
学位申請論文

植村 英明

Off-Diagonal Short Time Expansion of the Heat Kernel on a Certain Nilpotent Lie Group

Hideaki UEMURA

Department of Mathematics

Kochi University

0. Introduction.

Let $\mathcal{L}$ be a differential operator of Hörmander type ;

$$\mathcal{L} = \frac{1}{2} \sum_{\alpha=1}^r V_{\alpha}^2 + V_0 ,$$

where V_{α} , $\alpha = 0, 1, \dots, r$, are C^{∞} -vector fields on $\mathbb{R}^d$. Under the condition $(H.1)_{\infty}$ of these vector fields given in §2 below, the fundamental solution $p(t, x, y)$ of the heat equation $\frac{\partial u}{\partial t} = \mathcal{L}u$ exists. Its short time expansion of the form

$$(0.1) \quad p(t, x, y) \sim \exp\left(-\frac{d(x, y)^2}{2t}\right) t^{-N/2} (c_0 + c_1 t + \dots) \text{ as } t \downarrow 0$$

has been studied by many authors in both analytical and probabilistic methods, cf. e.g. J.-M.Bismut [7], T.J.S.Taylor [21], S.Kusuoka [11], S.Watanabe [24], R.Léandre [16], G.Ben Arous [3]. Among others, G.Ben Arous [3] has shown that (0.1) holds with $N = d$ when the pair

(x, y) of points x and y is out of the cut-locus, i.e. when

(i) there exists a unique $h_0 \in K_{\min}^{x, y}$,

(ii) the deterministic Malliavin covariance with respect to x and h_0 is non-degenerate,

(iii) x and y are not conjugate along h_0 (i.e. the Hessian of the mapping $h \in K^{x, y} \rightarrow \frac{1}{2} \|h\|_H^2$ is non-degenerate at h_0),

cf. §2 for the precise meaning of notions and notations like $K^{x,y}$, $K_{\min}^{x,y}$, the deterministic Malliavin covariance, etc. Also, $d(x,y)$ in (0.1) is the control metric or the Carnot-Caratheodory metric which coincides with the H -norm of elements in $K_{\min}^{x,y}$. Indeed, it was shown by R.Léandre [13],[14] and [15] that, under the assumption of $(H.1)_{\infty}$, it holds generally

$$(0.2) \quad \lim_{t \downarrow 0} 2t \log p(t,x,y) = -d(x,y)^2$$

When the pair (x,y) is in the cut-locus, we can still expect that (0.1) holds but the exponent N is usually greater than d . In the simplest case of $x = y$, the expansion (0.1) with $d(x,y) = 0$ has been obtained by G.Ben Arous [4], R.Léandre [16] and S.Takanobu [20] under some restriction on the drift vector field V_0 . If this restriction is violated, the situation is much more complicated, cf G.Ben Arous [5], G.Ben Arous-R.Léandre [6].

Consider the case (x,y) is in the cut-locus and $x \neq y$. First we consider the case when (i) is violated but (ii) and (iii) remain valid for every $h_0 \in K_{\min}^{x,y}$. Here, however, the definition of non-conjugacy in (iii) should be modified as :

(iii)' the Hessian of the mapping $h \in K^{x,y} \rightarrow \frac{1}{2} \|h\|_H^2$ is non-degenerate at h_0 in the direction normal to $K_{\min}^{x,y}$.

Then we can expect that (0.1) holds with $N = d + \dim K_{\min}^{x,y}$ just as in the case of the heat kernel on a sphere with $\mathcal{L}$ = a half of the Laplacian and y is antipodal to x . (cf. S.A.Molchanov [17]. Note that $K_{\min}^{x,y}$ is in one-to-one correspondence with the set of minimal geodesics (minimal horizontal curves given in §2) connecting x and y and hence $\dim K_{\min}^{x,y}$ = the dimension of the set of all minimal geodesics connecting x and y .) A typical example of this situation is the case of the Heisenberg group realized by $\mathbb{R}^3$ and $x = (0,0,0)$, $y = (0,0,\eta)$, $\eta \neq 0$ (cf B.Gaveau [9], R.Azencott

[2]). In this case, $K_{\min}^{x,y}$ constitutes a one-dimensional submanifold in the Cameron-Martin Hilbert space and $N = 4 = d + \dim K_{\min}^{x,y}$. If, furthermore, the condition (ii) is violated, i.e., the deterministic Malliavin covariance degenerates at $h \in K_{\min}^{x,y}$, we may still expect that (0.1) holds with $N > d + \dim K_{\min}^{x,y}$, however.

Purpose of this paper is to illustrate these situations in a concrete case of the nilpotent Lie group $N_{4,2}$ realized by $\mathbb{R}^{10}$. In this case, an explicit integral representation of the heat kernel was obtained by B.Gaveau [9] (cf. also M.Chaleyat-Maurel [8]) and the short time expansion (0.1) could be obtained directly from it. We follow here, however, a probabilistic approach given by H.Uemura-S.Watanabe [22] which can explain well the role of $\dim K_{\min}^{x,y}$ and the degeneracy of the Malliavin covariance in the determination of N and which may give some insight, we hope, in more general situations.

Finally, we explain briefly our method. First we represent the heat kernel as

$$(0.3) \quad p(\varepsilon^2, x, y) = E[\delta_y(X_1^\varepsilon)]$$

by a generalized expectation of a generalized Wiener functional in the sense of S.Watanabe [24] where X_t^ε is the solution of the following stochastic differential equation :

$$(0.4) \quad \begin{cases} dX_t = \varepsilon \sum_{\alpha=1}^r V_\alpha(X_t) \circ dw_t^\alpha + \varepsilon^2 V_0(X_t) dt \\ X_0 = x \end{cases} .$$

δ_y is, of course, the Dirac δ -function at $y \in \mathbb{R}^d$. We evaluate the generalized expectation in the right-hand side of (0.3) by appealing to the theory of large deviations and the theory of asymptotic expansions of Wiener functionals as developed in S.Watanabe [24].

Roughly, X_t^ε conditioned by $X_1^\varepsilon = y$ will be concentrated on the set $M^{x,y} = \{ c^{x,h} ; h \in K_{\min}^{x,y} \}$ of minimal horizontal curves connecting

x and y as $\varepsilon \downarrow 0$, actually will be distributed uniformly on this set. It will be shown clearly by our probabilistic method how this limiting behavior of tied-down trajectories X_t^ε is reflected on that of $p(\varepsilon^2, x, y)$ as $\varepsilon \downarrow 0$.

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1. Probabilistic preliminaries.

In this section we introduce some notions and results on asymptotic expansions of generalized Wiener functionals as are necessary in the future. The reader is referred to S.Watanabe [23], [24] for details.

Let (W, H, μ) be an abstract Wiener space. $D_p^S(E)$ ($s \in \mathbb{R}$, $1 \leq p < \infty$) be the completion of $\mathcal{P}(E)$ ($:= \{E\text{-valued polynomial Wiener functionals}\}$) by the norm $\|\cdot\|_{p,s} = \|(I-L)^{s/2} \cdot\|_p$, where L is the Ornstein-Uhlenbeck operator (the number operator), $\|\cdot\|_p$ is the L^p -norm with respect to the measure μ , and E is a separable Hilbert space. Especially when $E = \mathbb{R}$, we denote D_p^S instead of $D_p^S(\mathbb{R})$. Then it holds that $D_p^0(E) = L^p(E, \mu)$ and $D_p^S(E)^*$, the dual space of $D_p^S(E)$, coincides with $D_q^{-S}(E)$ under the identification of $D_2^0(E)^*$ ($= L^2(E, \mu)^*$) with itself, q being the conjugate exponent of p ; $1/p + 1/q = 1$.

We define H -derivative $D : \mathcal{P}(E) \rightarrow \mathcal{P}(H \otimes E)$ by $DF(u)[h] := \lim_{\varepsilon \downarrow 0} \frac{F(u+\varepsilon h) - F(u)}{\varepsilon}$, $h \in H$. Here $H \otimes E$ is a Hilbert space formed of all linear operators from H to E of Hilbert-Schmidt type endowed with the Hilbert-Schmidt inner product. D can be extended to a

bounded linear operator $D_p^S(E) \rightarrow D_p^{S-1}(H \otimes E)$ and we denote again this extended linear operator by D . If D^* is the dual operator of D , then D^* maps from $D_p^{S+1}(H \otimes E)$ to $D_p^S(E)$ and $L = -D^*D$. (See also N.Ikeda-S.Watanabe [10] or H.Sugita [19].)

Set $D^\infty(E) := \bigcap_{s \geq 0} \bigcap_{1 < p < \infty} D_p^S(E)$, $\tilde{D}^\infty(E) := \bigcap_{s \geq 0} \bigcap_{1 < p < \infty} D_p^S(E)$, $\tilde{D}^{-\infty}(E) := \bigcup_{s \geq 0} \bigcap_{1 < p < \infty} D_p^{-S}(E)$ and $D^{-\infty}(E) := \bigcup_{s \geq 0} \bigcap_{1 < p < \infty} D_p^{-S}(E)$. We call an element of $D^{-\infty}(E)$ a *generalized Wiener functional* in analogy with the Schwartz distribution theory. When $E = \mathbb{R}$ we denote them simply by D^∞ , $\tilde{D}^\infty$, $\tilde{D}^{-\infty}$, $D^{-\infty}$ respectively. For $G \in D^\infty$ and $\Phi \in D^{-\infty}$ (or $G \in \tilde{D}^\infty$ and $\Phi \in \tilde{D}^{-\infty}$), $G \cdot \Phi$ (= $\Phi \cdot G$) $\in D^{-\infty}$ is defined by $D^{-\infty} \langle G \cdot \Phi, F \rangle_{D^\infty} := D^{-\infty} \langle \Phi, G \cdot F \rangle_{D^\infty}$ [resp. $\tilde{D}^{-\infty} \langle \Phi, G \cdot F \rangle_{\tilde{D}^\infty} = 1$ for all $F \in D^\infty$].

For $F(w) = (F^1(w), \dots, F^d(w)) \in D^\infty(\mathbb{R}^d)$, i.e. $F^i(w) \in D^\infty$, $i = 1, \dots, d$, set $\sigma^{ij}(w) = \langle DF^i(w), DF^j(w) \rangle_H$, $i, j = 1, \dots, d$. Here $\langle \cdot, \cdot \rangle_H$ means the inner product of H . We call this $d \times d$ matrix valued Wiener functional $\sigma(w) = (\sigma^{ij}(w))_{i,j=1,\dots,d}$ the *Malliavin covariance of F* . If $\sigma(w)$ is positive definite for almost all w and furthermore $(\det \sigma(w))^{-1} \in \bigcap_{1 < p < \infty} L^p(\mu)$, we say that F is *non-degenerate* (in Malliavin's sense), and in this case, for any $T \in \mathcal{G}'(\mathbb{R}^d)$, a tempered Schwartz distribution on $\mathbb{R}^d$, its *pull-back* $T(F)$ is defined as an element of $\tilde{D}^{-\infty}$. For $G \in \tilde{D}^\infty$, we denote $\tilde{D}^{-\infty} \langle T(F), G \rangle_{\tilde{D}^\infty}$ (= $D^{-\infty} \langle G \cdot T(F), 1 \rangle_{D^\infty}$) by $E[T(F) \cdot G]$ or $E[G \cdot T(F)]$. Especially when $T = \delta_y$, the Dirac's δ -function at $y \in \mathbb{R}^d$, $E[G \cdot \delta_y(F)] = E[G|F=y] \cdot p(y)$, $p(y)$ being the C^∞ -density of F .

Let $F(\varepsilon, w) \in D_p^S(E)$ for all $\varepsilon \in (0, 1]$. If $\|F(\varepsilon, w)\|_{p,S} = o(\varepsilon^n)$ as $\varepsilon \downarrow 0$, we say $F(\varepsilon, w) = o(\varepsilon^n)$ as $\varepsilon \downarrow 0$ in $D_p^S(E)$. When $F(\varepsilon, w) \in D^\infty(E)$ for all $\varepsilon \in (0, 1]$, we say $F(\varepsilon, w) = o(\varepsilon^n)$ as $\varepsilon \downarrow 0$ in $D^\infty(E)$ if $F(\varepsilon, w) = o(\varepsilon^n)$ as $\varepsilon \downarrow 0$ in $D_p^S(E)$ for all $s > 0$.

and $p \in (1, \infty)$. Similarly we define $F(\varepsilon, w) = o(\varepsilon^n)$ in $\tilde{D}^\infty(E)$, in $\tilde{D}^{-\infty}(E)$ and in $D^{-\infty}(E)$.

Let $F(\varepsilon, w) \in D_p^S(E)$ for all $\varepsilon \in (0, 1]$. We say $F(\varepsilon, w)$ has the asymptotic expansion in $D_p^S(E)$:

$$F(\varepsilon, w) \sim f_0(w) + \varepsilon \cdot f_1(w) + \varepsilon^2 \cdot f_2(w) + \cdots \text{ as } \varepsilon \downarrow 0 \text{ in } D_p^S(E)$$

if $f_i(w) \in D_p^S(E)$, $i = 0, 1, 2, \dots$, and furthermore for all n ,

$$F(\varepsilon, w) - \sum_{i=0}^n \varepsilon^i \cdot f_i(w) = o(\varepsilon^n) \text{ as } \varepsilon \downarrow 0 \text{ in } D_p^S(E).$$

Similarly we define the asymptotic expansion in $D^\infty(E)$, in $\tilde{D}^\infty(E)$, in $\tilde{D}^{-\infty}(E)$ and in $D^{-\infty}(E)$. For example, we say $F(\varepsilon, w)$ has the asymptotic expansion in $\tilde{D}^\infty(E)$ when for all n and s , there exists $p = p(s, n)$ such that $f_i(w) \in D_p^S(E)$, $i = 0, 1, 2, \dots, n$, and

$$F(\varepsilon, w) - \sum_{i=0}^n \varepsilon^i \cdot f_i(w) = o(\varepsilon^n) \text{ as } \varepsilon \downarrow 0 \text{ in } D_p^S(E).$$

Let $F(\varepsilon, w) \in D^\infty(\mathbb{R}^d)$ for all $\varepsilon \in (0, 1]$ and $\sigma(\varepsilon, w)$ be its Malliavin covariance. We say $F(\varepsilon, w)$ is *uniformly non-degenerate* if $F(\varepsilon, w)$ is non-degenerate for all $\varepsilon \in (0, 1]$ and furthermore

$$\overline{\lim}_{\varepsilon \downarrow 0} \| \{\det \sigma(\varepsilon, w)\}^{-1} \|_p < \infty \quad \text{for all } p \in (1, \infty)$$

Here we give an important theorem concerning the asymptotic expansion of pull-backs.

Theorem 1.1. (S. Watanabe [24])

Let a family $F(\varepsilon, w) \in D^\infty(\mathbb{R}^d)$, $0 < \varepsilon \leq 1$, be uniformly non-degenerate and have the asymptotic expansion in $D^\infty(\mathbb{R}^d)$:

$$F(\varepsilon, w) \sim f_0(w) + \varepsilon \cdot f_1(w) + \cdots \text{ as } \varepsilon \downarrow 0 \text{ in } D^\infty(\mathbb{R}^d)$$

Then for all $T \in \mathcal{G}'(\mathbb{R}^d)$, its pull-back $T(F(\varepsilon, w)) \in \tilde{D}^{-\infty}$ and has the asymptotic expansion in $\tilde{D}^{-\infty}$:

$$T(F(\varepsilon, w)) \sim \Phi_0(w) + \varepsilon \cdot \Phi_1(w) + \cdots \text{ as } \varepsilon \downarrow 0 \text{ in } \tilde{D}^{-\infty}.$$

Furthermore, these coefficients $\Phi_i(w)$, $i = 0, 1, 2, \dots$, are obtained from the formal Taylor expansion of T , i.e. formally from

$$T(F(\varepsilon, w)) = T(f_0) + \partial T(f_0)(\varepsilon \cdot f_1 + \varepsilon^2 \cdot f_2 + \dots) + \frac{1}{2} \partial^2 T(f_0)(\varepsilon \cdot f_1 + \varepsilon^2 \cdot f_2 + \dots) \otimes (\varepsilon \cdot f_1 + \varepsilon^2 \cdot f_2 + \dots) + \dots,$$

namely $\Phi_i(w)$ is obtained by picking up all coefficients of ε^i in the right-hand side above. For example, $\Phi_0 = T(f_0)$, $\Phi_1 = \partial T(f_0)f_1$ and $\Phi_2 = \partial T(f_0)f_2 + \frac{1}{2} \partial^2 T(f_0) f_1 \otimes f_1$.

Corollary 1.2.

Under the same assumptions as in Theorem 1.1,

$$E[T(F(\varepsilon, w))] \sim E[\Phi_0(w)] + \varepsilon \cdot E[\Phi_1(w)] + \dots \text{ as } \varepsilon \downarrow 0.$$

2. Stochastic representation of heat kernels.

Here we discuss the stochastic representation of the fundamental solution of heat equations by using the above results. Consider the following differential operator $\mathcal{L}$ of Hörmander type on $\mathbb{R}^d$:

$$\mathcal{L} = \frac{1}{2} \sum_{\alpha=1}^r V_{\alpha}^2,$$

where $V_{\alpha}(x) = \sum_{i=1}^d V_{\alpha}^i(x) \frac{\partial}{\partial x_i}$, $\alpha = 1, \dots, r$, and we assume $V_{\alpha}^i(x) \in C_b^{\infty}(\mathbb{R}^d) :=$ the totality of C^{∞} -functions such that all derivatives are bounded. Let $p(t, x, y)$ be the fundamental solution of $\frac{\partial}{\partial t} - \mathcal{L}$, i.e.

$$\begin{cases} \frac{\partial}{\partial t} p(t, x, y) = \mathcal{L}_x p(t, x, y) \\ \lim_{t \downarrow 0} p(t, x, y) = \delta_x(y) \end{cases}.$$

$p(t, x, y)$ can be obtained probabilistically by the following way: Let (W_0^r, P) be an r -dimensional Wiener space, i.e. $W_0^r := \{w = (w_t) \in C([0, 1] \rightarrow \mathbb{R}^r); w_0 = 0\}$ is a Banach space endowed with the

norm $\|w\| := \sup_{t \in [0,1]} |w_t|$ and P is the Wiener measure on W_0^r . Let

H be the Cameron-Martin subspace of W_0^r , i.e. H is a Hilbert space consisted of all absolutely continuous functions on $[0,1]$ whose Radon-Nikodym derivatives are square integrable with the norm $\|h\|_H := \left(\int_0^1 \left| \frac{dh_t}{dt} \right|^2 dt \right)^{1/2}$. Then (W_0^r, H, P) is an abstract Wiener space. Now consider the following stochastic differential equation (abbr. S.D.E.) on $\mathbb{R}^d$:

$$(2.1) \quad \begin{cases} dX_t = \sum_{\alpha=1}^r V_{\alpha}(X_t) \circ dw_t^{\alpha} \\ X_0 = x \end{cases}.$$

Here $w_t = (w_t^1, \dots, w_t^r) \in W_0^r$ and $\circ dw_t^{\alpha}$ denotes the stochastic differential of Stratonovich type. We denote by X_t the solution of S.D.E. (2.1). We assume the following Hörmander-type condition on the vector fields V_{α} , $\alpha = 1, \dots, r$:

(H.1) $_{\infty}$ If we set

$$H(n) = \{ x \in \mathbb{R}^d ; \text{l.s.} \{ [V_{\alpha_1}, [V_{\alpha_2}, \dots, [V_{\alpha_{k-1}}, V_{\alpha_k}] \dots] (x) , \\ \alpha_i \in \{ 1, \dots, r \} , k \leq n \} = T_x(\mathbb{R}^d) \}$$

$$\text{then } \bigcup_{n=1}^{\infty} H(n) = \mathbb{R}^d$$

Here l.s. means the linear span. In the case $\bigcup_{n=1}^N H(n) (= H(N)) = \mathbb{R}^d$, we say the condition (H.1) $_N$ is fulfilled. From now on we always assume (H.1) $_{\infty}$. Then it is known (cf. S.Kusuoka-D.W.Stroock [12]) that the Malliavin covariance $\sigma(t)$ of $X_t \in D^{\infty}(\mathbb{R}^d)$ is non-degenerate for each fixed $t \in (0,1]$, more precisely positive constants $K_1 = K_1(p)$ and K_2 exist such that $E[|\det \sigma(t)|^{-p}]^{1/p} \leq K_1 t^{-K_2}$, $t \in (0,1]$, $p \in (1, \infty)$. Hence $\delta_y(X_t) \in \tilde{D}^{-\infty}$. Moreover we can see that

$$p(t, x, y) = E[\delta_y(X_t)] .$$

Let X_t^ε be a solution of the following S.D.E. (2.2) :

$$(2.2) \quad \begin{cases} dX_t = \varepsilon \sum_{\alpha=1}^r V_\alpha(X_t) \circ dw_t^\alpha \\ X_0 = x \end{cases} .$$

Then it is easy to see that $\{X_t^\varepsilon\} \stackrel{\mathcal{L}}{\approx} \{X_{\varepsilon^2 t}\}$, so the fundamental solution $p(t, x, y)$ can be expressed also by

$$p(\varepsilon^2, x, y) = E[\delta_y(X_1^\varepsilon)]$$

In the following we use this representation to study its asymptotic behavior as $\varepsilon \downarrow 0$.

For each $h \in H$, consider the following differential equation :

$$(2.3) \quad \begin{cases} \frac{dc(t)}{dt} = \sum_{\alpha=1}^r V_\alpha(c(t)) \cdot \frac{dh_t^\alpha}{dt} \\ c(0) = x \end{cases} .$$

We denote the solution by $c^{x,h}(t)$. Such a curve for some x and h is called a *horizontal curve with respect to* $\{V_\alpha\}$. For all $x, y \in \mathbb{R}^d$, set

$$K^{x,y} = \{ h \in H ; c^{x,h}(1) = y \} .$$

Then under the condition $(H.1)_\infty$, it is well-known that $K^{x,y} \neq \emptyset$ for all $x, y \in \mathbb{R}^d$ (cf. J.-M. Bismut [7], Th.1.14). Thus, for all $x, y \in \mathbb{R}^d$, we set

$$d(x, y) = \min \{ \|h\|_H ; h \in K^{x,y} \} .$$

This defines a metric called *the control metric* of x and y . Let

$$K_{\min}^{x,y} = \{ h \in K^{x,y} ; \|h\|_H = d(x, y) \} .$$

Then it is also well-known that $K_{\min}^{x,y} \neq \emptyset$ (cf. J.-M. Bismut [7], Th. 1.14). We define $M^{x,y}$ by

$$M^{x,y} = \{ c^{x,h} ; h \in K_{\min}^{x,y} \}$$

and call its element *the minimal horizontal curve* connecting x and y .

