} * & \longleftarrow & (S^{4n} \cup_2 e^{4n+1}) & \longrightarrow & * \\ \downarrow & & \downarrow i_1 & & \downarrow \\ S^{4n} & \xleftarrow{\text{pr}_2} & (S^{4n} \cup_2 e^{4n+1}) \times S^{4n} & \xrightarrow{\mu} & S^{4n} \\ \downarrow & & \downarrow \text{pr}_1 & & \downarrow \\ * & \longleftarrow & (S^{4n} \cup_2 e^{4n+1}) & \longrightarrow & * \end{array}$$

Note that, for each column, the composite of the two vertical maps is the identity. Then by taking the homotopy pushout of each row, we get $Q_{4n+2,2} \rightarrow \tilde{T}_{8n+2} \xrightarrow{p} Q_{4n+2,2}$, whose composite is homotopic to the identity map. Then the map $p: \tilde{T}_{8n+2} \rightarrow Q_{4n+2,2}$ admits a right homotopy inverse. This implies that $Q_{4n+2,2} \rightarrow W'$ is null-homotopic, and thus δ admits a left homotopy inverse. $\square$

Lemma 8.4 $\text{cat}(W') = 1$.

Proof By Lemma 8.3, we have $\text{cat}(W') \leq \text{cat}(\Sigma(\tilde{T}_{8n+2})) = 1$. Thus since W' is not contractible, the statement is proved. $\square$

Furthermore, the pushout construction of a spherical fibration in the proof of Proposition 3.4 can be generalized; there is a map $\kappa: S^{8n+2} \times S^{4n} \rightarrow \tilde{T}_{8n+2}$ with $\kappa|_{* \times S^{4n}}$ being the inclusion of the bottom cell, such that there is a homotopy commutative diagram

$$\begin{array}{ccccc}
 S^{4n} & \xleftarrow{\text{pr}_2} & S^{8n+2} \times S^{4n} & \xrightarrow{\kappa} & \tilde{T}_{8n+2} \\
 \downarrow & & \downarrow \text{pr}_1 & & \downarrow p \\
 * & \xleftarrow{\quad} & S^{8n+2} & \xrightarrow{\quad} & Q_{4n+2,2} \\
 \downarrow & & \downarrow & & \downarrow \\
 S^{4n+1} & \xleftarrow{\text{pr}_2} & S^{8n+2} \times S^{4n+1} & \xrightarrow{\tilde{\kappa}} & W' \\
 \parallel & & \downarrow & & \downarrow \delta \\
 S^{4n+1} & \xleftarrow{\Sigma \text{pr}_2} & \Sigma(S^{8n+2} \times S^{4n}) & \xrightarrow{\Sigma \kappa} & \Sigma(\tilde{T}_{8n+2})
 \end{array}$$

where all columns are homotopy cofiber sequences and $p: \tilde{T} \rightarrow T$ is the map between the homotopy pushouts of the top two rows. Notice that colimits commute with colimits; then by taking the homotopy pushouts of the rows, we get a cofiber sequence $\tilde{T} \xrightarrow{p} T \rightarrow W(0) \xrightarrow{\delta} \Sigma \tilde{T}$. Then by the argument as in the proof of Proposition 4.9, $W(0)$ has the homotopy type of the cofiber of the composite

$$S^{12n+3} = S^{8n+2} \star S^{4n} \xrightarrow{s} S^{8n+2} \times S^{4n+1} \xrightarrow{\tilde{\kappa}} W',$$

where s is the map as in (10). Note that $\kappa|_{S^{8n+2} \times *}$ corresponds to the attaching map ψ of the $(8n+3)$ -cell of $\tilde{T}$.

Lemma 8.5 *The composite*

$$S^{8n+2} \star S^{4n} \xrightarrow{s} S^{8n+2} \times S^{4n+1} \xrightarrow{\tilde{\kappa}} W' \xrightarrow{q} W'/W'_{4n+1} = S^{8n+2} \cup_2 e^{8n+3}$$

is trivial.