Consider the following differential equation on $d \times d$ matrix :

$$(2.4) \quad \begin{cases} \frac{dY(t)}{dt} = \sum_{\alpha=1}^r \partial V_{\alpha}(c(t)) Y(t) \cdot \frac{dh_t^{\alpha}}{dt} \\ Y(0) = I \end{cases},$$

where $c(t)$ is the solution of (2.3) and $\partial V_{\alpha}(x)$ is a $d \times d$ matrix whose (i,j) -component is $\partial V_{\alpha}^i(x) / \partial x_j$. This solution is denoted by $Y^{x,h}(t)$. With this solution we define a $d \times d$ matrix $\sigma^{x,h}$ by

$$\sigma^{x,h} = \sum_{\alpha=1}^r \int_0^1 Y^{x,h}(1) Y^{x,h}(t)^{-1} V_{\alpha}(c^{x,h}(t)) \otimes Y^{x,h}(1) Y^{x,h}(t)^{-1} V_{\alpha}(c^{x,h}(t)) dt.$$

This $\sigma^{x,h}$ is called *the deterministic Malliavin covariance* with respect to x and h and plays an important role later when we discuss the minimal horizontal curve.

We define *the Hamiltonian function* associated to the vector fields V_{α} , $\alpha = 1, \dots, r$, by

$$H(p,x) = \frac{1}{2} \sum_{\alpha=1}^r \langle p, V_{\alpha}(x) \rangle^2, \quad (p,x) \in T^*(\mathbb{R}^d),$$

where $\langle \cdot, \cdot \rangle$ denotes the coupling of elements in $T_x^*(\mathbb{R}^d)$ and $T_x(\mathbb{R}^d)$. Consider the following *Hamilton equation* with respect to $H(p,x)$ above :

$$(2.5) \quad \begin{cases} \dot{x}_t = \frac{\partial H}{\partial p}(p_t, x_t) \\ \dot{p}_t = - \frac{\partial H}{\partial x}(p_t, x_t) \end{cases},$$

where $\dot{\cdot}$ denotes the time derivative $\frac{d}{dt}$. The solution of this equation (2.5) is called a *bicharacteristic*. We denote the bicharacteristic with an initial value (p_0, x_0) by $(p_t(p_0, x_0), x_t(p_0, x_0))$. Now we summarize some results concerning to the bicharacteristic. Refer to J.-M. Bismut [7] for details.

$$(2.6-I) \quad \text{Let } p_t := p_t(p_0, x_0), \quad x_t := x_t(p_0, x_0) \quad \text{and} \quad \dot{h}_t :=$$

($\langle p_t, V_1(x_t) \rangle$, $\dots$, $\langle p_t, V_r(x_t) \rangle$) Then

$$c^{x_0, h}(t) = x_t(p_0, x_0)$$

(2.6-II) If the deterministic Malliavin covariance $\sigma^{x_0, h}$, $h \in K_{\min}^{x_0, y}$, is non-degenerate. i.e. $\det \sigma^{x_0, h} > 0$, then there exists a unique p_0 such that

$$c^{x_0, h}(t) = x_t(p_0, x_0)$$

(2.6-III) The following (H.2) is a sufficient condition on vector fields V_α , $\alpha = 1, \dots, r$, for the non-degeneracy of its deterministic Malliavin covariance :

(H.2) $V_1(x_0)$, $\dots$, $V_r(x_0)$ are linearly independent and
 l.s { $V_1(x_0)$, $\dots$, $V_r(x_0)$, $[V_1, Y](x_0)$, $\dots$,
 $[V_r, Y](x_0)$ } = $T_{x_0}(\mathbb{R}^d)$

for every fixed $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathbb{R}^r \setminus \{0\}$ setting

$$Y = \sum_{\alpha=1}^r \lambda_\alpha V_\alpha .$$

Namely, if (H.2) is satisfied at $x_0 \in \mathbb{R}^d$, then

$\det \sigma^{x_0, h} > 0$ for every $h \in H$ such that $h \neq 0$.

3. Nilpotent Lie groups of order r with n -generators.

In this section we introduce a nilpotent Lie group which will be the main subject of this paper (cf. B.Gaveau [9]) Let $V_1, \dots, V_n$ be C^∞ -vector fields. For $I = (i_1, \dots, i_k) \in \{1, \dots, n\}^k$ we define $V_{[I]}$ and V_I by

$$V_{[I]} = [V_{i_1}, [V_{i_2}, \dots [V_{i_{k-1}}, V_{i_k}] \dots]] ,$$

$$V_I = V_{i_1} \cdot V_{i_2} \cdot \dots \cdot V_{i_k} ,$$

and let $|I|$ be the length of I . (In this case $|I| = k$.) It is easy to show that there exist constants A_{IJ} such that

$$V_{[I]} = \sum A_{IJ} \cdot V_J$$

and $A_{IJ} = 0$ if $|I| \neq |J|$.

Definition 3.1.

We say that a system of vector fields $\{V_1, \dots, V_n\}$ is free of order r at x if $\sum_{|I| \leq r} a_I \cdot V_{[I]}(x) = 0$, $a_I \in \mathbb{R}$, implies $\sum_{|I| \leq r} a_I \cdot A_{IJ} = 0$ for all J satisfying $|J| \leq r$. Let $V =$
 $\ell.s. \{V_1, \dots, V_n\}$. We say the vector space V is free of order r if $\{V_1, \dots, V_n\}$ is free of order r for all x .

Definition 3.2.

Let $\mathfrak{g}$ be a Lie algebra.

i) $\mathfrak{g}$ is said to be nilpotent of order r if $\mathfrak{g} = V^1 \oplus \dots \oplus V^r$ where V^i , $i = 1, \dots, r$, are vector subspaces of $\mathfrak{g}$ satisfying $V^2 = [V^1, V^1]$, $V^3 = [V^1, V^2]$, $\dots$, $V^r = [V^1, V^{r-1}]$, $[V^1, V^r] = \{0\}$ and $[V^i, V^j] \subset V^{i+j}$.

ii) Furthermore $\mathfrak{g}$ is said to have n generators if $\dim V^1 = n$ and moreover V^1 is free of order r .

We say $\mathfrak{g}$ is a nilpotent Lie algebra of order r with n -generators if i) and ii) above are satisfied and denote it by $\mathfrak{n}_{n,r}$. Let $N_{n,r}$ be a Lie group corresponding to $\mathfrak{n}_{n,r}$. This $N_{n,r}$ is called a nilpotent Lie group of order r with n -generators. From now on, we assume $r = 2$.

Proposition 3.1.

Let $\mathfrak{n}_{n,2} = V^1 \oplus V^2$, $\{V_i, i = 1, \dots, n\}$ be a base of V^1 ,

and $V_{jk} := [V_j, V_k]$. Then a system $\{V_i, V_{jk}; 1 \leq i \leq n, 1 \leq j < k \leq n\}$ is a base of $n_{n,2}$.

Proof.

Set $\sum_{|I| \leq 2} a_I \cdot V_{[I]}(x) = 0$ where $a_I = 0$ if $I = (i_1, i_2)$ satisfies $i_1 > i_2$. Since V^1 is free, $\sum_{|I| \leq 2} a_I \cdot A_{IJ} = 0$ for all J . Therefore by taking $J = i$, $i = 1, \dots, n$, or $J = (j, k)$, $1 \leq j < k \leq n$, we see easily that $a_J = 0$, i.e. $\{V_i, V_{jk}; 1 \leq i \leq n, 1 \leq j < k \leq n\}$ is linearly independent. Since $V_{[(i_1, i_2)]} = -V_{[(i_2, i_1)]}$, it is clear that the above system is a base. //

With this base we can introduce a canonical coordinate on $N_{n,2}$ as follows :

$$(x_i, x_{(jk)})_{\substack{1 \leq i \leq n \\ 1 \leq j < k \leq n}} \longleftrightarrow \exp \left(\sum_{i=1}^n x_i \cdot V_i + \sum_{1 \leq j < k \leq n} x_{(jk)} \cdot V_{jk} \right) \in N_{n,2}.$$

Hence $N_{n,2}$ is realized by $\mathbb{R}^{n(n+1)/2}$ under this coordinate and the group action is given as follows by Campbell-Hausdorff's theorem :

$$(x_i, x_{(jk)}) \cdot (y_i, y_{(jk)}) = (x_i + y_i, x_{(jk)} + y_{(jk)} + \frac{1}{2}(x_j y_k - x_k y_j)).$$

Define mappings $L_{(x_i, x_{(jk)})}$ and $R_{(y_i, y_{(jk)})}$ on $\mathbb{R}^{n(n+1)/2}$ by

$$L_{(x_i, x_{(jk)})}(z_i, z_{(jk)}) = (x_i, x_{(jk)}) \cdot (z_i, z_{(jk)}),$$

and

$$R_{(y_i, y_{(jk)})}(z_i, z_{(jk)}) = (z_i, z_{(jk)}) \cdot (y_i, y_{(jk)}).$$

Then both $L_{(x_i, x_{(jk)})}$ and $R_{(y_i, y_{(jk)})}$ are affine mappings with the determinants 1 and so the Haar measure of $N_{n,2}$ is the Lebesgue measure. Under this coordinate V_i is expressed as follows :

$$(3.1) \quad V_i = \frac{\partial}{\partial x_i} + \frac{1}{2} \left(\sum_{k < i} x_k \frac{\partial}{\partial x_{(ki)}} - \sum_{k > i} x_k \frac{\partial}{\partial x_{(ik)}} \right).$$

Set

$$\Delta_{n,2} = \sum_{i=1}^n V_i^2$$

Obviously $\{V_i, 1 \leq i \leq n\}$ satisfies $(H.1)_2$. The group $N_{2,2}$ is called *the 3-dimensional Heisenberg group* (cf. B.Gaveau [9], H.Uemura-S.Watanabe [22]), and the group $N_{3,2}$ does not play a different role from $N_{2,2}$ in our future considerations. Thus, in this paper, we assume $n = 4$ and study the group $N_{4,2}$ exclusively.

Notations. (cf. H.Uemura-S.Watanabe [22])

i) $x \in \mathbb{R}^{10}$ is denoted by $x = (x_i, x_{(jk)})_{\substack{i=1, \dots, 4 \\ 1 \leq j < k \leq 4}}$ or by $[x, X]$

where $x \in \mathbb{R}^4$ and $X \in \mathfrak{o}(4) :=$ the totality of 4×4 real skew-symmetric matrices, defined by $x = (x_1, \dots, x_4)$ and

$$X_{ij} = \begin{cases} x_{(ij)} & \text{if } i < j, \\ -x_{(ji)} & \text{if } i > j, \\ 0 & \text{otherwise.} \end{cases}$$

We also denote such X by $\sum_{i < j} x_{(ij)} \delta_{ij} - \sum_{i > j} x_{(ji)} \delta_{ij}$

ii) For every $\Omega \in O(4)$ we define a mapping $T(\Omega)$ on $\mathbb{R}^{10}$ by

$$T(\Omega)x = [\Omega x, \Omega X^t \Omega]$$

iii) For $X, Y \in \mathfrak{o}(4)$, define $X \sim Y$ if and only if $X = \Omega Y^t \Omega$ for some $\Omega \in O(4)$

Remark 3.1.

Noting that $\Omega X^t \Omega \in \mathfrak{o}(4)$ and that $\|X\| = \|\Omega X^t \Omega\|$, $\|\cdot\|$ being a 16-dimensional Euclidean norm by regarding X as an element of 16-dimensional Euclidean space, we know $T(\Omega) \in O(10)$. And it is easy to see that ${}^t T(\Omega) = T({}^t \Omega)$

4. Computation of minimal horizontal curves.

In this section we determine all the minimal horizontal curves on $N_{4,2}$ connecting the origin 0 and $x = [0, X]$. For each $h \in K^{0,x}$, the horizontal curve $c^h(t) = (c^{h,i}(t), c^{h,(jk)}(t))_{\substack{i=1,\dots,4 \\ 1 \leq j < k \leq 4}}$

and the deterministic Malliavin covariance

$$\sigma (= \sigma(h)) = \begin{pmatrix} \sigma^{ij} & \sigma^{i(mn)} \\ \sigma^{(kl)j} & \sigma^{(kl)(mn)} \end{pmatrix}_{1 \leq i, j \leq 4, 1 \leq k < l \leq 4, 1 \leq m < n \leq 4}$$

are given as follows :

$$\begin{aligned} c^{h,i}(t) &= h_t^i, \quad i = 1, \dots, 4, \\ c^{h,(jk)}(t) &= \frac{1}{2} \int_0^t \{ h_s^j \cdot \dot{h}_s^k - h_s^k \cdot \dot{h}_s^j \} ds, \quad 1 \leq j < k \leq 4, \end{aligned}$$

where $\dot{\cdot} = \frac{d}{dt}$ and $c^h(1) = [0, X]$ and

$$\begin{aligned} \sigma^{ij} &= \delta_{ij}, \quad 1 \leq i, j \leq 4, \\ \sigma^{(kl)j} &= \sigma^{j(kl)} = 0 \quad \text{if } k \neq j \text{ and } l \neq j, \\ \sigma^{(kl)k} &= \sigma^{k(kl)} = - \int_0^1 h_t^l dt, \\ \sigma^{(kl)l} &= \sigma^{l(kl)} = \int_0^1 h_t^k dt, \\ \sigma^{(kl)(mn)} &= 0 \quad \text{if } \{k, l, m, n\} = \{1, 2, 3, 4\}, \\ \sigma^{(kl)(kl)} &= \int_0^1 \{ (h_t^k)^2 + (h_t^l)^2 \} dt, \\ \sigma^{(kl)(kn)} &= \int_0^1 h_t^l \cdot h_t^n dt, \\ \sigma^{(kl)(mk)} &= \sigma^{(mk)(kl)} = - \int_0^1 h_t^l \cdot h_t^m dt \end{aligned}$$

and

$$\sigma^{(kl)(ml)} = \int_0^1 h_t^k \cdot h_t^m dt.$$

Proposition 4.1.

If rank $X = 4$, the above deterministic Malliavin covariance σ is non-degenerate.

Proof.

For all $X \in \mathfrak{o}(4)$, there exists $U \in \mathfrak{o}(4)$ such that $U = u_1(\delta_{12} - \delta_{21}) + u_2(\delta_{34} - \delta_{43})$ and $X \sim U$. If $\text{rank } X = 4$, then $\text{rank } U = 4$, i.e. $u_1, u_2 \neq 0$. It is enough to prove in the case $X = U$ because

$$\sigma(\Omega h) = T(\Omega) \sigma(h) {}^t T(\Omega), \quad \Omega \in O(4),$$

which is easily obtained by that

$$Y^{0, \Omega h}(t) = T(\Omega) Y^{0, h}(t) {}^t T(\Omega)$$

and that

$$\begin{aligned} T(\Omega) & \sum_{\alpha=1}^4 V_{\alpha}(c^h(t)) \otimes V_{\alpha}(c^h(t)) {}^t T(\Omega) \\ &= \sum_{\alpha=1}^4 V_{\alpha}(c^{\Omega h}(t)) \otimes V_{\alpha}(c^{\Omega h}(t)), \end{aligned}$$

$Y^{x, h}$ and V_{α} being as in (2.4) and (3.1) respectively.

Since $h \in K^{0, [0, U]}$,

$$(4.1) \quad \begin{cases} \int_0^1 \dot{h}_t^i dt = 0, \quad i = 1, \dots, 4, \\ \frac{1}{2} \int_0^1 (\dot{h}_t^1 \cdot \dot{h}_t^2 - \dot{h}_t^2 \cdot \dot{h}_t^1) dt = u_1 \quad (\neq 0), \\ \frac{1}{2} \int_0^1 (\dot{h}_t^3 \cdot \dot{h}_t^4 - \dot{h}_t^4 \cdot \dot{h}_t^3) dt = u_2 \quad (\neq 0), \\ \frac{1}{2} \int_0^1 (\dot{h}_t^i \cdot \dot{h}_t^j - \dot{h}_t^j \cdot \dot{h}_t^i) dt = 0 \quad \text{if } (i, j) \neq (1, 2), (3, 4). \end{cases}$$

It is easy to show that σ is transformed into the following θ by a general linear mapping: $\theta^{ij} = \delta_{ij}$, $\theta^{(ij)k} = \theta^{k(ij)} = 0$ for all $1 \leq i < j \leq 4$, $k = 1, \dots, 4$, and $\theta^{(ij)(kl)}$ are given by

replacing h with $\bar{h}$ in $\sigma^{(ij)(kl)}$, where $\bar{h}_t^i := h_t^i - \int_0^1 h_s^i ds$.

Clearly (4.1) remains valid under replacing h with $\bar{h}$.

Now it is enough to show that ${}^t \xi \theta \xi = 0$ implies $\xi = 0$ where we set $\theta = (\theta^{(ij)(kl)})_{\substack{1 \leq i < j \leq 4 \\ 1 \leq k < l \leq 4}}$ and $\xi = {}^t (\xi_{12}, \xi_{13}, \xi_{14}, \xi_{23}, \xi_{24}, \xi_{34})$.

Since

$$\begin{aligned} {}^t\xi\theta\xi = \int_0^1 \{ & (-\xi_{12}\cdot\bar{h}_t^1 + \xi_{23}\cdot\bar{h}_t^3 + \xi_{24}\cdot\bar{h}_t^4)^2 \\ & + (\xi_{12}\cdot\bar{h}_t^2 + \xi_{13}\cdot\bar{h}_t^3 + \xi_{14}\cdot\bar{h}_t^4)^2 \\ & + (\xi_{13}\cdot\bar{h}_t^1 + \xi_{23}\cdot\bar{h}_t^2 - \xi_{34}\cdot\bar{h}_t^4)^2 \\ & + (\xi_{14}\cdot\bar{h}_t^1 + \xi_{24}\cdot\bar{h}_t^2 + \xi_{34}\cdot\bar{h}_t^3)^2 \} dt, \end{aligned}$$

we see that ${}^t\xi\theta\xi = 0$ is equivalent to the following (4.2) :

$$(4.2) \quad \begin{cases} -\xi_{12}\cdot\bar{h}_t^1 + \xi_{23}\cdot\bar{h}_t^3 + \xi_{24}\cdot\bar{h}_t^4 = 0, \\ \xi_{12}\cdot\bar{h}_t^2 + \xi_{13}\cdot\bar{h}_t^3 + \xi_{14}\cdot\bar{h}_t^4 = 0, \\ \xi_{13}\cdot\bar{h}_t^1 + \xi_{23}\cdot\bar{h}_t^2 - \xi_{34}\cdot\bar{h}_t^4 = 0, \\ \xi_{14}\cdot\bar{h}_t^1 + \xi_{24}\cdot\bar{h}_t^2 + \xi_{34}\cdot\bar{h}_t^3 = 0. \end{cases}$$

Then substituting (4.2) into (4.1), we can easily show $\xi = 0$. This completes the proof. //

Thus, in view of (2.6-II), the minimal horizontal curve in this case is obtained from bicharacteristics. This is also true in the case $\text{rank } X = 2$, because we can reduce this case to that of Heisenberg group.

Now we determine the bicharacteristics on $N_{4,2}$. Substituting (3.1), the Hamilton equation (2.5) is given by

$$(4.3) \quad \begin{cases} \dot{x}_t^i = p_t^i + \frac{1}{2} \left\{ \sum_{k < i} x_t^k \cdot p_t^{(ki)} - \sum_{k > i} x_t^k \cdot p_t^{(ik)} \right\}, \\ \dot{x}_t^{(ij)} = \frac{1}{2} (x_t^i \cdot \dot{x}_t^j - x_t^j \cdot \dot{x}_t^i), \\ \dot{p}_t^i = -\frac{1}{2} \left\{ \sum_{i < j} p_t^j \cdot p_t^{(ij)} - \sum_{i > j} p_t^j \cdot p_t^{(ji)} \right\} \\ \quad - \frac{1}{4} \left\{ \sum_{\substack{l < j \\ i < j}} x_t^l \cdot p_t^{(ij)} \cdot p_t^{(lj)} + \sum_{\substack{l > j \\ i > j}} x_t^l \cdot p_t^{(ji)} \cdot p_t^{(jl)} \right. \\ \quad \left. - \sum_{i < j < l} x_t^l \cdot p_t^{(ij)} \cdot p_t^{(jl)} - \sum_{k < j < i} x_t^k \cdot p_t^{(kj)} \cdot p_t^{(ji)} \right\}, \\ \dot{p}_t^{(ij)} = 0. \end{cases}$$

Moreover it is easy to show that

$$(4.4) \quad \dot{h}_t^i \quad (:= \langle p_t, V_i(x_t) \rangle) = \dot{x}_t^i \quad .$$

Since $p_t^{(ij)} = p_0^{(ij)}$ and $\{x_t^{(ij)}\}$ are obtained by $\{x_t^i\}$, setting $x_t = {}^t(x_t^1, \dots, x_t^4)$ and $p_t = {}^t(p_t^1, \dots, p_t^4)$, we must solve the following equation :

$$(4.5) \quad \frac{d}{dt} \begin{pmatrix} x_t \\ p_t \end{pmatrix} = \begin{pmatrix} -A & I \\ A^2 & -A \end{pmatrix} \begin{pmatrix} x_t \\ p_t \end{pmatrix} \quad .$$

Here $A = (a_{ij})_{i,j=1,\dots,4} \in \mathfrak{o}(4)$ is given by

$$a_{ij} = \begin{cases} \frac{1}{2} \cdot p_0^{(ij)} & , \quad i < j \quad , \\ -\frac{1}{2} \cdot p_0^{(ji)} & , \quad i > j \quad , \\ 0 & , \quad i = j \quad , \end{cases}$$

and I denotes the 4×4 identity matrix.

Proposition 4.2.

For all $\Omega \in O(4)$,

$$\begin{cases} p_t(T(\Omega)p_0, T(\Omega)x_0) = T(\Omega)p_t(p_0, x_0) \\ x_t(T(\Omega)p_0, T(\Omega)x_0) = T(\Omega)x_t(p_0, x_0) \end{cases} \quad .$$

Proof.

It is easy to see that

$$\frac{d}{dt} \begin{pmatrix} \Omega x_t \\ \Omega p_t \end{pmatrix} = \begin{pmatrix} -\Omega A^t \Omega & I \\ (\Omega A^t \Omega)^2 & -\Omega A^t \Omega \end{pmatrix} \begin{pmatrix} \Omega x_t \\ \Omega p_t \end{pmatrix} \quad ,$$

so the assertion of this proposition is obvious. //

Remark 4.1.

We know that for all $A \in \mathfrak{o}(4)$, there exist $\Omega \in O(4)$ and $Q \in Q(4) := \{ q_1(\delta_{12} - \delta_{21}) + q_2(\delta_{34} - \delta_{43}) \in \mathfrak{o}(4) ; 0 \leq q_1 \leq q_2 \}$ such that

$$A = \Omega Q^t \Omega$$

Thus, by the proposition above, we can conclude that determining all the minimal horizontal curves connecting 0 and $x = [0, X]$ is

equivalent to determining all $(\tilde{p}_0 (= [\tilde{p}_0, 2Q]), \Omega) \in \mathbb{R}^{10} \times O(4)$, $Q \in Q(4)$, such that the H -norm of h , given by (4.4) from the solution of (4.3) with the initial value $(\tilde{p}_0, 0)$ satisfying $x_1(\tilde{p}_0, 0) = T({}^t\Omega)x$, takes a minimum.

Replacing A by $Q \in Q(4)$ in (4.5), we have

$$(4.6) \quad \begin{cases} \dot{x}_t^{2i-1} = -q_i \cdot x_t^{2i} + p_t^{2i-1} \\ \dot{x}_t^{2i} = q_i \cdot x_t^{2i-1} + p_t^{2i} \\ \dot{p}_t^{2i-1} = -q_i^2 \cdot x_t^{2i-1} - q_i \cdot p_t^{2i} \\ \dot{p}_t^{2i} = -q_i^2 \cdot x_t^{2i} + q_i \cdot p_t^{2i-1} \end{cases}, \quad i = 1, 2,$$

with initial value $(x_0, p_0) := (0, \tilde{p}_0)$. We denote the solution of (4.6) by $(x_t(\tilde{p}_0), p_t(\tilde{p}_0))$. (In the following we always assume $x_0 = 0$, so we always omit x_0 .) In this case clearly $h_t = x_t$ and the solution of (4.6) is :

a) if $q_i = 0$

$$\begin{cases} x_t^{2i-1}(\tilde{p}_0) = \tilde{p}_0^{2i-1} t \\ x_t^{2i}(\tilde{p}_0) = \tilde{p}_0^{2i} t \\ p_t^{2i-1}(\tilde{p}_0) = \tilde{p}_0^{2i-1} \\ p_t^{2i}(\tilde{p}_0) = \tilde{p}_0^{2i} \end{cases},$$

b) if $q_i > 0$

$$\begin{cases} x_t^{2i-1}(\tilde{p}_0) = (\tilde{p}_0^{2i-1}/2q_i) \sin 2q_i t + (\tilde{p}_0^{2i}/2q_i) (\cos 2q_i t - 1) \\ x_t^{2i}(\tilde{p}_0) = -(\tilde{p}_0^{2i-1}/2q_i) (\cos 2q_i t - 1) + (\tilde{p}_0^{2i}/2q_i) \sin 2q_i t \\ p_t^{2i-1}(\tilde{p}_0) = (\tilde{p}_0^{2i-1}/2) (\cos 2q_i t + 1) - (\tilde{p}_0^{2i}/2) \sin 2q_i t \\ p_t^{2i}(\tilde{p}_0) = (\tilde{p}_0^{2i-1}/2) \sin 2q_i t + (\tilde{p}_0^{2i}/2) (\cos 2q_i t + 1) \end{cases},$$

thus, always, $\frac{1}{2} \cdot \|h\|_H^2 = \frac{1}{2} \cdot \sum_{i=1}^4 (\tilde{p}_0^i)^2$

By the condition $x_1^i(\tilde{p}_0) = 0$, $i = 1, \dots, 4$, we must have that

$$q_i = r_i \pi, \quad r_i \in \mathbb{N} \quad \text{if} \quad (\tilde{p}_0^{2i-1}, \tilde{p}_0^{2i}) \neq (0, 0),$$

and we set $r_i = 0$ when $\tilde{p}_0^{2i-1} = \tilde{p}_0^{2i} = 0$

$x_1^{(ij)}(\tilde{\mathbf{p}}_0)$ and $\frac{1}{2} \cdot \|h\|_H^2$ are computed as follows :

i) In the case $0 = r_1 = r_2$,

$$x_1^{(ij)}(\tilde{\mathbf{p}}_0) = 0 \quad \text{and} \quad \frac{1}{2} \cdot \|h\|_H^2 = 0 .$$

ii) In the case $0 = r_1 < r_2$,

$$\begin{aligned} x_1^{(ij)}(\tilde{\mathbf{p}}_0) &= 0 \quad \text{if} \quad (ij) \neq (34) , \\ x_1^{(34)}(\tilde{\mathbf{p}}_0) &= \frac{1}{4r_2\pi} \cdot \{ (\tilde{p}_0^3)^2 + (\tilde{p}_0^4)^2 \} \end{aligned}$$

and

$$\frac{1}{2} \cdot \|h\|_H^2 = 2r_2\pi \cdot x_1^{(34)}(\tilde{\mathbf{p}}_0) .$$

ii)' In the case $0 < r_1 = r_2 = r$,

$$\begin{aligned} x_1^{(12)}(\tilde{\mathbf{p}}_0) &= \frac{1}{4r\pi} \cdot \{ (\tilde{p}_0^1)^2 + (\tilde{p}_0^2)^2 \} , \\ x_1^{(13)}(\tilde{\mathbf{p}}_0) &= \frac{1}{4r\pi} \cdot \{ \tilde{p}_0^2 \cdot \tilde{p}_0^3 - \tilde{p}_0^1 \cdot \tilde{p}_0^4 \} , \\ x_1^{(14)}(\tilde{\mathbf{p}}_0) &= \frac{1}{4r\pi} \cdot \{ \tilde{p}_0^1 \cdot \tilde{p}_0^3 + \tilde{p}_0^2 \cdot \tilde{p}_0^4 \} , \\ x_1^{(23)}(\tilde{\mathbf{p}}_0) &= \frac{-1}{4r\pi} \cdot \{ \tilde{p}_0^1 \cdot \tilde{p}_0^3 + \tilde{p}_0^2 \cdot \tilde{p}_0^4 \} , \\ x_1^{(24)}(\tilde{\mathbf{p}}_0) &= \frac{1}{4r\pi} \cdot \{ \tilde{p}_0^2 \cdot \tilde{p}_0^3 - \tilde{p}_0^1 \cdot \tilde{p}_0^4 \} , \\ x_1^{(34)}(\tilde{\mathbf{p}}_0) &= \frac{1}{4r\pi} \cdot \{ (\tilde{p}_0^3)^2 + (\tilde{p}_0^4)^2 \} \end{aligned}$$

and

$$\frac{1}{2} \cdot \|h\|_H^2 = 2r\pi \cdot \{ x_1^{(12)}(\tilde{\mathbf{p}}_0) + x_1^{(34)}(\tilde{\mathbf{p}}_0) \} .$$

iii) In the case $0 < r_1 < r_2$,

$$\begin{aligned} x_1^{(12)}(\tilde{\mathbf{p}}_0) &= \frac{1}{4r_1\pi} \cdot \{ (\tilde{p}_0^1)^2 + (\tilde{p}_0^2)^2 \} , \\ x_1^{(34)}(\tilde{\mathbf{p}}_0) &= \frac{1}{4r_2\pi} \cdot \{ (\tilde{p}_0^3)^2 + (\tilde{p}_0^4)^2 \} , \\ x_1^{(ij)}(\tilde{\mathbf{p}}_0) &= 0 \quad \text{otherwise} \end{aligned}$$

and

$$\frac{1}{2} \cdot \|h\|_H^2 = 2r_1\pi \cdot x_1^{(12)}(\tilde{\mathbf{p}}_0) + 2r_2\pi \cdot x_1^{(34)}(\tilde{\mathbf{p}}_0) .$$

Thus, by setting $x_1(\tilde{\mathbf{p}}_0) = [0, X(\tilde{\mathbf{p}}_0)]$, we know that :

in the case i) , $\text{rank } X(\tilde{\mathbf{p}}_0) = 0$,

in the case ii) or ii)' , $\text{rank } X(\tilde{\mathbf{p}}_0) = 2$

and

in the case iii) , $\text{rank } X(\tilde{p}_0) = 4$.

Therefore the cases that the given matrix X is rank 0 (i.e. $X = 0$), rank 2 and rank 4 correspond respectively to the case i), the case ii) or ii)' and the case iii).

Finally we find the *minimal* horizontal curves x_t connecting 0 and $x = [0, X]$ Equivalently we determine all $h \in K_{\min}^{0, X}$

I. The case of $\text{rank } X = 0$, i.e. $X = 0$.

In this case clearly $x_t = 0$ and $h = 0$

II. The case of $\text{rank } X = 2$.

First of all we show that the case ii)' can be reduced to the case ii).

Define $\theta_\theta \in O(2)$ and $A_\theta^{(4)} \in O(4)$ by

$$\theta_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} , \quad \theta \in \mathbb{R} ,$$

and

$$(4.7) \quad A_\theta^{(4)} = \begin{pmatrix} \cos\theta_1 \cdot \theta_{\theta_2} & -\sin\theta_1 \cdot \theta_{\theta_3} \\ \sin\theta_1 \cdot \theta_{\theta_4} & \cos\theta_1 \cdot \theta_{\theta_5} \end{pmatrix}$$

where $\underline{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4)$ and $\theta_5 = -\theta_2 + \theta_3 + \theta_4$ Then it is easy to

see that for given $Q \in Q(4)$ such that $q_1 = q_2$, $\Omega \in O(4)$

satisfies $\Omega Q^t \Omega = Q$ if and only if $\Omega = A_\theta^{(4)}$, and that for all $z \in \mathbb{R}^4$,

there exists $\underline{\theta}$ such that $A_\theta^{(4)} z = {}^t(0, 0, \tilde{z}_3, \tilde{z}_4)$. Thus if

$(\tilde{p}'_0 (= [\tilde{p}'_0, \tilde{Q}]), \Omega')$ attains the minimal horizontal curve and furthermore $\tilde{Q}$ is as in ii)', there exists $\underline{\theta}$ such that

$$T(A_\theta^{(4)})\tilde{p}'_0 = [\tilde{p}'_0, \tilde{Q}] , \quad \tilde{p}'_0 = {}^t(0, 0, \tilde{p}'_0{}^3, \tilde{p}'_0{}^4) .$$

So, by Proposition 4.2 and the invariance of H -norms under the

orthogonal mapping, the case ii)' is reduced to the case ii) (Recall

that we set $q_i = 0$ if $\tilde{p}_0^{2i-1} = \tilde{p}_0^{2i} = 0$) . Therefore we only

consider the case ii).

Let $U_1 = u(\delta_{34} - \delta_{43})$, $u > 0$, be the matrix satisfying $X \sim U_1$, thus there exists $\Omega \in O(4)$ such that ${}^t\Omega X \Omega = U_1$. All of such Ω are obtained by $\{ \Omega_1 A_{\underline{\theta}}^{(2)} ; \underline{\theta} = (\theta_1, \theta_2) \in [0, 2\pi)^2 \}$, where Ω_1 is an element of $O(4)$ satisfying ${}^t\Omega_1 X \Omega_1 = U_1$ and

$$(4.8) \quad A_{\underline{\theta}}^{(2)} = \begin{pmatrix} \theta_{\theta_1} & 0 \\ 0 & \theta_{\theta_2} \end{pmatrix}.$$

This is easily seen from the fact that

$${}^t\Omega U_1 \Omega = U_1 \quad \text{if and only if} \quad \Omega = A_{\underline{\theta}}^{(2)} \quad \text{for some } \underline{\theta}$$

and that ${}^t\Omega_1 X \Omega_1 = U_1$ implies ${}^t\Omega_1 \Omega_1 U_1 {}^t\Omega_1 \Omega_1 = U_1$

Since $x_1^{(3,4)}(\tilde{p}_0) = u$, $\frac{1}{2} \cdot \|h\|_H^2 = 2r_2 u \pi$ and this takes a minimum when $r_2 = 1$. So $x_1^{(3,4)}(\tilde{p}_0) = \frac{1}{4\pi} \cdot \{(\tilde{p}_0^3)^2 + (\tilde{p}_0^4)^2\} = u$, i.e.

$$(\tilde{p}_0^3)^2 + (\tilde{p}_0^4)^2 = 4\pi u.$$

Thus, for some $\alpha \in [0, 2\pi)$, we can write

$$\begin{cases} \tilde{p}_0^3 = \sqrt{4\pi u} \cdot \cos \alpha \\ \tilde{p}_0^4 = \sqrt{4\pi u} \cdot \sin \alpha \end{cases}.$$

Therefore

$$h_t^1(\tilde{p}_0) = h_t^2(\tilde{p}_0) = 0$$

and

$$\begin{pmatrix} h_t^3(\tilde{p}_0) \\ h_t^4(\tilde{p}_0) \end{pmatrix} = \theta_\alpha \cdot \begin{pmatrix} \sqrt{u/\pi} \cdot \sin 2\pi t \\ \sqrt{u/\pi} \cdot (1 - \cos 2\pi t) \end{pmatrix}$$

Noting that $\theta_\theta \cdot \theta_\alpha = \theta_{\theta+\alpha}$, every element of $K_{\min}^{0,x}$ is obtained by

$$h_{\underline{\theta}} = \Omega_1 A_{\underline{\theta}}^{(2)} \tilde{h},$$

where

$$(4.9) \quad \tilde{h}_t = {}^t(0, 0, \sqrt{u/\pi} \cdot \sin 2\pi t, \sqrt{u/\pi} \cdot (1 - \cos 2\pi t))$$

Since $\tilde{h}_t^1 = \tilde{h}_t^2 = 0$, we can change $A_{\underline{\theta}}^{(2)}$ to the following $A_{\underline{\theta}}^{(1)}$:

$$(4.10) \quad A_{\theta}^{(1)} = \begin{pmatrix} I & 0 \\ 0 & \theta_{\theta} \end{pmatrix}, \quad \theta \in [0, 2\pi) .$$

Thus every element of $K_{\min}^{0, X}$ is obtained by

$$(4.11) \quad h^{\theta} = \Omega_1 A_{\theta}^{(1)} \tilde{h}, \quad \theta \in [0, 2\pi) .$$

III. The case $\text{rank } X = 4$.

III-a) The case $X \sim U_2 = u_1(\delta_{12} - \delta_{21}) + u_2(\delta_{34} - \delta_{43})$, $u_1 > u_2 > 0$.

Similarly to the case II, we know that all $\Omega \in O(4)$ satisfying ${}^t\Omega X \Omega = U_2$ are obtained as in the form $\Omega = \Omega_2 A_{\underline{\theta}}^{(2)}$, $\underline{\theta} = (\theta_1, \theta_2) \in [0, 2\pi)^2$, Ω_2 being any fixed element of $O(4)$ such that ${}^t\Omega_2 X \Omega_2 = U_2$. Also $\frac{1}{2} \cdot \|h\|_H^2 = 2r_1 u_1 \pi + 2r_2 u_2 \pi$, so it takes a minimum when $r_1 = 1$ and $r_2 = 2$. Therefore every element of $K_{\min}^{0, X}$ is obtained by

$$(4.12) \quad h^{\underline{\theta}} = \Omega_2 A_{\underline{\theta}}^{(2)} h$$

where

$$(4.13) \quad h_t = {}^t(\sqrt{u_1/\pi} \cdot \sin 2\pi t, \sqrt{u_1/\pi} \cdot (1 - \cos 2\pi t), \\ \sqrt{u_2/2\pi} \cdot \sin 4\pi t, \sqrt{u_2/2\pi} \cdot (1 - \cos 4\pi t)) ,$$

and $A_{\underline{\theta}}^{(2)}$ is as in (4.8) .

III-b) The case $X \sim U_3 = u(\delta_{12} - \delta_{21} + \delta_{34} - \delta_{43})$, $u > 0$.

Similarly to the case II or III-a) we know that all Ω satisfying ${}^t\Omega X \Omega = U_3$ are obtained by $\Omega = \Omega_3 A_{\underline{\theta}}^{(4)}$ where Ω_3 is any fixed element of $O(4)$ satisfying ${}^t\Omega_3 X \Omega_3 = U_3$. After all every element of $K_{\min}^{0, X}$ is obtained by

$$(4.14) \quad h^{\underline{\theta}} = \Omega_3 A_{\underline{\theta}}^{(4)} h, \quad \underline{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4) \in [0, \pi/2] \times [0, 2\pi)^3 ,$$

where $A_{\underline{\theta}}^{(4)}$ is as in (4.7) and h is given by

$$(4.15) \quad h_t = {}^t(\sqrt{u/\pi} \cdot \sin 2\pi t, \sqrt{u/\pi} \cdot (1 - \cos 2\pi t), \\ \sqrt{u/2\pi} \cdot \sin 4\pi t, \sqrt{u/2\pi} \cdot (1 - \cos 4\pi t)) .$$

5. Asymptotic expansion of the heat kernel on $N_{4,2}$

Here we compute the asymptotic behavior of the heat kernel $p(\varepsilon^2, 0, x)$, $x = [0, U] \neq 0$. x is classified into the following three cases (cf. §4).

(Case A) $U \sim u(\delta_{34} - \delta_{43})$, $u > 0$.

(Case B) $U \sim u_1(\delta_{12} - \delta_{21}) + u_2(\delta_{34} - \delta_{43})$, $u_1 > u_2 > 0$

(Case C) $U \sim u(\delta_{12} - \delta_{21} + \delta_{34} - \delta_{43})$, $u > 0$.

Now consider the following S.D.E. associated to $\mathcal{L}$ on the 4-dimensional Wiener space :

$$(5.1) \quad \begin{cases} dX_t = \varepsilon \sum_{\alpha=1}^4 V_{\alpha}(X_t) \circ dw_t^{\alpha} \\ X_0 = 0 \end{cases}$$

where V_{α} , $\alpha = 1, \dots, 4$, are given in (3.1). We denote the solution by $X_t^{\varepsilon} = (X_t^{\varepsilon, i}, X_t^{\varepsilon, (jk)})_{i=1, \dots, 4, 1 \leq j < k \leq 4}$. Then X_t^{ε} is obtained in the

following concrete form ;

$$\begin{cases} X_t^{\varepsilon, i} = \varepsilon w_t^i, & i = 1, \dots, 4, \\ X_t^{\varepsilon, (jk)} = \varepsilon^2 S^{jk}(t, w), & 1 \leq j < k \leq 4, \end{cases}$$

where

$$S^{jk}(t, w) = \frac{1}{2} \int_0^t (w_s^j dw_s^k - w_s^k dw_s^j) .$$

Define an $\mathfrak{o}(4)$ -valued process $S(t, w)$ by

$$S(t, w) = \sum_{i < j} S^{ij}(t, w) \delta_{ij} - \sum_{i > j} S^{ji}(t, w) \delta_{ij} .$$

Then

$$\begin{aligned} p(\varepsilon^2, 0, x) &= E[\delta_x(X_1^{\varepsilon})] \\ &= E[\delta_{[0, U]}([\varepsilon w_1, \varepsilon^2 S(1, w)])] \end{aligned}$$

For every $\Omega \in O(4)$, set $U' = {}^t \Omega U \Omega$. Then, recalling Remark 3.1,

we see

$$\begin{aligned}
& E[\delta_{[0,U]}([\varepsilon w_1, \varepsilon^2 S(1, w)])] \\
&= E[\delta_{T(t_\Omega)[0,U]}([\varepsilon w_1, \varepsilon^2 S(1, w)])] \\
&= E[\delta_{[0,U]}(T(\Omega)[\varepsilon w_1, \varepsilon^2 S(1, w)])] \\
&= E[\delta_{[0,U]}([\varepsilon \Omega w_1, \varepsilon^2 S(1, \Omega w)])] \\
&= E[\delta_{[0,U]}([\varepsilon w_1, \varepsilon^2 S(1, w)])] .
\end{aligned}$$

Therefore, it is sufficient to treat the following three cases :

$$(\text{Case A}) \quad U = u(\delta_{34} - \delta_{43}) \quad , \quad u > 0$$

$$(\text{Case B}) \quad U = u_1(\delta_{12} - \delta_{21}) + u_2(\delta_{34} - \delta_{43}) \quad , \quad u_1 > u_2 > 0$$

$$(\text{Case C}) \quad U = u(\delta_{12} - \delta_{21} + \delta_{34} - \delta_{43}) \quad , \quad u > 0 \quad .$$

$$(\text{Case A}) \quad U = u(\delta_{34} - \delta_{43}) \quad , \quad u > 0 \quad .$$

In this case every element h^θ of $K_{\min}^{0,x}$ is obtained as in (4.11) :

$$h^\theta = A_\theta^{(1)} \tilde{h} \quad , \quad \theta \in [0, 2\pi) \quad ,$$

where $A_\theta^{(1)}$ and $\tilde{h}$ are given in (4.10) and (4.9), respectively.

We want to obtain the asymptotic behavior of the heat kernel $p(\varepsilon^2, 0, x)$ as $\varepsilon \downarrow 0$ through the expression $p(\varepsilon^2, 0, x) = E[\delta_x(X_1^\varepsilon)]$ by evaluating the generalized expectation of the right-hand side. Roughly, the family of diffusions $\{X_t^\varepsilon\}$ conditioned by $X_1^\varepsilon = x$ will be concentrated on the family $M^{0,x}$, actually, will be distributed uniformly on $M^{0,x}$ as $\varepsilon \downarrow 0$. To see how this fact will be reflected on the asymptotic behavior of $p(\varepsilon^2, 0, x)$, we will proceed as in H.Uemura-S.Watanabe [22].

First, we need the following lemma.