Proof Consider the homotopy commutative diagram

$$\begin{array}{ccccc}
 Q_{4n+2,3} & \longrightarrow & \tilde{T}_{8n+2} & \xrightarrow{q} & S^{8n+1} \cup_2 e^{8n+2} \\
 \downarrow & & \downarrow p & & \downarrow \\
 Q_{4n+2,2} & \xlongequal{\quad} & Q_{4n+2,2} & \longrightarrow & * \\
 \downarrow & & \downarrow & & \downarrow \\
 S^{4n+1} & \longrightarrow & W' & \xrightarrow{q} & S^{8n+2} \cup_2 e^{8n+3} \\
 \downarrow & & \downarrow \delta & & \parallel \\
 \Sigma Q_{4n+2,3} & \longrightarrow & \Sigma(\tilde{T}_{8n+2}) & \xrightarrow{q} & S^{8n+2} \cup_2 e^{8n+3}
 \end{array}$$

where all columns and rows are homotopy cofiber sequences. Then the composite in the statement corresponds to the attaching map of the $(12n+4)$ -cell of $\Sigma\tilde{T}/\Sigma Q_{4n+2,3}$. On the other hand, by [21], there is a stable homotopy equivalence

$$\tilde{T} \simeq_S Q_{4n+2,3} \vee (S^{8n+1} \cup_2 e^{8n+2}) \vee S^{8n+3} \vee S^{12n+3}. \quad (15)$$

Then by the Freudenthal suspension theorem, we get

$$\Sigma\tilde{T}/\Sigma Q_{4n+2,3} = (S^{8n+2} \cup_2 e^{8n+3}) \vee S^{8n+4} \vee S^{12n+4}.$$

Thus the attaching map of the $(12n+4)$ -cell of $\Sigma\tilde{T}/\Sigma Q_{4n+2,3}$ is trivial, completing the proof. $\square$

We are now ready to prove Proposition 1.6.

Proof of Proposition 1.6 Recall that there is a homotopy commutative diagram

$$\begin{array}{ccccc}
 S^{12n+3} & \xrightarrow{s} & S^{8n+2} \ltimes S^{4n+1} & \xrightarrow{\tilde{\kappa}} & W' \\
 \parallel & & \downarrow & & \downarrow \delta \\
 S^{12n+3} & \xrightarrow{\iota} & \Sigma(S^{8n+2} \times S^{4n}) & \xrightarrow{\Sigma\kappa} & \Sigma(\tilde{T}_{8n+2}).
 \end{array}$$

By [5, Theorem 3.16], we have

$$\overline{H}_\nabla(\Sigma\kappa \circ \iota) = \overline{H}_\nabla(\delta \circ \tilde{\kappa} \circ s) = \delta^{\wedge 2} \circ \overline{H}_\nabla(\tilde{\kappa} \circ s) + \overline{H}_\nabla(\delta) \circ \Sigma(\tilde{\kappa} \circ s).$$

We aim to prove $\overline{H}_\nabla(\tilde{\kappa} \circ s) = 0$. Then since $\text{cat}(W') = 1$ by Lemma 8.4, it follows that $\text{wcat}(W(0)) = 1$ by Theorem 5.2 as stated.

First, we prove that $\overline{H}_\nabla(\delta) \circ \Sigma(\tilde{\kappa} \circ s) = 0$. Since $(\Sigma(\tilde{T}_{8n+2}))^{\wedge 2}$ is $(8n+1)$ -connected, $\overline{H}_\nabla(\delta)$ has an extension

$$\tilde{\delta}: \Sigma W'/S^{4n+2} = S^{8n+3} \cup_2 e^{8n+4} \rightarrow (\Sigma(\tilde{T}_{8n+2}))^{\wedge 2}.$$

On the other hand, the composite

$$S^{12n+4} \xrightarrow{\Sigma s} \Sigma(S^{8n+2} \ltimes S^{4n+1}) \xrightarrow{\Sigma\tilde{\kappa}} \Sigma W' \xrightarrow{q} S^{8n+3} \cup_2 e^{8n+4}$$

is trivial by Lemma 8.5. Then we get

$$\overline{H}_\nabla(\delta) \circ \Sigma(\tilde{\kappa} \circ s) = \tilde{\delta} \circ q \circ \Sigma(\tilde{\kappa} \circ s) = 0.$$