Lemma 5.1.A. (cf. H.Uemura-S.Watanabe [22])

For every fixed $\theta_0 \in [0, 2\pi)$, there exists $\eta_0 > 0$, such that

for each η , $0 < \eta < \eta_0$, there exists $\gamma = \gamma(\eta) > 0$ satisfying

$$\int_{|\theta - \theta_0| < \eta} \delta_0 \left(\frac{d}{d\theta} \langle A_{\theta}^{(1)} \tilde{h}, w \rangle_H \right) \cdot \left(- \frac{d^2}{d\theta^2} \langle A_{\theta}^{(1)} \tilde{h}, w \rangle_H \right) d\theta = 1$$

on $\{ w ; \|w - A_{\theta_0}^{(1)} \tilde{h}\|_2 < \gamma \}$

and

$$(5.2) \quad \{ \theta ; \|A_{\theta}^{(1)} \tilde{h} - A_{\theta_0}^{(1)} \tilde{h}\|_2 < \gamma \} \subset \{ \theta ; |\theta - \theta_0| < \eta \} .$$

Here $\langle h, w \rangle_H$ is the extended H -inner product of $h \in H$ and $w \in W_0^4$ defined by

$$\langle h, w \rangle_H = \sum_{i=1}^4 \int_0^1 \dot{h}_t^i dw_t^i ,$$

and $\|\cdot\|_2$ is defined by

$$\|w\|_2^2 = |w_1|^2 + \int_0^1 |w_t|^2 dt , \quad w \in W_0^4$$

Proof

Let $F(\theta, w) = \frac{d}{d\theta} \langle A_{\theta}^{(1)} \tilde{h}, w \rangle_H$ and its Jacobian $\frac{d^2}{d\theta^2} \langle A_{\theta}^{(1)} \tilde{h}, w \rangle_H$ be denoted by $J(\theta, w)$. Clearly $J(\theta, w)$ is continuous with respect to the norm $|\theta| + \|w\|_2$ and it is easy to check that

$$J(\theta_0, A_{\theta_0}^{(1)} \tilde{h}) = -4\pi u \quad (\neq 0) .$$

So we can find η_0 and γ_0 such that $J(\theta, w) < 0$ for all $(\theta, w) \in$

$$\{ \theta ; |\theta - \theta_0| < \eta_0 \} \times \{ w ; \|w - A_{\theta_0}^{(1)} \tilde{h}\|_2 < \gamma_0 \} .$$

Furthermore for any $\eta < \eta_0$, we can choose $\gamma = \gamma(\eta) < \gamma_0$ such that for every $w \in \{ w ; \|w - A_{\theta_0}^{(1)} \tilde{h}\|_2 < \gamma \}$ there exists some $\theta_w \in \{ \theta ; |\theta - \theta_0| < \eta \}$

satisfying $F(\theta_w, w) = 0$. The reason is as follows :

$$\text{Let } W_{\eta} = \{ \theta ; |\theta - \theta_0| < \eta \} \text{ and } F_{\eta}^{\eta} = \{ F(\theta, w) ; \theta \in W_{\eta} \} .$$

That $0 \in F_{\theta_0}^{\eta}$ is easily seen from that $F(\theta_0, A_{\theta_0}^{(1)} \tilde{h}) = 0$. On the

other hand it is easy to show that if $x \in F_{w_0}^{\eta} \cap \left(\bigcup_{w_n \neq w_0, w_n \rightarrow w_0} F_{w_n}^{\eta} \right)^c$

, then $x \in \partial F_{w_0}^{\eta}$. But $F_{w_0}^{\eta}$ is open and hence if $x \in F_{w_0}^{\eta}$, there

exists $\gamma(\eta) > 0$ such that $x \in F_w^\eta$ for all w satisfying $\|w - w_0\|_2 < \gamma(\eta)$. Setting $w_0 = A_{\theta_0}^{(1)} \tilde{h}$ and $x = 0$, we conclude the above statement.

Let $G(w) \in \tilde{D}^\infty$ be a Wiener functional whose support is contained in $\{w; \|w - A_{\theta_0}^{(1)} \tilde{h}\|_2 < \gamma\}$. Then

$$\begin{aligned} & E \left[\int_{|\theta - \theta_0| < \eta} \delta_0(F(\theta, w)) \cdot (-J(\theta, w)) d\theta G(w) \right] \\ &= \int_{|\theta - \theta_0| < \eta} E[\delta_0(F(\theta, w)) \cdot (-J(\theta, w)) \cdot G(w)] d\theta \\ &= \lim_{n \uparrow \infty} \int_{|\theta - \theta_0| < \eta} E[\varphi_n(F(\theta, w)) \cdot (-J(\theta, w)) \cdot G(w)] d\theta. \end{aligned}$$

Here (φ_n) is a sequence in $\mathcal{S}(\mathbb{R}^d)$ ($\mathcal{S} :=$ the Schwartz space of rapidly decreasing C^∞ -functions on $\mathbb{R}^d$) which converges to δ_0 in the distribution sense. Now clearly (5.2) is satisfied for all γ small enough. Note that the support of $G(w)$ is contained in $\{w; \|w - A_{\theta_0}^{(1)} \tilde{h}\|_2 < \gamma\}$. Thus, by the change of variable $x = F(\theta, w)$, the above is equal to

$$\begin{aligned} & \lim_{n \uparrow \infty} E \left[\int_{F_w^\eta} \varphi_n(x) dx G(w) \right] \\ &= E[G(w)], \end{aligned}$$

and this completes the proof. //

Remark 5.1.

We can easily show that

$$-\frac{d^2}{d\theta^2} \langle A_{\theta}^{(1)} \tilde{h}, w \rangle_H = \langle A_{\theta}^{(1)} \tilde{h}, w \rangle_H$$

and

$$\frac{d}{d\theta} \langle A_{\theta}^{(1)} \tilde{h}, w \rangle_H = \langle A_{\theta + (\pi/2)}^{(1)} \tilde{h}, w \rangle_H,$$

so the equality in Lemma 5.1.A is equivalent to

$$\int_{|\theta - \theta_0| < \eta} \delta_0(\langle A_{\theta + (\pi/2)}^{(1)} \tilde{h}, w \rangle_H) \cdot \langle A_{\theta}^{(1)} \tilde{h}, w \rangle_H d\theta = 1$$

Since $K_1 := K_{\min}^{0,x}$ is compact, for all $\gamma > 0$, there exist $(h^{\tilde{\theta}_1}, \dots, h^{\tilde{\theta}_n}) \subset K_1$ such that $K_1 \subset \bigcup_{i=1}^n V_i$ where

$$V_i = \{ w \in W_0^4 ; \|w - h^{\tilde{\theta}_i}\|_2^2 < \gamma^2/2 \} .$$

Set

$$U_i = \{ w \in W_0^4 ; \|w - h^{\tilde{\theta}_i}\|_2^2 < \gamma^2 \} \supset V_i .$$

Let $\psi(\xi) \in C^\infty(\mathbb{R})$ satisfy $0 \leq \psi \leq 1$, $\psi(\xi) = 1$ on $|\xi| \leq \gamma^2/2$ and $\psi(\xi) = 0$ on $|\xi| \geq \gamma^2$. Set $\Psi_i(w) = \psi(\|w - h^{\tilde{\theta}_i}\|_2^2)$. Then it is easy to see that $\Psi_i \in D^\infty$ and

$$I_{U_i}(w) \geq \Psi_i(w) \geq I_{V_i}(w)$$

Setting $\Phi(w) = 1 - \prod_{i=1}^n (1 - \Psi_i(w))$, we see clearly

$$1 - \Phi(w) \leq I_{\bigcap_{i=1}^n V_i^c}$$

and $\bigcap_{i=1}^n V_i^c$ is a closed set which is disjoint from K_1 . Now

$$\begin{aligned} p(\varepsilon^2, 0, x) &= E[\delta_x(X_1^\varepsilon)] \\ &= E[\delta_x(X_1^\varepsilon)(1 - \Phi(\varepsilon w))] + E[\delta_x(X_1^\varepsilon)\Phi(\varepsilon w)] \\ &= J_1^{(1)} + J_2^{(1)} . \end{aligned}$$

Here γ which appears in the definition of Φ is the constant $\gamma(\eta)$ in Lemma 5.1.A associated with η which will be decided in Lemma 5.4 below.

Lemma 5.2. (cf. S.Watanabe [24] Lemma 3.3)

There exists a constant $c > 0$ such that

$$J_1^{(1)} (= E[\delta_x(X_1^\varepsilon)(1 - \Phi(\varepsilon w))]) = O(\exp\{-(\|\tilde{h}\|_H^2 + c)/2\varepsilon^2\})$$

Proof

Clearly for every $\delta > 0$,

$$\begin{aligned} & E[\delta_X(X_1^\varepsilon)(1 - \Phi(\varepsilon w))] \\ &= E[\delta_X(X_1^\varepsilon) \cdot \psi(|X_1^\varepsilon - x|^2/\delta^2) (1 - \Phi(\varepsilon w))] . \end{aligned}$$

By an integration by parts, the above integral can be given in the form

$$\sum E[P_k(\varepsilon, w) \psi^{(l)}(|X_1^\varepsilon - x|^2/\delta^2) \prod_{i=1}^n (1 - \psi)^{(m_i)}(\|\varepsilon w - h^{\theta_i}\|_2^2) \varphi(X_1^\varepsilon)] ,$$

where $P_k(\varepsilon, w)$ is a polynomial of X_1^ε , $|X_1^\varepsilon - x|^2$, $\|\varepsilon w - h^{\theta_i}\|_2^2$, $\gamma(\varepsilon)$ ($\gamma :=$ the inverse of the Malliavin covariance of X_1^ε) and their derivatives, and φ is a bounded continuous function on $\mathbb{R}^{10}$.

Appealing to S.Kusuoka-D.W.Stroock [12], we know

$$E[|P_k(\varepsilon, w)|^p]^{1/p} = O(\varepsilon^{-k}) \quad \text{for some } k \in \mathbb{N} .$$

Thus there exists a constant M such that

$$J_1^{(1)} \leq \varepsilon^{-l_M \cdot P} [|X_1^\varepsilon - x| \leq \delta\gamma , \varepsilon w \in \prod_{i=1}^n V_i^C]^{1/q} ,$$

where $\frac{1}{p} + \frac{1}{q} = 1$. By R.Azencott [1], we have

$$\begin{aligned} & \lim_{\varepsilon \downarrow 0} \varepsilon^2 \log P[|X_1^\varepsilon - x| \leq \delta\gamma , \varepsilon w \in \prod_{i=1}^n V_i^C] \\ & \leq - \inf \{ \frac{1}{2} \cdot \|h\|_H^2 ; |c^{0,h}(1) - x| \leq \delta\gamma , h \in \prod_{i=1}^n V_i^C \} . \end{aligned}$$

Now the right-hand side of the above inequality is strictly less than $-\frac{1}{2} \cdot \|\tilde{h}\|_H^2$ by taking δ small enough, because, otherwise, by taking

$\delta = 1/m$, there exist $h_m \in H$ satisfying $|c^{0,h_m}(1) - x| \leq \gamma/m$, $h_m \in \prod_{i=1}^n V_i^C$ and $\lim \|h_m\|_H^2 \leq \|\tilde{h}\|_H^2$. Then taking a subsequence $\{h_m\}$ of $\{h_m\}$, there exists $\bar{h}$ such that $h_m \rightarrow \bar{h}$ weakly. Such $\bar{h}$

satisfies $\|\bar{h}\|_H^2 \leq \|\tilde{h}\|_H^2$, $c^{0,\bar{h}}(1) = x$ and $\bar{h} \in \prod_{i=1}^n V_i^C$. Therefore $\bar{h} \in K_1$ and this is a contradiction because $\prod_{i=1}^n V_i^C$ and K_1 are disjoint.

This completes the proof. //

In the following, therefore, we consider $J_2^{(1)}$. Let

$$\Phi = 1 - \prod_{i=1}^n (1 - \Psi_i) = \sum_{i=1}^n \Phi_i ,$$

where $\Phi_1 = \Psi_1$, $\Phi_2 = \Psi_2(1-\Psi_1)$, $\Phi_3 = \Psi_3(1-\Psi_1)(1-\Psi_2)$, $\dots$. Then clearly $\Phi_i \cdot I_{U_i} = \Phi_i$, $i = 1, \dots, n$. By Lemma 5.1.A and Remark 5.1 ,

$$\int_{|\theta - \theta_i| < \eta} \delta_0(\langle A_{\theta + (\pi/2)}^{(1)} \tilde{h}, w \rangle_H) \langle A_{\theta}^{(1)} \tilde{h}, w \rangle_H d\theta \cdot \Phi_i(w) = \Phi_i(w) ,$$

$$i = 1, \dots, n$$

So

$$\begin{aligned} J_2^{(1)} &= E[\delta_X(X_1^\varepsilon) \Phi(\varepsilon w)] \\ &= \sum_{i=1}^n E[\delta_X(X_1^\varepsilon) \Phi_i(\varepsilon w)] \\ &= \sum_{i=1}^n E[\delta_{[0, U]}([\varepsilon w_1, \varepsilon^2 S(1, w)]) \Phi_i(\varepsilon w)] \\ &= \sum_{i=1}^n \int_{|\theta - \theta_i| < \eta} E[\delta_{[0, U]}([\varepsilon w_1, \varepsilon^2 S(1, w)]) \delta_0(\langle h^{\theta + (\pi/2)}, \varepsilon w \rangle_H) \\ &\quad \cdot \langle h^\theta, \varepsilon w \rangle_H \cdot \Phi_i(\varepsilon w)] d\theta \\ &= \sum_{i=1}^n \int_{|\theta - \theta_i| < \eta} \exp(-\|h^\theta\|_H^2 / 2\varepsilon^2) \cdot E[\exp(-\langle h^\theta, w \rangle_H / \varepsilon) \\ &\quad \times \delta_{[0, 0]}((\varepsilon w_1, \varepsilon \int_0^1 (h_s^{\theta, i} dw_s^j - h_s^{\theta, j} dw_s^i) + \varepsilon^2 S^{ij}(1, w))) \\ &\quad \times \delta_0(\langle h^{\theta + (\pi/2)}, h^\theta + \varepsilon w \rangle_H) \cdot \langle h^\theta, h^\theta + \varepsilon w \rangle_H \cdot \Phi_i(h^\theta + \varepsilon w)] d\theta , \end{aligned}$$

where the last equality is due to the Cameron-Martin transformation (abbr C-M transformation) $w \rightarrow w + (h^\theta / \varepsilon)$. Now we give some notations.

Notations.

For $w, \tilde{w} \in W_0^4$, we define 4×4 matrices $w \otimes \tilde{w}$, $\dot{w} \otimes \tilde{w}$ and $w \otimes \dot{\tilde{w}}$ as follows :

$$(w \otimes \tilde{w})_{ij} = \int_0^1 w_t^i \cdot \tilde{w}_t^j dt ,$$

$$(\dot{w} \otimes \tilde{w})_{ij} = \int_0^1 \tilde{w}_t^j dw_t^i$$

and

$$(w \otimes \dot{w})_{ij} = \int_0^1 w_t^i d\tilde{w}_t^j .$$

Of course, we define them only when the right-hand sides have meaning as ordinary or stochastic integrals.

Remark 5.2.

It is easy to see that

$$S(1, w) = \frac{1}{2} (w \otimes \dot{w} - \dot{w} \otimes w)$$

and that, for every 4×4 matrix A ,

$$\begin{aligned} (Aw) \otimes \tilde{w} &= A(w \otimes \tilde{w}) , \quad w \otimes (A\tilde{w}) = (w \otimes \tilde{w})^t A , \\ (A\dot{w}) \otimes \tilde{w} &= A(\dot{w} \otimes \tilde{w}) , \quad \dot{w} \otimes (A\tilde{w}) = (\dot{w} \otimes \tilde{w})^t A , \\ (Aw) \otimes \dot{\tilde{w}} &= A(w \otimes \dot{\tilde{w}}) \quad \text{and} \quad w \otimes (A\dot{\tilde{w}}) = (w \otimes \dot{\tilde{w}})^t A . \end{aligned}$$

Then

$$\begin{aligned} J_2^{(1)} &= \sum_{i=1}^n \int_{|\theta - \tilde{\theta}_i| < \eta} \exp(-\|A_{\theta}^{(1)} \tilde{h}\|_H^2 / 2\varepsilon^2) \cdot E[\exp(-\langle A_{\theta}^{(1)} \tilde{h}, w \rangle_H / \varepsilon) \\ &\quad \times \delta_0(\varepsilon w_1) \delta_0(\varepsilon(A_{\theta}^{(1)} \tilde{h} \otimes \dot{w} - \dot{w} \otimes A_{\theta}^{(1)} \tilde{h}) + \frac{\varepsilon^2}{2}(w \otimes \dot{w} - \dot{w} \otimes w)) \\ &\quad \times \delta_0(\langle A_{\theta+(\pi/2)}^{(1)} \tilde{h} , A_{\theta}^{(1)} \tilde{h} + \varepsilon w \rangle_H) \\ &\quad \times \langle A_{\theta}^{(1)} \tilde{h} , A_{\theta}^{(1)} \tilde{h} + \varepsilon w \rangle_H \cdot \Phi_i(A_{\theta}^{(1)} \tilde{h} + \varepsilon w)] d\theta \end{aligned}$$

and noting that $A_{\theta}^{(1)} \in O(4)$ and Remark 5.2, this is equal to

$$\begin{aligned} &\exp(-\|\tilde{h}\|_H^2 / 2\varepsilon^2) \cdot \sum_{i=1}^n \int_{|\theta - \tilde{\theta}_i| < \eta} E[\exp(-\langle \tilde{h}, {}^t A_{\theta}^{(1)} w \rangle_H / \varepsilon) \\ &\quad \times \delta_0(\varepsilon A_{\theta}^{(1)} \cdot {}^t A_{\theta}^{(1)} w_1) \\ &\quad \times \delta_0(A_{\theta}^{(1)} \{ \varepsilon(\tilde{h} \otimes {}^t A_{\theta}^{(1)} \dot{w} - {}^t A_{\theta}^{(1)} \dot{w} \otimes \tilde{h}) \\ &\quad \quad + \frac{\varepsilon^2}{2}({}^t A_{\theta}^{(1)} w \otimes {}^t A_{\theta}^{(1)} \dot{w} - {}^t A_{\theta}^{(1)} \dot{w} \otimes {}^t A_{\theta}^{(1)} w) \}) {}^t A_{\theta}^{(1)}) \\ &\quad \times \delta_0(\langle {}^t A_{\theta}^{(1)} \cdot A_{\theta+(\pi/2)}^{(1)} \tilde{h} , \tilde{h} + \varepsilon \cdot {}^t A_{\theta}^{(1)} w \rangle_H) \\ &\quad \times \langle \tilde{h} , \tilde{h} + \varepsilon \cdot {}^t A_{\theta}^{(1)} w \rangle_H \cdot \Phi_i(A_{\theta}^{(1)} (\tilde{h} + \varepsilon \cdot {}^t A_{\theta}^{(1)} w))] d\theta . \end{aligned}$$

By the invariance of Wiener measure under an orthogonal

transformation, we see, noting also that ${}^t A_{\theta}^{(1)} \cdot A_{\theta+(\pi/2)}^{(1)} = A_{\pi/2}^{(1)}$,

$$\begin{aligned}
J_2^{(1)} &= \exp(-\|\tilde{h}\|_H^2/2\varepsilon^2) \cdot \sum_{i=1}^n \int_{|\theta-\tilde{\theta}_i|<\eta} E[\exp(-\langle \tilde{h}, w \rangle_H / \varepsilon) \\
&\quad \times \delta_{[0,0]}(T(A_\theta^{(1)})[\varepsilon w_1, \varepsilon(\tilde{h} \otimes \dot{w} - \dot{w} \otimes \tilde{h}) + \frac{\varepsilon^2}{2}(\overline{w} \otimes \dot{w} - \dot{w} \otimes w)]) \\
&\quad \times \delta_0(\langle A_{\pi/2}^{(1)} \tilde{h}, \tilde{h} + \varepsilon w \rangle_H) \\
&\quad \times \langle \tilde{h}, \tilde{h} + \varepsilon w \rangle_H \cdot \Phi_i(A_\theta^{(1)} \tilde{h} + \varepsilon A_\theta^{(1)} w)] d\theta .
\end{aligned}$$

Since $\langle A_{\pi/2}^{(1)} \tilde{h}, \tilde{h} \rangle_H = 0$, $-\langle \tilde{h}, w \rangle_H / \varepsilon = 2\pi \cdot S^{3,4}(1, w)$ under the condition that $(\tilde{h} \otimes \dot{w} - \dot{w} \otimes \tilde{h}) + \frac{\varepsilon}{2}(\overline{w} \otimes \dot{w} - \dot{w} \otimes w) = 0$ and that $A_\theta^{(1)} w_1 = 0$ (note that $\tilde{h}^1 = \tilde{h}^2 \equiv 0$) and $T(A_\theta^{(1)}) \in O(10)$, we have finally,

$$\begin{aligned}
J_2^{(1)} &= \exp(-\|\tilde{h}\|_H^2/2\varepsilon^2) \cdot \sum_{i=1}^n \int_{|\theta-\tilde{\theta}_i|<\eta} E[\exp\{2\pi \cdot S^{3,4}(1, w)\} \\
&\quad \times \delta_0(\varepsilon w_1) \delta_0(\varepsilon(\tilde{h} \otimes \dot{w} - \dot{w} \otimes \tilde{h}) + \frac{\varepsilon^2}{2}(\overline{w} \otimes \dot{w} - \dot{w} \otimes w)) \\
&\quad \times \delta_0(\langle A_{\pi/2}^{(1)} \tilde{h}, \varepsilon w \rangle_H) \\
&\quad \times \langle \tilde{h}, \tilde{h} + \varepsilon w \rangle_H \cdot \Phi_i(A_\theta^{(1)} \tilde{h} + \varepsilon A_\theta^{(1)} w)] d\theta .
\end{aligned}$$

Define $\mathbb{R}^{11}$ -valued Wiener functional $g_0^{(1)}(w)$ by

$$\begin{aligned}
(5.3) \quad g_0^{(1)}(w) &= (w_1, S^{1,2}(1, w), (\tilde{h} \otimes \dot{w} - \dot{w} \otimes \tilde{h})_{ij, 1 \leq i < j \leq 4}, \\
&\quad (i, j) \neq (1, 2) \\
&\quad \langle A_{\pi/2}^{(1)} \tilde{h}, w \rangle_H) ,
\end{aligned}$$

then by Lemma 5.4 and Lemma 5.5, given below, we can conclude that

$$\begin{aligned}
(5.4) \quad J_2^{(1)} &\sim \exp(-\|\tilde{h}\|_H^2/2\varepsilon^2) \cdot \varepsilon^{-12} \sum_{i=1}^n \int_{|\theta-\tilde{\theta}_i|<\eta} \Phi_i(A_\theta^{(1)} \tilde{h}) d\theta \\
&\quad E[\exp\{2\pi \cdot S^{3,4}(1, w)\} \delta_0(g_0^{(1)}(w))] \cdot \|\tilde{h}\|_H^2 \quad \text{as } \varepsilon \downarrow 0 .
\end{aligned}$$

Lemma 5.3.A.

$$E[\exp\{2\pi S^{3,4}(1, w)\} \delta_0(g_0^{(1)}(w))] = \frac{3}{2^7 \pi^3 u^3} .$$

Proof.

Define $\xi_k^{(i)}$, $\eta_k^{(i)}$, $k = 1, 2, \dots$, and $\eta_0^{(i)}$, $i = 1, \dots, 4$, by

$$\begin{aligned}
(5.5) \quad \xi_k^{(i)} &= \sqrt{2} \int_0^1 \sin 2\pi k t dw_t^i, \quad k = 1, 2, \dots, \quad i = 1, \dots, 4, \\
\eta_k^{(i)} &= \sqrt{2} \int_0^1 \cos 2\pi k t dw_t^i, \quad k = 1, 2, \dots, \quad i = 1, \dots, 4,
\end{aligned}$$

and

$$\eta_0^{(i)} = w_1^i, \quad i = 1, \dots, 4.$$

Then we can easily show that

$$S^{ij}(1, w) = \frac{1}{2\pi} \left[\sum_{k=1}^{\infty} \frac{1}{k} (\xi_k^{(j)} (\eta_k^{(i)} - \sqrt{2} \cdot \eta_0^{(i)}) - \xi_k^{(i)} (\eta_k^{(j)} - \sqrt{2} \cdot \eta_0^{(j)})) \right] \\ , \quad 1 \leq i < j \leq 4,$$

$$\begin{aligned} (\tilde{h} \otimes \dot{w} - \dot{w} \otimes \tilde{h})_{13} &= -\sqrt{u/2\pi} \xi_1^{(1)}, \\ (\tilde{h} \otimes \dot{w} - \dot{w} \otimes \tilde{h})_{14} &= \sqrt{u/2\pi} (\eta_1^{(1)} - \sqrt{2} \cdot \eta_0^{(1)}), \\ (\tilde{h} \otimes \dot{w} - \dot{w} \otimes \tilde{h})_{23} &= -\sqrt{u/2\pi} \xi_1^{(2)}, \\ (\tilde{h} \otimes \dot{w} - \dot{w} \otimes \tilde{h})_{24} &= \sqrt{u/2\pi} (\eta_1^{(2)} - \sqrt{2} \cdot \eta_0^{(2)}), \\ (\tilde{h} \otimes \dot{w} - \dot{w} \otimes \tilde{h})_{34} &= \sqrt{u/2\pi} \{ \xi_1^{(4)} + (\eta_1^{(3)} - \sqrt{2} \cdot \eta_0^{(3)}) \} \end{aligned}$$

and

$$\langle A_{\pi/2}^{(1)} \tilde{h}, w \rangle_H = \sqrt{2\pi u} (\xi_1^{(3)} - \eta_1^{(4)})$$

Thus

$$\begin{aligned} &E[\exp(2\pi S^{34}(1, w)) \delta_0(g_0^{(1)}(w))] \\ &= E[\delta_0(\eta_0^{(1)}, \eta_0^{(2)}, S^{12}(1, w), -\sqrt{u/2\pi} \cdot \xi_1^{(1)}, \sqrt{u/2\pi}(\eta_1^{(1)} - \sqrt{2} \cdot \eta_0^{(1)}), \\ &\quad -\sqrt{u/2\pi} \cdot \xi_1^{(2)}, \sqrt{u/2\pi}(\eta_1^{(2)} - \sqrt{2} \cdot \eta_0^{(2)}))] \\ &\quad \times E[\exp\{\sum_{k=1}^{\infty} \frac{1}{k} (\xi_k^{(4)} (\eta_k^{(3)} - \sqrt{2} \cdot \eta_0^{(3)}) - \xi_k^{(3)} (\eta_k^{(4)} - \sqrt{2} \cdot \eta_0^{(4)}))\} \\ &\quad \delta_0(\eta_0^{(3)}, \eta_0^{(4)}, \sqrt{u/2\pi} \{ \xi_1^{(4)} + (\eta_1^{(3)} - \sqrt{2} \cdot \eta_0^{(3)}) \}, \sqrt{2\pi u} (\xi_1^{(3)} - \eta_1^{(4)}))] \\ &= J_3^{(1)} \times J_4^{(1)}. \end{aligned}$$

By Proposition 5.1 below, we see that $J_3^{(1)} = \frac{3}{16\pi u^2}$. On the other hand,

$$\begin{aligned} J_4^{(1)} &= E[\exp\{\sum_{k=2}^{\infty} \frac{1}{k} (\xi_k^{(4)} \eta_k^{(3)} - \xi_k^{(3)} \eta_k^{(4)})\}] \\ &\quad \times E[\exp(\xi_1^{(4)} \eta_1^{(3)} - \xi_1^{(3)} \eta_1^{(4)}) | \eta_0^{(3)} = \eta_0^{(4)} = \xi_1^{(4)} + \eta_1^{(3)} = \xi_1^{(3)} - \eta_1^{(4)} = 0] \\ &\quad \times \{(2\pi)^2 \cdot \sqrt{\det C}\}^{-1}, \end{aligned}$$

where C is the covariant matrix of $(\eta_0^{(3)}, \eta_0^{(4)}, \sqrt{u/2\pi}(\xi_1^{(4)} + \eta_1^{(3)}), \sqrt{2\pi u}(\xi_1^{(3)} - \eta_1^{(4)}))$ and it is easy to see that $\det C = 4u^2$. So, by a slight computation, we have

$$\begin{aligned}
J_4^{(1)} &= \prod_{k=2}^{\infty} \left(1 - \frac{1}{k^2}\right)^{-1} \\
&\times E[\exp(-\frac{1}{2}((\xi_1^{(4)} - \eta_1^{(3)})/\sqrt{2})^2 + ((\xi_1^{(3)} + \eta_1^{(4)})/\sqrt{2})^2))] \times \frac{1}{8\pi^2 u} \\
&= \prod_{k=2}^{\infty} \left(1 - \frac{1}{k^2}\right)^{-1} \times \left((1/\sqrt{2\pi}) \int_{-\infty}^{\infty} e^{-x^2} dx\right)^2 \times \frac{1}{8\pi^2 u} \\
&= \frac{1}{8\pi^2 u} .
\end{aligned}$$

Thus the assertion of this lemma is concluded. //

Proposition 5.1.

Let $J_3^{(1)}$ be as in the proof of above lemma. Then

$$J_3^{(1)} = \frac{3}{16\pi u^2} .$$

Proof.

Noting that

$$\int_{-\infty}^{\infty} e^{-2\pi i t x} dt = \delta_x ,$$

it is easy to see that

$$\begin{aligned}
J_3^{(1)} &= \int_{-\infty}^{\infty} E[\exp\{-2\pi i t \cdot \sum_{k=1}^{\infty} \frac{1}{k} (\xi_k^{(1)} \eta_k^{(2)} - \xi_k^{(2)} \eta_k^{(1)})\} \\
&\quad | \eta_0^{(1)} = \eta_0^{(2)} = \xi_1^{(1)} = \xi_1^{(2)} = \eta_1^{(1)} = \eta_1^{(2)} = 0] p_1(0) dt ,
\end{aligned}$$

where $p_1(x)$ is the density of the law of $(\eta_0^{(1)}, \eta_0^{(2)}, -\sqrt{u/2\pi} \cdot \xi_1^{(1)}, \sqrt{u/2\pi} \cdot (\eta_1^{(1)} - \sqrt{2} \cdot \eta_0^{(1)}), -\sqrt{u/2\pi} \cdot \xi_1^{(2)}, \sqrt{u/2\pi} \cdot (\eta_1^{(2)} - \sqrt{2} \cdot \eta_0^{(2)}))$ and $p_1(0) = \frac{1}{2\pi u^2}$. By a slight computation, the above conditional expectation is equal to

$$\begin{aligned}
&\prod_{k=2}^{\infty} E[\exp\{-2\pi i \cdot \frac{t}{k} (\xi_k^{(1)} \eta_k^{(2)} - \xi_k^{(2)} \eta_k^{(1)})\}] \\
&= \prod_{k=2}^{\infty} \frac{k^2}{4\pi^2 t^2 + k^2} \\
&= (1 + 4\pi^2 t^2) \frac{2\pi^2 t}{\sinh 2\pi^2 t} .
\end{aligned}$$

Thus

$$J_3^{(1)} = \frac{1}{2\pi u^2} \int_{-\infty}^{\infty} (1 + 4\pi^2 t^2) \frac{2\pi^2 t}{\sinh 2\pi^2 t} dt$$

$$= \frac{3}{16\pi u^2} \quad . \quad //$$

It is easy to see that $\|\tilde{h}\|_H^2 = 4\pi u$ and hence,

$$J_2^{(1)} \sim \exp\left(-\frac{2\pi u}{\varepsilon^2}\right) \varepsilon^{-1/2} \frac{3}{2^5 \pi^2 u^2} \sum_{i=1}^n \int_{|\theta - \theta_i| < \eta} \Phi_i(A_{\theta}^{(1)} \tilde{h}) \, d\theta \quad .$$

Now

$$\begin{aligned} & \sum_{i=1}^n \int_{|\theta - \theta_i| < \eta} \Phi_i(A_{\theta}^{(1)} \tilde{h}) \, d\theta \\ &= \sum_{i=1}^n \int_{|\theta - \theta_i| < \eta} I_{U_i}(A_{\theta}^{(1)} \tilde{h}) \cdot \Phi_i(A_{\theta}^{(1)} \tilde{h}) \, d\theta \\ &= \sum_{i=1}^n \int_0^{2\pi} I_{U_i}(A_{\theta}^{(1)} \tilde{h}) \cdot \Phi_i(A_{\theta}^{(1)} \tilde{h}) \, d\theta \\ &= \sum_{i=1}^n \int_0^{2\pi} \Phi_i(A_{\theta}^{(1)} \tilde{h}) \, d\theta = 2\pi \quad . \end{aligned}$$

We have, therefore,

$$J_2^{(1)} \sim \exp\left(-\frac{2\pi u}{\varepsilon^2}\right) \varepsilon^{-1/2} \frac{3}{16\pi u^2} \quad \text{as } \varepsilon \downarrow 0 \quad .$$

Therefore, we can now conclude the following.

Theorem 5.1.A.

In Case A, i.e., $x = [0, U]$, $U \sim u(\delta_{34} - \delta_{43})$, $u > 0$,

$$p(\varepsilon^2, 0, x) \sim \exp\left(-\frac{2\pi u}{\varepsilon^2}\right) \varepsilon^{-1/2} \frac{3}{16\pi u^2} \quad \text{as } \varepsilon \downarrow 0 \quad .$$

(Case B) $U = u_1(\delta_{12} - \delta_{21}) + u_2(\delta_{34} - \delta_{43})$, $u_1 > u_2 > 0$.

In this case every element $h_{\underline{\theta}}^{\underline{\theta}}$ of $K_{\min}^{0, x}$ is obtained as in (4.12) :

$$h_{\underline{\theta}}^{\underline{\theta}} = A_{\underline{\theta}}^{(2)} h \quad , \quad \underline{\theta} = (\theta_1, \theta_2) \in [0, 2\pi)^2 \quad ,$$

where $A_{\underline{\theta}}^{(2)}$ and h are as in (4.8) and (4.13), respectively. We set $h^{[1]}$ and $h^{[2]}$ by

$$h_t^{[1]} = t \left(\sqrt{u_1/\pi} \sin 2\pi t, \sqrt{u_1/\pi} (1 - \cos 2\pi t), 0, 0 \right)$$

and

$$h_t^{[2]} = t (0 , 0 , \sqrt{u_2/2\pi} \sin 4\pi t , \sqrt{u_2/2\pi} (1 - \cos 4\pi t)).$$

Similarly as in Case A, we need only to evaluate

$$J_2^{(2)} := E[\delta_X(X_1^\varepsilon)\Phi(\varepsilon w)]$$

where X_t^ε is a solution of S.D.E. (5.1) and Φ is defined as in Case A associated with $K_2 := K_{\min}^{0,x}$, $x = [0, U]$. Again $\gamma(\eta)$ used in the definition of Φ is given by the following lemma with η determined by Lemma 5.4 below. In the following we use the same notations as in Case A.

Lemma 5.1.B.

For every $\underline{\theta}_0 \in [0, 2\pi)^2$, there exists $\eta_0 > 0$ such that for each $\eta \in (0, \eta_0)$ there exists $\gamma(\eta) > 0$ satisfying

$$\begin{aligned} \int_{|\underline{\theta} - \underline{\theta}_0| < \eta} \delta_0 \left(\left(\frac{\partial}{\partial \theta_i} \langle A_{\underline{\theta}}^{(2)} h, w \rangle_H \right)_{i=1,2} \right) \\ \det \left(\left(\frac{\partial^2}{\partial \theta_i \partial \theta_j} \langle A_{\underline{\theta}}^{(2)} h, w \rangle_H \right)_{i,j=1,2} \right) d\theta_1 d\theta_2 = 1 \\ \text{on } \{ w ; \|w - A_{\underline{\theta}_0}^{(2)} h\|_2 < \gamma \} \end{aligned}$$

and

$$\{ \underline{\theta} ; \|A_{\underline{\theta}}^{(2)} h - A_{\underline{\theta}_0}^{(2)} h\|_2 < \gamma \} \subset \{ \underline{\theta} ; |\underline{\theta} - \underline{\theta}_0| < \eta \}.$$

Proof is similar to Lemma 5.1.A and omitted.

Remark 5.3.

It is easy to see that

$$\begin{aligned} \frac{\partial}{\partial \theta_1} \langle A_{\underline{\theta}}^{(2)} h, w \rangle_H &= \langle A_{(\theta_1 + (\pi/2), \theta_2)}^{(2)} h^{[1]}, w \rangle_H, \\ \frac{\partial}{\partial \theta_2} \langle A_{\underline{\theta}}^{(2)} h, w \rangle_H &= \langle A_{(\theta_1, \theta_2 + (\pi/2))}^{(2)} h^{[2]}, w \rangle_H \end{aligned}$$

and

$$\det \left(\left(\frac{\partial^2}{\partial \theta_i \partial \theta_j} \langle A_{\underline{\theta}}^{(2)} h, w \rangle_H \right)_{i,j=1,2} \right)$$

$$= \langle A_{\underline{\theta}}^{(2)} h^{[1]}, w \rangle_H \cdot \langle A_{\underline{\theta}}^{(2)} h^{[2]}, w \rangle_H$$

Thus, denoting $d\underline{\theta} = d\theta_1 d\theta_2$,

$$\begin{aligned} J_2^{(2)} &= E[\delta_x(X_1^\varepsilon) \Phi(\varepsilon w)] \\ &= \sum_{i=1}^n E[\delta_x(X_1^\varepsilon) \Phi_i(\varepsilon w)] \\ &= \sum_{i=1}^n \int_{|\underline{\theta} - \underline{\theta}_i| < \eta} E[\delta_0(\varepsilon w_1) \delta_U(\varepsilon^2 S(1, w)) \\ &\quad \times \delta_0(\langle A_{(\theta_1 + (\pi/2), \theta_2)}^{(2)} h^{[1]}, \varepsilon w \rangle_H) \cdot \langle A_{\underline{\theta}}^{(2)} h^{[1]}, \varepsilon w \rangle_H \\ &\quad \times \delta_0(\langle A_{(\theta_1, \theta_2 + (\pi/2))}^{(2)} h^{[2]}, \varepsilon w \rangle_H) \cdot \langle A_{\underline{\theta}}^{(2)} h^{[2]}, \varepsilon w \rangle_H \cdot \Phi_i(\varepsilon w)] d\underline{\theta} . \end{aligned}$$

Note that $\langle A_{\underline{\theta}}^{(2)} h^{[1]}, \varepsilon w \rangle_H$ is a function of θ_1 and (w^1, w^2) , and $\langle A_{\underline{\theta}}^{(2)} h^{[2]}, \varepsilon w \rangle_H$ that of θ_2 and (w^3, w^4) .

By the C-M transformation $w \rightarrow w + (A_{\underline{\theta}}^{(2)} h / \varepsilon)$,

$$\begin{aligned} J_2^{(2)} &= \sum_{i=1}^n \int_{|\underline{\theta} - \underline{\theta}_i| < \eta} \exp(-\|A_{\underline{\theta}}^{(2)} h\|_H^2 / 2\varepsilon^2) E[\exp(-\langle A_{\underline{\theta}}^{(2)} h, w \rangle_H / \varepsilon) \\ &\quad \times \delta_0(\varepsilon w_1) \delta_O(\varepsilon \langle A_{\underline{\theta}}^{(2)} h \otimes \dot{w} - \dot{w} \otimes A_{\underline{\theta}}^{(2)} h \rangle + \frac{\varepsilon^2}{2} (w \otimes \dot{w} - \dot{w} \otimes w)) \\ &\quad \times \delta_0(\langle A_{(\theta_1 + (\pi/2), \theta_2)}^{(2)} h^{[1]}, A_{\underline{\theta}}^{(2)} h + \varepsilon w \rangle_H) \\ &\quad \times \delta_0(\langle A_{(\theta_1, \theta_2 + (\pi/2))}^{(2)} h^{[2]}, A_{\underline{\theta}}^{(2)} h + \varepsilon w \rangle_H) \\ &\quad \times \langle A_{\underline{\theta}}^{(2)} h^{[1]}, A_{\underline{\theta}}^{(2)} h + \varepsilon w \rangle_H \cdot \langle A_{\underline{\theta}}^{(2)} h^{[2]}, A_{\underline{\theta}}^{(2)} h + \varepsilon w \rangle_H \\ &\quad \times \Phi_i(A_{\underline{\theta}}^{(2)} h + \varepsilon w)] d\underline{\theta} \end{aligned}$$

and noting that $A_{\underline{\theta}}^{(2)} \in O(4)$ and Remark 5.2, this is equal to

$$\begin{aligned} &\exp(-\|h\|_H^2 / 2\varepsilon^2) \sum_{i=1}^n \int_{|\underline{\theta} - \underline{\theta}_i| < \eta} E[\exp(-\langle h, {}^t A_{\underline{\theta}}^{(2)} w \rangle_H / \varepsilon) \\ &\quad \times \delta_0(\varepsilon A_{\underline{\theta}}^{(2)} \cdot {}^t A_{\underline{\theta}}^{(2)} w_1) \\ &\quad \times \delta_O(A_{\underline{\theta}}^{(2)} \{ \varepsilon (h \otimes {}^t A_{\underline{\theta}}^{(2)} w - {}^t A_{\underline{\theta}}^{(2)} w \otimes h) \\ &\quad \quad + \frac{\varepsilon^2}{2} ({}^t A_{\underline{\theta}}^{(2)} w \otimes {}^t A_{\underline{\theta}}^{(2)} \dot{w} - {}^t A_{\underline{\theta}}^{(2)} \dot{w} \otimes {}^t A_{\underline{\theta}}^{(2)} w) \} {}^t A_{\underline{\theta}}^{(2)}) \\ &\quad \times \delta_0(\langle {}^t A_{\underline{\theta}}^{(2)} \cdot A_{(\theta_1 + (\pi/2), \theta_2)}^{(2)} h^{[1]}, h + \varepsilon {}^t A_{\underline{\theta}}^{(2)} w \rangle_H) \\ &\quad \times \delta_0(\langle {}^t A_{\underline{\theta}}^{(2)} \cdot A_{(\theta_1, \theta_2 + (\pi/2))}^{(2)} h^{[2]}, h + \varepsilon {}^t A_{\underline{\theta}}^{(2)} w \rangle_H) \end{aligned}$$

$$\begin{aligned} & \times \langle h^{[1]}, h + \varepsilon^t A_{\underline{\theta}}^{(2)} w \rangle_H \cdot \langle h^{[2]}, h + \varepsilon^t A_{\underline{\theta}}^{(2)} w \rangle_H \\ & \times \Phi_i(A_{\underline{\theta}}^{(2)}(h + \varepsilon^t A_{\underline{\theta}}^{(2)} w))] d\underline{\theta} . \end{aligned}$$

By the invariance of Wiener measure under an orthogonal transformation, we see, noting also that $\langle A_{(\pi/2, 0)}^{(2)} h^{[1]}, h \rangle_H = \langle A_{(0, \pi/2)}^{(2)} h^{[2]}, h \rangle_H = 0$,

$$\begin{aligned} J_2^{(2)} &= \exp(-\|h\|_H^2/2\varepsilon^2) \sum_{i=1}^n \int_{|\underline{\theta}-\underline{\theta}_i|<\eta} E[\exp(-\langle h, w \rangle_H/\varepsilon) \\ & \times \delta_{[0,0]}(T(A_{\underline{\theta}}^{(2)})[\varepsilon w_1, \varepsilon(h \otimes \dot{w} - \dot{w} \otimes h) + \frac{\varepsilon^2}{2}(w \otimes \dot{w} - \dot{w} \otimes w)]) \\ & \times \delta_0(\varepsilon \langle A_{(\pi/2, 0)}^{(2)} h^{[1]}, w \rangle_H) \delta_0(\varepsilon \langle A_{(0, \pi/2)}^{(2)} h^{[2]}, w \rangle_H) \\ & \times \langle h^{[1]}, h + \varepsilon w \rangle_H \cdot \langle h^{[2]}, h + \varepsilon w \rangle_H \cdot \Phi_i(A_{\underline{\theta}}^{(2)}(h + \varepsilon w))] d\underline{\theta} . \end{aligned}$$

Since $T(A_{\underline{\theta}}^{(2)}) \in O(10)$ and $-\langle h, w \rangle_H/\varepsilon = 2\pi \cdot S^{1^2}(1, w) + 4\pi \cdot S^{3^4}(1, w)$ under the condition that $(h \otimes \dot{w} - \dot{w} \otimes h) + \frac{\varepsilon}{2}(w \otimes \dot{w} - \dot{w} \otimes w) = 0$ and that $A_{\underline{\theta}}^{(2)} w_1 = 0$, we have finally,

$$\begin{aligned} J_2^{(2)} &= \exp(-\|h\|_H^2/2\varepsilon^2) \sum_{i=1}^n \int_{|\underline{\theta}-\underline{\theta}_i|<\eta} E[\exp\{2\pi S^{1^2}(1, w) + 4\pi S^{3^4}(1, w)\} \delta_0(\varepsilon w_1) \\ & \times \delta_0(\varepsilon(h \otimes \dot{w} - \dot{w} \otimes h) + \frac{\varepsilon^2}{2}(w \otimes \dot{w} - \dot{w} \otimes w)) \\ & \times \delta_0(\varepsilon \langle A_{(\pi/2, 0)}^{(2)} h^{[1]}, w \rangle_H) \delta_0(\varepsilon \langle A_{(0, \pi/2)}^{(2)} h^{[2]}, w \rangle_H) \\ & \times \langle h^{[1]}, h + \varepsilon w \rangle_H \cdot \langle h^{[2]}, h + \varepsilon w \rangle_H \cdot \Phi_i(A_{\underline{\theta}}^{(2)}(h + \varepsilon w))] d\underline{\theta} . \end{aligned}$$

Therefore, setting R^{1^2} -valued Wiener functional $g_0^{(2)}(w)$ by

$$(5.6) \quad g_0^{(2)}(w) = (w_1, (h \otimes \dot{w} - \dot{w} \otimes h)_{ij}, 1 \leq i < j \leq 4, \langle A_{(\pi/2, 0)}^{(2)} h^{[1]}, w \rangle_H, \langle A_{(0, \pi/2)}^{(2)} h^{[2]}, w \rangle_H) ,$$

we have by Lemma 5.4 and Lemma 5.5 below,

$$\begin{aligned} (5.7) \quad J_2^{(2)} &\sim \exp(-\|h\|_H^2/2\varepsilon^2) \cdot \varepsilon^{-1^2} \cdot \|h^{[1]}\|_H^2 \cdot \|h^{[2]}\|_H^2 \\ & \times \sum_{i=1}^n \int_{|\underline{\theta}-\underline{\theta}_i|<\eta} \Phi_i(A_{\underline{\theta}}^{(2)} h) d\underline{\theta} \\ & \times E[\exp\{2\pi S^{1^2}(1, w) + 4\pi S^{3^4}(1, w)\} \delta_0(g_0^{(2)}(w))] \text{ as } \varepsilon \downarrow 0 . \end{aligned}$$

Lemma 5.3.B.

$$\begin{aligned} E[\exp\{2\pi S^{12}(1, w) + 4\pi S^{34}(1, w)\} \delta_0(g_0^{(2)}(w))] \\ = \frac{3}{64\pi^4 u_1 u_2 (u_1^2 - u_2^2)} \end{aligned}$$

Proof.