We now have $\overline{H}_\nabla(\Sigma\kappa \circ \iota) = \delta^{\wedge 2} \circ \overline{H}_\nabla(\tilde{\kappa} \circ s)$. Since $\kappa|_{S^{8n+2} \vee S^{4n}} = \psi + i$, where i is the bottom cell inclusion, we get

$$\overline{H}_\nabla(\Sigma\kappa \circ \iota) = \overline{H}_\nabla(\mathcal{H}(\kappa)) = \pm(\Sigma\psi \wedge \Sigma i) = \pm(1_{\Sigma(\tilde{T}_{8n+2})} \wedge \Sigma i) \circ (\Sigma\psi \wedge 1_{S^{4n+1}}).$$

On the other hand, it follows that $\Sigma\psi \wedge 1_{S^{4n+1}} = \Sigma^{4n+2}\psi$ is trivial from the stable splitting (15) and the Freudenthal suspension theorem. Then we get $\overline{H}_\nabla(\Sigma\kappa \circ \iota) = 0$. Thus by Lemma 8.3, we obtain $\overline{H}_\nabla(\tilde{\kappa} \circ s) = 0$, completing the proof. $\square$

9 Open problems

We conclude this paper with a list of questions regarding the parametrized topological complexity of spherical fibrations over spheres. We hope that these problems precipitate further research.

First, we consider a homotopy fibration $S^{2n+1} \rightarrow X \rightarrow S^{m+1}$ with $1 \leq n$ and $2n + 1 \leq m \leq 4n - 1$. Note that, except for the cases in Sect. 8, we have considered only $\text{wcat}(W(0)_{m+2n+2})$ as a lower bound.

Problem 9.1 Does there exist a homotopy fibration $S^{2n+1} \rightarrow X \rightarrow S^{m+1}$ such that $\text{wcat}(W(0)_{m+2n+2}) = 1$ but $\text{wcat}(W(0)) = 2$?

In Sect. 8, we have considered the unit tangent bundle $S^{4n+1} \rightarrow T \rightarrow S^{4n+2}$ and proved $\text{wcat}(W(0)) = 1$ as an example of the limit of the lower bound by the weak category.

Problem 9.2 For $n \geq 2$, is $\text{TC}[T \rightarrow S^{4n+2}] = 1$ or 2?

Next, we consider a homotopy fibration $S^{2n} \rightarrow X \rightarrow S^{m+1}$ with $2n \leq m \leq 4n - 3$. Then by Lemma 4.7, the space W in Corollary 4.2 can be described as a 4-cell complex $S^{2n} \cup e^{4n} \cup e^{m+2n+1} \cup e^{m+4n+1}$. However, the computation of $\text{wcat}(W)$ is considerably more difficult. We have $\text{wcat}(W_{4n}) = 2$ since $-[1_{S^{2n}}, 1_{S^{2n}}]$ is not a suspension. Then we get $\text{wcat}(W_{m+2n+1}) = 2$ by degree reason.

Problem 9.3 Does there exist a homotopy fibration $S^{2n} \rightarrow X \rightarrow S^{m+1}$ such that $\text{wcat}(W) = 3$?

We also need to improve an upper bound; the upper bound in Proposition 3.3 can only be applied to the unit sphere bundles, and the computation is not easy for $m > 2n$ cases.

Problem 9.4 Generalize the upper bound in Proposition 3.3 to homotopy fibrations $S^{2n} \rightarrow X \rightarrow S^{m+1}$.

Finally, we consider a homotopy fibration $S^n \rightarrow X \rightarrow S^{m+1}$ with $2n - 3 < m$. Then the argument about the crude Hopf invariants in Sect. 5 cannot be applied.

Problem 9.5 Compute $\text{wcat}(W)$ and $\text{TC}[X \rightarrow S^{m+1}]$ for homotopy fibrations $S^n \rightarrow X \rightarrow S^{m+1}$ with $2n - 3 < m$.

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