Let $p_2(x)$ be the density of the law of $g_0^{(2)}(w)$ Then

$$\begin{aligned} E[\exp\{2\pi S^{12}(1, w) + 4\pi S^{34}(1, w)\} \delta_0(g_0^{(2)}(w))] \\ = E[\exp\{2\pi S^{12}(1, w) + 4\pi S^{34}(1, w)\} | g_0^{(2)}(w)=0] \cdot p_2(0) , \end{aligned}$$

and it is easy to see that $p_2(0) = \frac{1}{16\pi^4 u_1 u_2 (2u_1 + u_2)^2}$.

Let $\Xi_{ij}^{(2)}$, $1 \leq i < j \leq 4$, be the (i, j) -component of $(h \otimes \dot{w} - \dot{w} \otimes h)$.

Then

$$\begin{aligned} \Xi_{12}^{(2)} &= \sqrt{u_1/2\pi} \xi_1^{(2)} + \sqrt{u_1/2\pi} (\eta_1^{(1)} - \sqrt{2} \cdot \eta_0^{(1)}) , \\ \Xi_{13}^{(2)} &= \sqrt{u_1/2\pi} \xi_1^{(3)} - \sqrt{u_2/4\pi} \xi_2^{(1)} , \\ \Xi_{14}^{(2)} &= \sqrt{u_1/2\pi} \xi_1^{(4)} + \sqrt{u_2/4\pi} (\eta_2^{(1)} - \sqrt{2} \cdot \eta_0^{(1)}) , \\ \Xi_{23}^{(2)} &= -\sqrt{u_1/2\pi} (\eta_1^{(3)} - \sqrt{2} \cdot \eta_0^{(3)}) - \sqrt{u_2/4\pi} \xi_2^{(2)} , \\ \Xi_{24}^{(2)} &= -\sqrt{u_1/2\pi} (\eta_1^{(4)} - \sqrt{2} \cdot \eta_0^{(4)}) + \sqrt{u_2/4\pi} (\eta_2^{(2)} - \sqrt{2} \cdot \eta_0^{(2)}) \end{aligned}$$

and

$$\Xi_{34}^{(2)} = \sqrt{u_2/4\pi} \xi_2^{(4)} + \sqrt{u_2/4\pi} (\eta_2^{(3)} - \sqrt{2} \cdot \eta_0^{(3)}) .$$

Here $\xi_j^{(i)}$, $\eta_l^{(k)}$ are as in (5.5) Set

$$\Xi_1^{(2)} := \langle A_{(\pi/2, 0)}^{(2)} h^{[1]}, w \rangle_H = -\sqrt{2\pi u_1} (\xi_1^{(1)} - \eta_1^{(2)})$$

and

$$\Xi_2^{(2)} := \langle A_{(0, \pi/2)}^{(2)} h^{[2]}, w \rangle_H = -\sqrt{4\pi u_2} (\xi_2^{(3)} - \eta_2^{(4)})$$

Then

$$\begin{aligned} E[\exp\{2\pi S^{12}(1, w) + 4\pi S^{34}(1, w)\} | g_0^{(2)}(w)=0] \\ = E[\exp\left(\sum_{k=1}^2 \sum_{m=1}^{\infty} \frac{k}{m} \{ \xi_m^{(2k)} (\eta_m^{(2k-1)} - \sqrt{2} \cdot \eta_0^{(2k-1)}) \right. \\ \left. - \xi_m^{(2k-1)} (\eta_m^{(2k)} - \sqrt{2} \cdot \eta_0^{(2k)}) \} \right) | g_0^{(2)}(w)=0] \end{aligned}$$

$$\begin{aligned}
&= \prod_{k=1}^2 E[\exp(\xi_k^{(2k)} \eta_k^{(2k-1)} - \xi_k^{(2k-1)} \eta_k^{(2k)}) | \tilde{\Xi}_{2k-1, 2k=0}^{(2)}, \Xi_k^{(2)}=0] \\
&\quad \times E[\exp\{\frac{1}{2}(\xi_2^{(2)} \eta_2^{(1)} - \xi_2^{(1)} \eta_2^{(2)}) + 2(\xi_1^{(4)} \eta_1^{(3)} - \xi_1^{(3)} \eta_1^{(4)})\} \\
&\quad | \tilde{\Xi}_{13}^{(2)} = \tilde{\Xi}_{14}^{(2)} = \tilde{\Xi}_{23}^{(2)} = \tilde{\Xi}_{24}^{(2)} = 0] \\
&\quad \times \prod_{k=1}^2 \prod_{m=3}^{\infty} E[\exp\{\frac{k}{m}(\xi_m^{(2k)} \eta_m^{(2k-1)} - \xi_m^{(2k-1)} \eta_m^{(2k)})\}] \\
&= I_1 \times I_2 \times I_3 .
\end{aligned}$$

Here $\tilde{\Xi}_{ij}^{(2)}$, $1 \leq i < j \leq 4$, denote random variables constructed by excluding the terms $\eta_0^{(k)}$ from $\Xi_{ij}^{(2)}$.

We see easily that $I_1 = \frac{1}{4}$ and that $I_3 = \prod_{k=1}^2 \prod_{m=3}^{\infty} \left(1 - \frac{k^2}{m^2}\right)^{-1} = 9$.

So all we must do is to compute I_2 .

Define $X_i^{(2)}$, $i = 1, \dots, 4$, by

$$\begin{aligned}
X_1^{(2)} &= -\sqrt{u_2/2} \cdot \eta_1^{(3)} + \sqrt{u_1} \cdot \xi_2^{(2)}, \\
X_2^{(2)} &= \sqrt{u_2/2} \cdot \xi_1^{(4)} - \sqrt{u_1} \cdot \eta_2^{(1)}, \\
X_3^{(2)} &= \sqrt{u_2/2} \cdot \xi_1^{(3)} + \sqrt{u_1} \cdot \xi_2^{(1)}
\end{aligned}$$

and

$$X_4^{(2)} = -\sqrt{u_2/2} \cdot \eta_1^{(4)} - \sqrt{u_1} \cdot \eta_2^{(2)} .$$

Then

$$\begin{aligned}
&\exp\{\frac{1}{2}(\xi_2^{(2)} \eta_2^{(1)} - \xi_2^{(1)} \eta_2^{(2)}) + 2(\xi_1^{(4)} \eta_1^{(3)} - \xi_1^{(3)} \eta_1^{(4)})\} \\
&= \exp\{(2(u_1+2u_2)/(2u_1+u_2)^2)(-X_1^{(2)}X_2^{(2)} + X_3^{(2)}X_4^{(2)}) + P_2(\Xi)\}
\end{aligned}$$

where $P_2(\Xi)$ is a polynomial of degree 2 in 4 variables $\Xi = (\tilde{\Xi}_{13}^{(2)}, \tilde{\Xi}_{14}^{(2)}, \tilde{\Xi}_{23}^{(2)}, \tilde{\Xi}_{24}^{(2)})$ whose constant term is 0. This equality is obtained by the orthogonal decomposition in $L^2(P)$ of $\xi_j^{(i)}$ and $\eta_l^{(k)}$ with respect to $\ell.s.(\tilde{\Xi}_{13}^{(2)}, \tilde{\Xi}_{14}^{(2)}, \tilde{\Xi}_{23}^{(2)}, \tilde{\Xi}_{24}^{(2)})$, for example, $\xi_2^{(2)}$ is decomposed by

$$\xi_2^{(2)} = \frac{2}{(2u_1+u_2)} (\sqrt{u_1} \cdot X_1^{(2)} - \sqrt{\pi u_2} \cdot \tilde{\Xi}_{23}^{(2)}) .$$

Noting that $X_i^{(2)} \sim N(0, u_1+(u_2/2))$, $i = 1, \dots, 4$,

$$I_2 = E[\exp\{(2(u_1+2u_2)/(2u_1+u_2)^2)(-X_1^{(2)}X_2^{(2)} + X_3^{(2)}X_4^{(2)})\}]$$

$$= (2u_1 + u_2)^2 / 3(u_1^2 - u_2^2) \quad .$$

Combined I_1 , I_2 and I_3 with $p_2(0)$, we conclude this lemma. //

It is easy to compute that $\|h\|_H^2 = 4\pi u_1 + 8\pi u_2$, $\|h^{[1]}\|_H^2 = 4\pi u_1$ and $\|h^{[2]}\|_H^2 = 8\pi u_2$, and we can show that

$$\sum_{i=1}^n \int_{|\underline{\theta} - \underline{\theta}_i| < \eta} \Phi_i(A_{\underline{\theta}}^{(2)} h) d\underline{\theta} = 4\pi^2$$

in the same way as in Case A. Therefore we have

$$J_2^{(2)} \sim \exp(-2\pi(u_1 + 2u_2)/\varepsilon^2) \varepsilon^{-12} \frac{6}{u_1^2 - u_2^2} \quad \text{as } \varepsilon \downarrow 0 \quad .$$

In conclusion, we have

Theorem 5.1.B.

In Case B, i.e., $x = [0, U]$, $U \sim u_1(\delta_{12} - \delta_{21}) + u_2(\delta_{34} - \delta_{43})$,
 $u_1 > u_2 > 0$,

$$p(\varepsilon^2 \cdot 0, x) \sim \exp(-2\pi(u_1 + 2u_2)/\varepsilon^2) \varepsilon^{-12} \frac{6}{u_1^2 - u_2^2} \quad \text{as } \varepsilon \downarrow 0 \quad .$$

(Case C) $U = u(\delta_{12} - \delta_{21} + \delta_{34} - \delta_{43})$, $u > 0$

In this case every element $h_{\underline{\theta}}^{\theta}$ of $K_{\min}^{\theta, x}$ is obtained as in (4.14) :

$$h_{\underline{\theta}}^{\theta} = A_{\underline{\theta}}^{(4)} h \quad , \quad \underline{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4) \in [0, \pi/2] \times [0, 2\pi)^3 \quad ,$$

where $A_{\underline{\theta}}^{(4)}$ and h are as in (4.7) and (4.15), respectively.

Now we need a lemma corresponding to Lemma 5.1.A or 5.1.B, but we must take care that the Malliavin covariance Σ of

$$\left(\frac{\partial}{\partial \theta_i} \langle A_{\underline{\theta}}^{(4)} h, w \rangle_H \right)_{i=1, \dots, 4} \quad \text{is degenerate at } \theta_1 = 0 \quad \text{or } \theta_1 = \pi/2$$

since $\det \Sigma = 3^2 \cdot 2^9 \cdot \pi^4 \cdot u^4 \cdot \cos^2 \theta_1 \cdot \sin^2 \theta_1$ Thus the corresponding

lemma is as follows.

Lemma 5.1.C.

For every $\underline{\theta}_0 \in (0, \pi/2) \times [0, 2\pi)^3$, there exists $\eta_0 > 0$ such that for each $\eta \in (0, \eta_0)$, there exists $\gamma = \gamma(\eta) > 0$ satisfying

$$\int_{|\underline{\theta} - \underline{\theta}_0| < \eta} \delta_0 \left(\left(\frac{\partial}{\partial \theta_i} \langle A_{\underline{\theta}}^{(4)} h, w \rangle_H \right)_{i=1, \dots, 4} \right) \\ \det \left(\left(\frac{\partial^2}{\partial \theta_i \partial \theta_j} \langle A_{\underline{\theta}}^{(4)} h, w \rangle_H \right)_{i, j=1, \dots, 4} \right) d\underline{\theta} = 1 \\ \text{on } \{ w ; \|w - A_{\underline{\theta}_0}^{(4)} h\|_2 < \gamma \} ,$$

and

$$\{ \underline{\theta} ; \|A_{\underline{\theta}}^{(4)} h - A_{\underline{\theta}_0}^{(4)} h\|_2 < \gamma \} \subset \{ \underline{\theta} ; |\underline{\theta} - \underline{\theta}_0| < \eta \}$$

where $d\underline{\theta} = d\theta_1 d\theta_2 d\theta_3 d\theta_4$.

Remark 5.4.

Now we fix $\underline{\theta}_0 \in (0, \pi/2) \times [0, 2\pi)^3$, then for every 4×4 matrix A we have

$$\int_{|\underline{\theta} - \underline{\theta}_0| < \eta} \delta_0 \left(\left(\frac{\partial}{\partial \theta_i} \langle A_{\underline{\theta}}^{(4)} h, Aw \rangle_H \right)_{i=1, \dots, 4} \right) \\ \det \left(\left(\frac{\partial^2}{\partial \theta_i \partial \theta_j} \langle A_{\underline{\theta}}^{(4)} h, Aw \rangle_H \right)_{i, j=1, \dots, 4} \right) d\underline{\theta} = 1 \\ \text{on } \{ w ; \|Aw - A_{\underline{\theta}_0}^{(4)} h\|_2 < \gamma \} .$$

Moreover if $A \in O(4)$,

$$(5.8) \quad \int_{|\underline{\theta} - \underline{\theta}_0| < \eta} \delta_0 \left(\left(\frac{\partial}{\partial \theta_i} \langle {}^t A \cdot A_{\underline{\theta}}^{(4)} h, w \rangle_H \right)_{i=1, \dots, 4} \right) \\ \det \left(\left(\frac{\partial^2}{\partial \theta_i \partial \theta_j} \langle {}^t A \cdot A_{\underline{\theta}}^{(4)} h, w \rangle_H \right)_{i, j=1, \dots, 4} \right) d\underline{\theta} = 1 \\ \text{on } \{ w ; \|w - {}^t A \cdot A_{\underline{\theta}_0}^{(4)} h\|_2 < \gamma \} .$$

Especially let A be $A_{\underline{\theta}_0}^{(4)} \cdot A_{\underline{\theta}'}^{(4)}$, $\underline{\theta}' \in [0, \pi/2] \times [0, 2\pi)^3$. Then

${}^t A \cdot A_{\underline{\theta}_0}^{(4)} h = A_{\underline{\theta}'}^{(4)} h$. Therefore Lemma 5.1.C is extended in the form

(5.8) for all elements of $K_{\min}^{0,x}$.

Now we define Φ , Φ_i , etc. in the same way as in Case A or in Case B, and it is enough to treat

$$J_2^{(3)} := E[\delta_X(X_1^\varepsilon) \Phi(\varepsilon w)] .$$

By (5.8), the definition of $\Phi_i(w)$ and the transformation $w \rightarrow A_{\underline{\theta}_i}^{(4)} \cdot {}^t A_{\underline{\theta}_0}^{(4)} w$, we have

$$\begin{aligned} & \int_{|\underline{\theta} - \underline{\theta}_0| < \eta} \delta_0 \left(\left(\frac{\partial}{\partial \theta_i} \langle A_{\underline{\theta}}^{(4)} h, w \rangle_H \right)_{i=1, \dots, 4} \right) \\ & \det \left\{ \left(\frac{\partial^2}{\partial \theta_i \partial \theta_j} \langle A_{\underline{\theta}}^{(4)} h, w \rangle_H \right)_{i,j=1, \dots, 4} \right\} d\underline{\theta} \cdot \Phi_i(A_{\underline{\theta}_i}^{(4)} \cdot {}^t A_{\underline{\theta}_0}^{(4)} w) \\ & = \Phi_i(A_{\underline{\theta}_i}^{(4)} \cdot {}^t A_{\underline{\theta}_0}^{(4)} w) . \end{aligned}$$

So

$$\begin{aligned} J_2^{(3)} &= \sum_{i=1}^n E[\delta_X(X_1^\varepsilon) \Phi_i(\varepsilon w)] \\ &= \sum_{i=1}^n \int_{|\underline{\theta} - \underline{\theta}_0| < \eta} E[\delta_0(\varepsilon w_1) \delta_U(\varepsilon^2 S(1, w)) \\ & \quad \times \delta_0 \left(\left(\frac{\partial}{\partial \theta_i} \langle A_{\underline{\theta}}^{(4)} h, \varepsilon w \rangle_H \right)_{i=1, \dots, 4} \right) \\ & \quad \times \det \left\{ \left(\frac{\partial^2}{\partial \theta_i \partial \theta_j} \langle A_{\underline{\theta}}^{(4)} h, \varepsilon w \rangle_H \right)_{i,j=1, \dots, 4} \right\} \\ & \quad \times \Phi_i(\varepsilon A_{\underline{\theta}_i}^{(4)} \cdot {}^t A_{\underline{\theta}_0}^{(4)} w)] d\underline{\theta} \\ &= \sum_{i=1}^n \int_{|\underline{\theta} - \underline{\theta}_0| < \eta} \exp(-\|A_{\underline{\theta}}^{(4)} h\|_H^2 / 2\varepsilon^2) E[\exp(-\langle A_{\underline{\theta}}^{(4)} h, w \rangle_H / \varepsilon) \\ & \quad \times \delta_0(\varepsilon w_1) \delta_0(\varepsilon(A_{\underline{\theta}}^{(4)} h \otimes \dot{w} - \dot{w} \otimes A_{\underline{\theta}}^{(4)} h) + \frac{\varepsilon^2}{2}(w \otimes \dot{w} - \dot{w} \otimes w)) \\ & \quad \times \delta_0 \left(\left(\frac{\partial}{\partial \theta_i} \langle A_{\underline{\theta}}^{(4)} h, h' \rangle_H \right)_{i=1, \dots, 4} \right) \\ & \quad \times \det \left\{ \left(\frac{\partial^2}{\partial \theta_i \partial \theta_j} \langle A_{\underline{\theta}}^{(4)} h, h' \rangle_H \right)_{i,j=1, \dots, 4} \right\} \\ & \quad \times \Phi_i(A_{\underline{\theta}_i}^{(4)} \cdot {}^t A_{\underline{\theta}_0}^{(4)} \cdot A_{\underline{\theta}}^{(4)} h + \varepsilon A_{\underline{\theta}_i}^{(4)} \cdot {}^t A_{\underline{\theta}_0}^{(4)} w)] d\underline{\theta} . \end{aligned}$$

where the last equality is obtained by a C-M transformation $w \rightarrow$

$w + A_{\underline{\theta}}^{(4)} h / \varepsilon$. Noting that $A_{\underline{\theta}}^{(4)} \in O(4)$ and Remark 5.2, this is

equal to

$$\begin{aligned}
& \sum_{i=1}^n \int_{|\underline{\theta}-\underline{\theta}_0|<\eta} \exp(-\|h\|_H^2/2\varepsilon^2) E[\exp(-\langle h, {}^t A_{\underline{\theta}}^{(4)} w \rangle_H / \varepsilon) \\
& \quad \times \delta_0(\varepsilon A_{\underline{\theta}}^{(4)} \cdot {}^t A_{\underline{\theta}}^{(4)} w_1) \\
& \quad \times \delta_0(A_{\underline{\theta}}^{(4)} \{ \varepsilon(h \otimes {}^t A_{\underline{\theta}}^{(4)} \dot{w} - {}^t A_{\underline{\theta}}^{(4)} \dot{w} \otimes h) \\
& \quad \quad + \frac{\varepsilon^2}{2} ({}^t A_{\underline{\theta}}^{(4)} w \otimes {}^t A_{\underline{\theta}}^{(4)} \dot{w} - {}^t A_{\underline{\theta}}^{(4)} \dot{w} \otimes {}^t A_{\underline{\theta}}^{(4)} w) \} {}^t A_{\underline{\theta}}^{(4)}) \\
& \quad \times \delta_0 \left(\left(\frac{\partial}{\partial \theta_i} \langle A_{\underline{\theta}}^{(4)} h, h' \rangle_H \right) \Big|_{h' = A_{\underline{\theta}}^{(4)} h + \varepsilon A_{\underline{\theta}}^{(4)} \cdot {}^t A_{\underline{\theta}}^{(4)} w} \right)_{i=1, \dots, 4} \\
& \times \det \left(\left(\frac{\partial^2}{\partial \theta_i \partial \theta_j} \langle A_{\underline{\theta}}^{(4)} h, h' \rangle_H \right) \Big|_{h' = A_{\underline{\theta}}^{(4)} h + \varepsilon A_{\underline{\theta}}^{(4)} \cdot {}^t A_{\underline{\theta}}^{(4)} w} \right)_{i,j=1, \dots, 4} \\
& \quad \times \Phi_i(A_{\underline{\theta}_i}^{(4)} \cdot {}^t A_{\underline{\theta}_0}^{(4)} \cdot A_{\underline{\theta}}^{(4)} h + \varepsilon A_{\underline{\theta}_i}^{(4)} \cdot {}^t A_{\underline{\theta}_0}^{(4)} \cdot A_{\underline{\theta}}^{(4)} \cdot {}^t A_{\underline{\theta}}^{(4)} w)] d\underline{\theta}
\end{aligned}$$

By the invariance of Wiener measure under an orthogonal

transformation, $\frac{\partial}{\partial \theta_i} \langle A_{\underline{\theta}}^{(4)} h, h' \rangle_H \Big|_{h' = A_{\underline{\theta}}^{(4)} h} = 0$ and $T(A_{\underline{\theta}}^{(4)}) \in O(10)$,

we finally see, noting that $-\langle h, w \rangle_H / \varepsilon = 2\pi S^{12}(1, w) + 4\pi S^{34}(1, w)$

under the condition $(h \otimes \dot{w} - \dot{w} \otimes h) + \frac{\varepsilon}{2}(w \otimes \dot{w} - \dot{w} \otimes w) = 0$ and $w_1 = 0$,

$$\begin{aligned}
J_2^{(3)} &= \exp(-\|h\|_H^2/2\varepsilon^2) \sum_{i=1}^n \int_{|\underline{\theta}-\underline{\theta}_0|<\eta} E[\exp\{2\pi S^{12}(1, w) + 4\pi S^{34}(1, w)\} \\
& \quad \times \delta_0(\varepsilon w_1) \delta_0(\varepsilon(h \otimes \dot{w} - w \otimes h) + \frac{\varepsilon^2}{2}(w \otimes \dot{w} - \dot{w} \otimes w)) \\
& \quad \times \delta_0 \left(\left(\varepsilon \cdot \frac{\partial}{\partial \theta_i} \langle A_{\underline{\theta}}^{(4)} h, h' \rangle_H \right) \Big|_{h' = A_{\underline{\theta}}^{(4)} w} \right)_{i=1, \dots, 4} \\
& \times \det \left(\left(\frac{\partial^2}{\partial \theta_i \partial \theta_j} \langle A_{\underline{\theta}}^{(4)} h, h' \rangle_H \right) \Big|_{h' = A_{\underline{\theta}}^{(4)} h + \varepsilon A_{\underline{\theta}}^{(4)} \cdot {}^t A_{\underline{\theta}}^{(4)} w} \right)_{i,j=1, \dots, 4} \\
& \quad \times \Phi_i(A_{\underline{\theta}_i}^{(4)} \cdot {}^t A_{\underline{\theta}_0}^{(4)} \cdot A_{\underline{\theta}}^{(4)} h + \varepsilon A_{\underline{\theta}_i}^{(4)} \cdot {}^t A_{\underline{\theta}_0}^{(4)} \cdot A_{\underline{\theta}}^{(4)} \cdot {}^t A_{\underline{\theta}}^{(4)} w)] d\underline{\theta}
\end{aligned}$$

Define R^{14} -valued Wiener functional $g_{0,\theta}^{(3)}(w)$ by

$$\begin{aligned}
(5.9) \quad g_{0,\theta}^{(3)}(w) &= (w_1, (h \otimes \dot{w} - \dot{w} \otimes h)_{ij}, 1 \leq i < j \leq 4, \\
& \quad \left(\frac{\partial}{\partial \theta_i} \langle A_{\underline{\theta}}^{(4)} h, h' \rangle_H \right) \Big|_{h' = A_{\underline{\theta}}^{(4)} w} \Big)_{i=1, \dots, 4}
\end{aligned}$$

Then, by Lemma 5.4 and Lemma 5.5, given below, we have

$$\begin{aligned}
(5.10) \quad J_2^{(3)} &\sim \exp(-\|h\|_H^2/2\varepsilon^2) \varepsilon^{-14} \sum_{i=1}^n \int_{|\underline{\theta}-\underline{\theta}_0|<\eta} \\
& \quad E[\exp\{2\pi S^{12}(1, w) + 4\pi S^{34}(1, w)\} \delta_0(g_{0,\theta}^{(3)}(w))]
\end{aligned}$$

$$\begin{aligned} & \times \det \left\{ \left(\frac{\partial^2}{\partial \theta_i \partial \theta_j} \langle A_{\underline{\theta}}^{(4)} h, h' \rangle_H \middle| h' = A_{\underline{\theta}}^{(4)} h \right) \right\}_{i,j=1, \dots, 4} \\ & \times \Phi_i(A_{\underline{\theta}_i}^{(4)} \cdot {}^t A_{\underline{\theta}_0}^{(4)} \cdot A_{\underline{\theta}}^{(4)} h) d\underline{\theta} \end{aligned}$$

Lemma 5.3.C.

$$\begin{aligned} & E[\exp\{2\pi S^{12}(1, w) + 4\pi S^{34}(1, w)\} \delta_0(g_{0, \underline{\theta}}^{(3)}(w))] \\ & = \frac{1}{2^{10} \cdot 3 \cdot u^5 \cdot \pi^6 \sin \theta_1 \cdot \cos \theta_1} . \end{aligned}$$

Proof.

Let $p_{\underline{\theta}}^{\theta}(x)$ be the density of the law of $g_{0, \underline{\theta}}^{(3)}(w)$, then

$$\begin{aligned} & E[\exp\{2\pi S^{12}(1, w) + 4\pi S^{34}(1, w)\} \delta_0(g_{0, \underline{\theta}}^{(3)}(w))] \\ & = E[\exp\{2\pi S^{12}(1, w) + 4\pi S^{34}(1, w)\} | g_{0, \underline{\theta}}^{(3)}(w) = 0] \cdot p_{\underline{\theta}}^{\theta}(0) , \end{aligned}$$

and it is easy to see that $p_{\underline{\theta}}^{\theta}(0) = \frac{1}{2^{10} \cdot 3 \cdot u^5 \cdot \pi^6 \cdot \sin \theta_1 \cdot \cos \theta_1}$. Let

$\Xi_{ij}^{(3)} = (h \otimes \dot{w} - \dot{w} \otimes h)_{ij}$, $1 \leq i < j \leq 4$, and $\Xi_k^{(3)} = \frac{\partial}{\partial \theta_k} \langle A_{\underline{\theta}}^{(4)} h, h' \rangle_H \middle| h' = A_{\underline{\theta}}^{(4)} w$, $k = 1, \dots, 4$. Then using $\xi_j^{(i)}$, $\eta_l^{(k)}$ in (5.5), we have

$$\begin{aligned} \Xi_{12}^{(3)} &= \sqrt{u/2\pi} \xi_1^{(2)} + \sqrt{u/2\pi} (\eta_1^{(1)} - \sqrt{2} \cdot \eta_0^{(1)}) , \\ \Xi_{13}^{(3)} &= \sqrt{u/2\pi} \xi_1^{(3)} - \sqrt{u/4\pi} \xi_2^{(1)} , \\ \Xi_{14}^{(3)} &= \sqrt{u/2\pi} \xi_1^{(4)} + \sqrt{u/4\pi} (\eta_2^{(1)} - \sqrt{2} \cdot \eta_0^{(1)}) , \\ \Xi_{23}^{(3)} &= -\sqrt{u/2\pi} (\eta_1^{(3)} - \sqrt{2} \cdot \eta_0^{(3)}) - \sqrt{u/4\pi} \xi_2^{(2)} , \\ \Xi_{24}^{(3)} &= -\sqrt{u/2\pi} (\eta_1^{(4)} - \sqrt{2} \cdot \eta_0^{(4)}) + \sqrt{u/4\pi} (\eta_2^{(2)} - \sqrt{2} \cdot \eta_0^{(2)}) , \\ \Xi_{34}^{(3)} &= \sqrt{u/4\pi} \xi_2^{(4)} + \sqrt{u/4\pi} (\eta_2^{(3)} - \sqrt{2} \cdot \eta_0^{(3)}) , \\ \Xi_1^{(3)} &= -\sqrt{4\pi u} \cos(\theta_2 - \theta_3) (\eta_2^{(1)} + \xi_2^{(2)}) \\ &\quad - \sqrt{4\pi u} \sin(\theta_2 - \theta_3) (\xi_2^{(1)} - \eta_2^{(2)}) \\ &\quad + \sqrt{2\pi u} \cos(\theta_2 - \theta_3) (\eta_1^{(3)} + \xi_1^{(4)}) \\ &\quad - \sqrt{2\pi u} \sin(\theta_2 - \theta_3) (\xi_1^{(3)} - \eta_1^{(4)}) , \\ \Xi_2^{(3)} &= -\sqrt{2\pi u} \cos^2 \theta_1 (\xi_1^{(1)} - \eta_1^{(2)}) \\ &\quad - \sqrt{4\pi u} \sin \theta_1 \cos \theta_1 \sin(\theta_2 - \theta_3) (\eta_2^{(1)} + \xi_2^{(2)}) \\ &\quad + \sqrt{4\pi u} \sin \theta_1 \cos \theta_1 \cos(\theta_2 - \theta_3) (\xi_2^{(1)} - \eta_2^{(2)}) \\ &\quad + \sqrt{2\pi u} \sin \theta_1 \cos \theta_1 \sin(\theta_2 - \theta_3) (\eta_1^{(3)} + \xi_1^{(4)}) \end{aligned}$$

$$\begin{aligned}
& + \sqrt{2\pi u} \sin \theta_1 \cos \theta_1 \cos(\theta_2 - \theta_3) (\xi_1^{(3)} - \eta_1^{(4)}) \\
& + \sqrt{4\pi u} \cos^2 \theta_1 (\xi_2^{(3)} - \eta_2^{(4)}) , \\
\Xi_3^{(3)} = & - \sqrt{4\pi u} (\xi_2^{(3)} - \eta_2^{(4)})
\end{aligned}$$

and

$$\begin{aligned}
\Xi_4^{(3)} = & - \sqrt{2\pi u} \sin^2 \theta_1 (\xi_1^{(1)} - \eta_1^{(2)}) \\
& + \sqrt{4\pi u} \sin \theta_1 \cos \theta_1 \sin(\theta_2 - \theta_3) (\eta_2^{(1)} + \xi_2^{(2)}) \\
& - \sqrt{4\pi u} \sin \theta_1 \cos \theta_1 \cos(\theta_2 - \theta_3) (\xi_2^{(1)} - \eta_2^{(2)}) \\
& - \sqrt{2\pi u} \sin \theta_1 \cos \theta_1 \sin(\theta_2 - \theta_3) (\eta_1^{(3)} + \xi_1^{(4)}) \\
& - \sqrt{2\pi u} \sin \theta_1 \cos \theta_1 \cos(\theta_2 - \theta_3) (\xi_1^{(3)} - \eta_1^{(4)}) \\
& - \sqrt{4\pi u} \cos^2 \theta_1 (\xi_2^{(3)} - \eta_2^{(4)}) .
\end{aligned}$$

Now set $\tilde{\Xi}_4^{(3)} = \Xi_2^{(3)} + \Xi_4^{(3)}$ and $\tilde{\Xi}_2^{(3)} = \Xi_2^{(3)} + \cos^2 \theta_1 (\Xi_3^{(3)} - \tilde{\Xi}_4^{(3)})$,
i.e.

$$\tilde{\Xi}_4^{(3)} = - \sqrt{2\pi u} (\xi_1^{(1)} - \eta_1^{(2)})$$

and

$$\begin{aligned}
\tilde{\Xi}_2^{(3)} = & - \sqrt{4\pi u} \sin \theta_1 \cos \theta_1 \sin(\theta_2 - \theta_3) (\eta_2^{(1)} + \xi_2^{(2)}) \\
& + \sqrt{4\pi u} \sin \theta_1 \cos \theta_1 \cos(\theta_2 - \theta_3) (\xi_2^{(1)} - \eta_2^{(2)}) \\
& + \sqrt{2\pi u} \sin \theta_1 \cos \theta_1 \sin(\theta_2 - \theta_3) (\eta_1^{(3)} + \xi_1^{(4)}) \\
& + \sqrt{2\pi u} \sin \theta_1 \cos \theta_1 \cos(\theta_2 - \theta_3) (\xi_1^{(3)} - \eta_1^{(4)}) ,
\end{aligned}$$

and let $\tilde{\Xi}_{ij}^{(3)}$, $1 \leq i < j \leq 4$, be random variables obtained by
excluding the terms $\eta_0^{(k)}$, $k = 1, \dots, 4$, from $\Xi_{ij}^{(3)}$. Then

$$\begin{aligned}
& E[\exp\{2\pi S^{12}(1, w) + 4\pi S^{34}(1, w)\} | g_{0, \theta}^{(3)}(w) = 0] \\
= & E[\exp\left(\sum_{k=1}^2 \sum_{m=1}^{\infty} \frac{k}{m} (\xi_m^{(2k)} (\eta_m^{(2k-1)} - \sqrt{2} \cdot \eta_0^{(2k-1)}) \right. \\
& \quad \left. - \xi_m^{(2k-1)} (\eta_m^{(2k)} - \sqrt{2} \cdot \eta_0^{(2k)})\right) | g_{0, \theta}^{(3)}(w) = 0] \\
= & E[\exp\left(\sum_{k=1}^2 \sum_{m=1}^{\infty} \frac{k}{m} (-\xi_m^{(2k-1)} \eta_m^{(2k)} + \xi_m^{(2k)} \eta_m^{(2k-1)})\right) | \\
& \quad \tilde{\Xi}_{ij}^{(3)} = 0, 1 \leq i < j \leq 4, \Xi_1^{(3)} = \tilde{\Xi}_2^{(3)} = \Xi_3^{(3)} = \tilde{\Xi}_4^{(3)} = 0] \\
= & E[\exp\{(-\xi_1^{(1)} \eta_1^{(2)} + \xi_1^{(2)} \eta_1^{(1)}) + (-\xi_2^{(3)} \eta_2^{(4)} + \xi_2^{(4)} \eta_2^{(3)})\} | \\
& \quad \tilde{\Xi}_4^{(3)} = \tilde{\Xi}_{12}^{(3)} = \tilde{\Xi}_{34}^{(3)} = \Xi_3^{(3)} = 0]
\end{aligned}$$

$$\begin{aligned}
& \times E[\exp\{\frac{1}{2}(\xi_2^{(2)}\eta_2^{(1)} - \xi_2^{(1)}\eta_2^{(2)}) + 2(\xi_1^{(4)}\eta_1^{(3)} - \xi_1^{(3)}\eta_1^{(4)})\} \\
& \quad | \tilde{\Xi}_1^{(3)} = \tilde{\Xi}_1^{(4)} = \tilde{\Xi}_2^{(3)} = \tilde{\Xi}_2^{(4)} = \Xi_1^{(3)} = \tilde{\Xi}_2^{(3)} = 0] \\
& \times \prod_{k=1}^2 \prod_{m=3}^{\infty} E[\exp\{\frac{k}{m}(\xi_m^{(2k)}\eta_m^{(2k-1)} - \xi_m^{(2k-1)}\eta_m^{(2k)})\}] \\
& = I_1 \times I_2 \times I_3 .
\end{aligned}$$

Here the second equality is obtained by that $\eta_0^{(i)} = 0$, $i = 1, \dots, 4$, and that $\Xi_2^{(3)} = \Xi_3^{(3)} = \Xi_4^{(3)} = 0$ if and only if $\tilde{\Xi}_2^{(3)} = \Xi_3^{(3)} = \tilde{\Xi}_4^{(3)} = 0$, and it is easy to see that $I_3 = \prod_{k=1}^2 \prod_{m=3}^{\infty} \left(1 - \frac{k^2}{m^2}\right)^{-1} = 9$ and that

$$\begin{aligned}
I_1 &= E[\exp(-\xi_1^{(1)}\eta_1^{(2)}) | \tilde{\Xi}_4^{(3)} = 0] \times E[\exp(\xi_1^{(2)}\eta_1^{(1)}) | \tilde{\Xi}_1^{(3)} = 0] \\
&\quad \times E[\exp(-\xi_2^{(3)}\eta_2^{(4)}) | \Xi_3^{(3)} = 0] \times E[\exp(\xi_2^{(4)}\eta_2^{(3)}) | \tilde{\Xi}_3^{(4)} = 0] \\
&= \left((1/\sqrt{2\pi}) \int_{-\infty}^{\infty} e^{-x^2} dx \right)^4 = \frac{1}{4} .
\end{aligned}$$

Define $X_i^{(3)}$, $i = 1, \dots, 4$, by

$$\begin{aligned}
X_1^{(3)} &= -\sqrt{2} \cdot \eta_1^{(3)} + 2 \cdot \xi_2^{(2)} , \\
X_2^{(3)} &= \sqrt{2} \cdot \xi_1^{(4)} - 2 \cdot \eta_2^{(1)} , \\
X_3^{(3)} &= \sqrt{2} \cdot \xi_1^{(3)} + 2 \cdot \xi_2^{(1)}
\end{aligned}$$

and

$$X_4^{(3)} = -\sqrt{2} \cdot \eta_1^{(4)} - 2 \cdot \eta_2^{(2)} .$$

Then

$$\begin{aligned}
(5.11) \quad & \exp\{\frac{1}{2}(\xi_2^{(2)}\eta_2^{(1)} - \xi_2^{(1)}\eta_2^{(2)}) + 2(\xi_1^{(4)}\eta_1^{(3)} - \xi_1^{(3)}\eta_1^{(4)})\} \\
& = \exp\{-\frac{1}{6}(X_1^{(3)}X_2^{(3)} - X_3^{(3)}X_4^{(3)}) + P_3(\Xi)\}
\end{aligned}$$

where $P_3(\Xi)$ is a polynomial of degree 2 in 4 variables $\Xi = (\tilde{\Xi}_1^{(3)}, \tilde{\Xi}_1^{(4)}, \tilde{\Xi}_2^{(3)}, \tilde{\Xi}_2^{(4)})$ whose constant term is 0. This equality is obtained in the same way as in Case B. Noting that

$$\Xi_1^{(3)} = \sqrt{\pi u} \{ (X_2^{(3)} - X_1^{(3)}) \cos(\theta_2 - \theta_3) - (X_3^{(3)} + X_4^{(3)}) \sin(\theta_2 - \theta_3) \}$$

and that

$$\tilde{\Xi}_2^{(3)} = \sqrt{\pi u} \sin \theta_1 \cos \theta_1$$

$$\times \{ (X_2^{(3)} - X_1^{(3)}) \sin(\theta_2 - \theta_3) + (X_3^{(3)} + X_4^{(3)}) \cos(\theta_2 - \theta_3) \},$$

we have $\Xi_1^{(3)} = \tilde{\Xi}_2^{(3)} = 0$ if and only if $X_1^{(3)} - X_2^{(3)} = 0$ and

$$X_3^{(3)} + X_4^{(3)} = 0. \quad \text{Thus}$$

$$\begin{aligned} I_2 &= E[\exp\{-\frac{1}{6}(X_1^{(3)}X_2^{(3)} - X_3^{(3)}X_4^{(3)})\} | \\ &\quad X_1^{(3)} - X_2^{(3)} = 0, X_3^{(3)} + X_4^{(3)} = 0] \\ &= E[\exp(-\frac{1}{6} X_1^{(3)}X_2^{(3)}) | X_1^{(3)} - X_2^{(3)} = 0] \\ &\quad \times E[\exp(\frac{1}{6} X_3^{(3)}X_4^{(3)}) | X_3^{(3)} + X_4^{(3)} = 0] \\ &= \frac{1}{2}, \end{aligned}$$

for $X_1^{(3)} + X_2^{(3)}, X_3^{(3)} - X_4^{(3)} \sim N(0, 12)$.

Combined I_1, I_2 and I_3 with $p_3^\theta(0)$, the proof is completed. //

It is easy to compute that

$$\begin{aligned} &\det \left(\left(\frac{\partial^2}{\partial \theta_i \partial \theta_j} \langle A_{\underline{\theta}}^{(4)} h, h \rangle_H \right) \Big|_{h=A_{\underline{\theta}}^{(4)} h} \right)_{i,j=1,\dots,4} \\ &= 3^2 \cdot 2^9 \cdot \pi^4 \cdot u^4 \cdot \cos^2 \theta_1 \sin^2 \theta_1 \end{aligned}$$

and that $\|h\|_H^2 = 12\pi u$, so we have

$$\begin{aligned} J_2^{(3)} &\sim \exp(-6\pi u/\varepsilon^2) \varepsilon^{-14} \frac{3}{2u\pi^2} \sum_{i=1}^n \int_{|\underline{\theta} - \underline{\theta}_0| < \eta} \\ &\quad \Phi_i(A_{\underline{\theta}_i}^{(4)} \cdot {}^t A_{\underline{\theta}_0}^{(4)} \cdot A_{\underline{\theta}}^{(4)} h) \cdot \sin \theta_1 \cos \theta_1 d\underline{\theta} \quad \text{as } \varepsilon \downarrow 0. \end{aligned}$$

Proposition 5.2.

Define a metric g on $K_{\min}^{0,x}$ by

$$g = \sum g_{ij} d\theta^i d\theta^j$$

where

$$g_{ij} = \langle \frac{\partial}{\partial \theta_i} A_{\underline{\theta}}^{(4)} h, \frac{\partial}{\partial \theta_j} A_{\underline{\theta}}^{(4)} h \rangle_H$$

If we introduce another metric g' on $K_{\min}^{0,x}$ by

$$g' = \sum g'_{ij} d\theta'^i d\theta'^j$$

where

$$g'_{ij} = \langle \frac{\partial}{\partial \theta'^i} A_{\underline{\theta}'}^{(4)} h, \frac{\partial}{\partial \theta'^j} A_{\underline{\theta}'}^{(4)} h \rangle_H$$

and, for some $\underline{\alpha} \in [0, \pi/2] \times [0, 2\pi)^3$, $A_{\underline{\theta}'}^{(4)} = A_{\underline{\alpha}}^{(4)} \cdot A_{\underline{\theta}}^{(4)}$, then $g = g'$.

Proof.

$$\begin{aligned} g'_{ij} &= \langle \sum_k \frac{\partial}{\partial \theta'_k} \frac{\partial \theta_k}{\partial \theta'_i} A_{\underline{\alpha}}^{(4)} \cdot A_{\underline{\theta}}^{(4)} h, \sum_l \frac{\partial}{\partial \theta'_l} \frac{\partial \theta_l}{\partial \theta'_j} A_{\underline{\alpha}}^{(4)} \cdot A_{\underline{\theta}}^{(4)} h \rangle_H \\ &= \sum_{k,l} \frac{\partial \theta_k}{\partial \theta'_i} \frac{\partial \theta_l}{\partial \theta'_j} \langle \frac{\partial}{\partial \theta_k} A_{\underline{\alpha}}^{(4)} \cdot A_{\underline{\theta}}^{(4)} h, \frac{\partial}{\partial \theta_l} A_{\underline{\alpha}}^{(4)} \cdot A_{\underline{\theta}}^{(4)} h \rangle_H \\ &= \sum_{k,l} \frac{\partial \theta_k}{\partial \theta'_i} \frac{\partial \theta_l}{\partial \theta'_j} \langle \frac{\partial}{\partial \theta_k} A_{\underline{\theta}}^{(4)} h, \frac{\partial}{\partial \theta_l} A_{\underline{\theta}}^{(4)} h \rangle_H \\ &= \sum_{k,l} \frac{\partial \theta_k}{\partial \theta'_i} \frac{\partial \theta_l}{\partial \theta'_j} g_{kl} . \end{aligned}$$

So it is easy to see that $g = g'$. //

Since $\|A_{\underline{\theta}}^{(4)} h\|_H^2$ is independent of $\underline{\theta}$, it is clear that

$$\langle \frac{\partial}{\partial \theta'_i} A_{\underline{\theta}'}^{(4)} h, \frac{\partial}{\partial \theta'_j} A_{\underline{\theta}'}^{(4)} h \rangle_H = - \frac{\partial^2}{\partial \theta'_i \partial \theta'_j} \langle A_{\underline{\theta}'}^{(4)} h, h' \rangle_H \Big|_{h' = A_{\underline{\theta}'}^{(4)} h} .$$

Thus

$$\begin{aligned} & \sum_{i=1}^n \int_{|\underline{\theta} - \underline{\theta}_0| < \eta} \Phi_i(A_{\underline{\theta}_i}^{(4)} \cdot {}^t A_{\underline{\theta}_0}^{(4)} \cdot A_{\underline{\theta}}^{(4)} h) \sin \theta_1 \cos \theta_1 d\underline{\theta} \\ &= \sum_{i=1}^n \int_{|\underline{\theta} - \underline{\theta}_0| < \eta} (I_{U_i} \cdot \Phi_i)(A_{\underline{\theta}_i}^{(4)} \cdot {}^t A_{\underline{\theta}_0}^{(4)} \cdot A_{\underline{\theta}}^{(4)} h) \sin \theta_1 \cos \theta_1 d\underline{\theta} \\ &= \sum_{i=1}^n \int_{\underline{\theta} \in [0, \pi/2] \times [0, 2\pi)^3} (I_{U_i} \cdot \Phi_i)(A_{\underline{\theta}_i}^{(4)} \cdot {}^t A_{\underline{\theta}_0}^{(4)} \cdot A_{\underline{\theta}}^{(4)} h) \\ & \quad \sin \theta_1 \cos \theta_1 d\underline{\theta} \\ &= \sum_{i=1}^n \int_{\underline{\theta} \in [0, \pi/2] \times [0, 2\pi)^3} (I_{U_i} \cdot \Phi_i)(A_{\underline{\theta}_i}^{(4)} h) \sin \theta_1 \cos \theta_1 d\underline{\theta} \\ &= \int_{\underline{\theta} \in [0, \pi/2] \times [0, 2\pi)^3} \sum_{i=1}^n \Phi_i(A_{\underline{\theta}_i}^{(4)} h) \sin \theta_1 \cos \theta_1 d\underline{\theta} \end{aligned}$$

$$= 4\pi^3 .$$

Here the third equality is due to Proposition 5.2. Therefore

$$J_2^{(3)} \sim \exp(-6\pi u/\varepsilon^2) \varepsilon^{-14} \frac{6\pi}{u} \quad \text{as } \varepsilon \downarrow 0$$

In conclusion, we have

Theorem 5.1.C.

In Case C, i.e., $x = [0, U]$, $U \sim u(\delta_{12} - \delta_{21} + \delta_{34} - \delta_{43})$, $u > 0$,

$$p(\varepsilon^2, 0, x) \sim \exp(-6\pi u/\varepsilon^2) \varepsilon^{-14} \frac{6\pi}{u} \quad \text{as } \varepsilon \downarrow 0$$

We finish this section by proving two lemmas quoted above which assured the asymptotics (5.4), (5.7) and (5.10).

Let $\chi_i : \mathbb{R}^{n(i)} \rightarrow \mathbb{R}$, $i = 1, 2, 3$, be C^∞ -functions satisfying $\text{Supp } \chi_i \subset \{ |x| \leq 1 \}$ where $n(1) = 11$, $n(2) = 12$ and $n(3) = 14$.

Lemma 5.4.

A) We can choose $\eta > 0$ such that for all $i = 1, \dots, n$,

$$\begin{aligned} & \exp\{2\pi S^{34}(1, w)\} \chi_1(g_{\varepsilon}^{(1)}(w)) \Phi_i(A_{\theta}^{(1)}(\tilde{h} + \varepsilon w)) \\ &= \exp\{2\pi S^{34}(1, w)\} \chi_1(g_0^{(1)}(w)) \Phi_i(A_{\theta}^{(1)}\tilde{h}) + O(\varepsilon) \\ & \quad \text{as } \varepsilon \downarrow 0 \text{ in } \tilde{D}^\infty \text{ if } |\theta - \tilde{\theta}_i| < \eta . \end{aligned}$$

Furthermore $O(\varepsilon)$ is uniform on $\{ \theta ; |\theta - \tilde{\theta}_i| < \eta \}$. Here Φ_i , $\tilde{\theta}_i$, $i = 1, \dots, n$, are as in the statement after Lemma 5.1.A.

B) We can choose $\eta > 0$ such that for all $i = 1, \dots, n$,

$$\begin{aligned} & \exp\{2\pi S^{12}(1, w) + 4\pi S^{34}(1, w)\} \chi_2(g_{\varepsilon}^{(2)}(w)) \Phi_i(A_{\underline{\theta}}^{(2)}(h + \varepsilon w)) \\ &= \exp\{2\pi S^{12}(1, w) + 4\pi S^{34}(1, w)\} \chi_2(g_0^{(2)}(w)) \Phi_i(A_{\underline{\theta}}^{(2)}h) + O(\varepsilon) \\ & \quad \text{as } \varepsilon \downarrow 0 \text{ in } \tilde{D}^\infty \text{ if } |\underline{\theta} - \tilde{\underline{\theta}}_i| < \eta . \end{aligned}$$

Furthermore $O(\varepsilon)$ is uniform on $\{ \underline{\theta} ; |\underline{\theta} - \tilde{\underline{\theta}}_i| < \eta \}$.

C) For all $\underline{\theta}_0 \in (0, \pi/2) \times [0, 2\pi)^3$ we can choose $\eta > 0$ such that

for all $i = 1, \dots, n$,

$$\begin{aligned}
 (5.12) \quad & \exp\{2\pi S^{12}(1, w) + 4\pi S^{34}(1, w)\} \chi_3(g_{\varepsilon, \underline{\theta}}^{(3)}(w)) \\
 & \cdot \Phi_i(A_{\underline{\theta}_i}^{(4)} {}^t A_{\underline{\theta}_0}^{(4)} \cdot A_{\underline{\theta}}^{(4)}(h + \varepsilon w)) \\
 & = \exp\{2\pi S^{12}(1, w) + 4\pi S^{34}(1, w)\} \chi_3(g_{0, \underline{\theta}}^{(3)}(w)) \\
 & \cdot \Phi_i(A_{\underline{\theta}_i}^{(4)} {}^t A_{\underline{\theta}_0}^{(4)} \cdot A_{\underline{\theta}}^{(4)} h) + O(\varepsilon) \\
 & \text{as } \varepsilon \downarrow 0 \text{ in } \tilde{D}^\infty \text{ if } |\underline{\theta} - \underline{\theta}_i| < \eta.
 \end{aligned}$$

Furthermore $O(\varepsilon)$ is uniform on $(\underline{\theta}; |\underline{\theta} - \underline{\theta}_i| < \eta)$

Here $g_0^{(1)}(w)$, $g_0^{(2)}(w)$ and $g_{0, \underline{\theta}}^{(3)}(w)$ are as in (5.3), (5.6) and (5.9), respectively, and we define $g_\varepsilon^{(1)}(w)$, $g_\varepsilon^{(2)}(w)$ and $g_{\varepsilon, \underline{\theta}}^{(3)}(w)$ by

$$\begin{aligned}
 g_\varepsilon^{(1)}(w) &= (w_1, S^{12}(1, w), \\
 & ((\dot{h} \otimes \dot{w} - \dot{w} \otimes \dot{h}) + \varepsilon S(1, w))_{ij}, 1 \leq i < j \leq 4, \quad \langle A_{\pi/2}^{(1)} \dot{h}, w \rangle_H), \\
 & \quad (i, j) \neq (1, 2) \\
 g_\varepsilon^{(2)}(w) &= (w_1, ((h \otimes \dot{w} - \dot{w} \otimes h) + \varepsilon S(1, w))_{ij}, 1 \leq i < j \leq 4, \\
 & \quad \langle A_{(\pi/2, 0)}^{(2)} h, w \rangle_H, \langle A_{(0, \pi/2)}^{(2)} h, w \rangle_H)
 \end{aligned}$$

and

$$\begin{aligned}
 g_{\varepsilon, \underline{\theta}}^{(3)}(w) &= (w_1, ((h \otimes \dot{w} - \dot{w} \otimes h) + \varepsilon S(1, w))_{ij}, 1 \leq i < j \leq 4, \\
 & \quad \left(\frac{\partial}{\partial \theta_i} \langle A_{\underline{\theta}}^{(4)} h, h' \rangle_H \Big|_{h' = A_{\underline{\theta}}^{(4)} w} \right)_{i=1, \dots, 4}).
 \end{aligned}$$

Proof.

We prove only C), the others being similarly proved. We use the same notations as in Lemma 5.3.C.

It is enough to prove that we can choose $\eta > 0$ such that

$$\begin{aligned}
 (5.13) \quad & \sup_{0 < \varepsilon \leq 1, |\underline{\theta} - \underline{\theta}_i| < \eta} \|\exp\{2\pi S^{12}(1, w) + 4\pi S^{34}(1, w)\} \chi_3(g_{\varepsilon, \underline{\theta}}^{(3)}(w)) \\
 & \cdot \Phi_i(A_{\underline{\theta}_i}^{(4)} {}^t A_{\underline{\theta}_0}^{(4)} A_{\underline{\theta}}^{(4)}(h + \varepsilon w))\|_{L^p(P)} < \infty
 \end{aligned}$$

for some $p > 1$. This is because the estimate (5.12) is true for almost all w and (5.13) guarantees the uniformly integrability,

thus (5.12) is valid in the sense of L^p for some $p > 1$: The L^p -estimate of its higher order H -derivatives can be obtained in the same way.

Using $\xi_j^{(i)}, \eta_j^{(i)}$ in (5.5) the integrand of (5.12) is expressed by

$$\begin{aligned}
& \exp\{-\xi_1^{(1)}(\eta_1^{(2)} - \sqrt{2} \cdot \eta_0^{(2)}) + \xi_1^{(2)}(\eta_1^{(1)} - \sqrt{2} \cdot \eta_0^{(1)}) \\
& \quad - \xi_2^{(3)}(\eta_2^{(4)} - \sqrt{2} \cdot \eta_0^{(4)}) + \xi_2^{(4)}(\eta_2^{(3)} - \sqrt{2} \cdot \eta_0^{(3)})\} \\
& \quad \cdot \chi_3(g_{\varepsilon, \theta}^{(3)}(w)) \cdot \Phi_i(A_{\underline{\theta}_i}^{(4)} \cdot {}^t A_{\underline{\theta}_0}^{(4)} \cdot A_{\underline{\theta}}^{(4)}(h + \varepsilon w)) \\
& \times \exp\left(\frac{1}{2}\{-\xi_2^{(1)}(\eta_2^{(2)} - \sqrt{2} \cdot \eta_0^{(2)}) + \xi_2^{(2)}(\eta_2^{(1)} - \sqrt{2} \cdot \eta_0^{(1)})\} \right. \\
& \quad \left. + 2\{-\xi_1^{(3)}(\eta_1^{(4)} - \sqrt{2} \cdot \eta_0^{(4)}) + \xi_1^{(4)}(\eta_1^{(3)} - \sqrt{2} \cdot \eta_0^{(3)})\}\right) \\
& \quad \cdot \chi_3(g_{\varepsilon, \theta}^{(3)}(w)) \cdot \Phi_i(A_{\underline{\theta}_i}^{(4)} \cdot {}^t A_{\underline{\theta}_0}^{(4)} \cdot A_{\underline{\theta}}^{(4)}(h + \varepsilon w)) \\
& \times \exp\left(\sum_{k=1}^2 \sum_{m=3}^{\infty} \frac{k}{m} \{\xi_m^{(2k)}(\eta_m^{(2k-1)} - \sqrt{2} \cdot \eta_0^{(2k-1)}) \right. \\
& \quad \left. - \xi_m^{(2k-1)}(\eta_m^{(2k)} - \sqrt{2} \cdot \eta_0^{(2k)})\}\right) \\
& \quad \cdot \chi_3(g_{\varepsilon, \theta}^{(3)}(w)) \\
& = I_1 \times I_2 \times I_3 .
\end{aligned}$$

It is easy to see that $\sup_{\varepsilon, \theta} E[I_3^p] < \infty$, $1 < p < 3/2$, thus all we must do is to verify

$$(5.14) \quad \sup_{\varepsilon, \theta} E[(I_1 \times I_2)^q] < \infty, \quad \text{for some } q > 3 .$$

If $\Phi_i(A_{\underline{\theta}_i}^{(4)} \cdot {}^t A_{\underline{\theta}_0}^{(4)} \cdot A_{\underline{\theta}}^{(4)}(h + \varepsilon w)) > 0$, then we have

$$\|A_{\underline{\theta}_i}^{(4)} \cdot {}^t A_{\underline{\theta}_0}^{(4)} \cdot A_{\underline{\theta}}^{(4)}(h + \varepsilon w) - A_{\underline{\theta}_i}^{(4)} h\|_2 < \gamma$$

where $\gamma = \gamma(\eta)$ is as in Lemma 5.1.C. Hence

$$\varepsilon^2 \int_0^1 |w_t|^2 dt - 2 \int_0^1 |(A_{\underline{\theta}_i}^{(4)} \cdot {}^t A_{\underline{\theta}_0}^{(4)} (A_{\underline{\theta}}^{(4)} - A_{\underline{\theta}_0}^{(4)}) h_t|^2 dt < 2\gamma^2 ,$$

i.e.

$$\varepsilon^2 \int_0^1 |w_t|^2 dt - 2 \int_0^1 |(A_{\underline{\theta}}^{(4)} - A_{\underline{\theta}_0}^{(4)}) h_t|^2 dt < 2\gamma^2 .$$

For all $\eta > 0$, there exist $\gamma' = \gamma'(\eta)$ such that $\|A_{\underline{\theta}}^{(4)} - A_{\underline{\theta}_0}^{(4)}\|_{\text{op}} < \gamma'$ if $|\underline{\theta} - \underline{\theta}_0| < \eta$ and $\gamma' \downarrow 0$ as $\eta \downarrow 0$. So there exists a constant $K > 0$ satisfying

$$(5.15) \quad \varepsilon^2 \int_0^1 |w_t|^2 dt < 2\gamma^2 + 2K\gamma'^2$$

for all $\varepsilon \in (0, 1]$. On the other hand, $\chi_3(g_{\varepsilon, \underline{\theta}}^{(3)}(w)) > 0$ implies that

$$\begin{aligned} |w_1| &< \delta, \\ |\Xi_{ij}^{(3)} + \varepsilon S^{ij}(1, w)| &< \delta \end{aligned}$$

and

$$\left| \frac{\partial}{\partial \theta_i} \langle A_{\underline{\theta}}^{(4)} h, h' \rangle_H \Big|_{h' = A_{\underline{\theta}}^{(4)} w} \right| < \delta$$

for some $\delta > 0$. Clearly, for any constant $c_1 \in \mathbb{R}$,

$$\begin{aligned} &\exp\left(c_1 \left\{ \xi_m^{(i)} (\eta_m^{(j)} - \sqrt{2} \cdot \eta_0^{(j)}) \right\}\right) \chi_3(g_{\varepsilon, \underline{\theta}}^{(3)}(w)) \\ &= \exp(c_1 \xi_m^{(i)} \eta_m^{(j)}) \exp(-\sqrt{2} c_1 \xi_m^{(i)} \eta_0^{(j)}) \chi_3(g_{\varepsilon, \underline{\theta}}^{(3)}(w)) \\ &\leq \exp(c_1 \xi_m^{(i)} \eta_m^{(j)}) \exp(|c_1| \cdot \sqrt{2} \cdot \delta \cdot \xi_m^{(i)}) \end{aligned}$$

and $\exp(|c_1| \cdot \sqrt{2} \cdot \delta \cdot \xi_m^{(j)}) \in L^q$ for all $q > 0$. Therefore we can assume $\eta_0^{(i)} = 0$ in (5.13).

First we treat with the term I_1 . Clearly

$$\begin{aligned} &\exp(-\xi_1^{(1)} \eta_1^{(2)}) \chi_3(g_{\varepsilon, \underline{\theta}}^{(3)}(w)) \\ &= \exp\left(\frac{1}{2} \left\{ \left((\xi_1^{(1)} - \eta_1^{(2)}) / \sqrt{2} \right)^2 - \left((\xi_1^{(1)} + \eta_1^{(2)}) / \sqrt{2} \right)^2 \right\}\right) \chi_3(g_{\varepsilon, \underline{\theta}}^{(3)}(w)) \\ &= \exp\left\{-\frac{1}{2} \left((\xi_1^{(1)} + \eta_1^{(2)}) / \sqrt{2} \right)^2\right\} \exp\{(\tilde{\Xi}_4^{(3)})^2 / 8\pi u\} \chi_3(g_{\varepsilon, \underline{\theta}}^{(3)}(w)) \\ &\leq \exp(\delta^2 / 2\pi u) \exp\left\{-\frac{1}{2} \left((\xi_1^{(1)} + \eta_1^{(2)}) / \sqrt{2} \right)^2\right\} \in L^q \text{ for all } q > 0. \end{aligned}$$

Similarly we can prove $\exp(-\xi_2^{(3)} \eta_2^{(4)}) \chi_3(g_{\varepsilon, \underline{\theta}}^{(3)}) \in L^q$ for all $q > 0$.

Next

$$\exp(\xi_1^{(2)} \eta_1^{(1)}) = \exp\left\{-\frac{1}{2} \left((\xi_1^{(2)} - \eta_1^{(1)}) / \sqrt{2} \right)^2\right\} \exp\left\{\frac{\pi}{u} (\tilde{\Xi}_1^{(3)})^2\right\}$$

and

$$\begin{aligned} \exp\left\{\frac{\pi}{u}(\tilde{\Xi}_1^{(3)})^2\right\} \Phi_i(A_{\underline{\theta}_i}^{(4)} \cdot {}^t A_{\underline{\theta}_0}^{(4)} \cdot A_{\underline{\theta}}^{(4)}(h+\varepsilon w)) \\ \leq \exp\left(\frac{\pi}{u}((1+\sqrt{u/\pi})\delta + |\varepsilon S^{12}(1,w)|^2)\right). \end{aligned}$$

It is easy to show that there exist a Brownian motion $B(t)$ on W_0^1 such that

$$\varepsilon S^{12}(1,w) = B\left(\varepsilon^2 \int_0^1 \{(w_t^1)^2 + (w_t^2)^2\} dt\right).$$

Appealing to (5.15), for each $q > 1$ we can choose η such that

$$\sup_{\varepsilon, \theta} E[\exp\{q \xi_1^{(2)} \eta_1^{(1)}\} \Phi_i(A_{\underline{\theta}_i}^{(4)} \cdot {}^t A_{\underline{\theta}_0}^{(4)} \cdot A_{\underline{\theta}}^{(4)}(h+\varepsilon w))] < \infty.$$

Similarly, for each $q > 1$ we can choose η such that

$$\sup_{\varepsilon, \theta} E[\exp\{q \xi_2^{(4)} \eta_2^{(3)}\} \Phi_i(A_{\underline{\theta}_i}^{(4)} \cdot {}^t A_{\underline{\theta}_0}^{(4)} \cdot A_{\underline{\theta}}^{(4)}(h+\varepsilon w))] < \infty$$

Therefore it is easy to check that

$$\sup_{\varepsilon, \theta} E[I_1^q] < \infty, \quad q > 1$$

As for the term I_2 it is enough by (5.11) to treat with the terms $\exp\{-\frac{1}{6}(X_1 X_2 - X_3 X_4)\}$ and $\exp\{P_3(\Xi)\}$. Clearly

$$\begin{aligned} |P_3(\Xi)| &\leq \sum c_2 |\tilde{\Xi}_{ij}^{(3)}| + \sum c_3 |\tilde{\Xi}_{ij}^{(3)}| |\tilde{\Xi}_{kl}^{(3)}| \\ &\leq \sum c_2 |\tilde{\Xi}_{ij}^{(3)}| + \sum c_3 \frac{1}{2}(|\tilde{\Xi}_{ij}^{(3)}|^2 + |\tilde{\Xi}_{kl}^{(3)}|^2). \end{aligned}$$

Noting that $|\tilde{\Xi}_{ij}^{(3)}| \leq \delta + |\varepsilon S^{ij}(1,w)|$ on $\chi_3 > 0$, we can control the term $\exp\{P_3(\Xi)\}$ in the same way as in I_1 . Furthermore, noting that $|\Xi_1^{(3)}| < \delta$ and $|\tilde{\Xi}_2^{(3)}| < \delta$ if $\chi_3 > 0$, we can easily show that $|X_1 - X_2|^2 + |X_3 + X_4|^2 < 2\delta^2$, the term $\exp\{-\frac{1}{6}(X_1 X_2 - X_3 X_4)\}$ is also controlled in the same way as in I_1 . Therefore

$$\sup_{\varepsilon, \theta} E[I_2^q] < \infty \quad \text{for all } q > 1$$

and this completes the proof. //

Lemma 5.5.

All of $g_{\varepsilon}^{(1)}$, $g_{\varepsilon}^{(2)}$ and $g_{\varepsilon,\theta}^{(3)}$ are uniformly non-degenerate.

Remark 5.5.

The above lemma ensures the asymptotic expansions of $\delta_0(g_{\varepsilon}^{(1)})$, $\delta_0(g_{\varepsilon}^{(2)})$ and $\delta_0(g_{\varepsilon,\theta}^{(3)})$, thus, combined with Lemma 5.4, we can justify the asymptotics (5.4), (5.7) and (5.10) and furthermore the asymptotic expansions of $J_2^{(1)}$, $J_2^{(2)}$ and $J_2^{(3)}$. Hence, we can conclude that $p(t,0,x)$ has the expansion of the form (0.1), the main term of which is given by Theorem 5.1.A, B and C respectively.

Proof of Lemma 5.5.

Here we treat only $g_{\varepsilon}^{(1)}$ since the others can be proved in a similar way.

Let $g_{\varepsilon,t}^{(1)}(w)$ be the R^{11} -valued Wiener process given by

$$g_{\varepsilon,t}^{(1)}(w) = \left(w_t, S^{12}(t,w), \left(\int_0^t (\tilde{h}_t^i dw_t^j - \tilde{h}_t^j dw_t^i) + \varepsilon S^{ij}(t,w) \right)_{1 \leq i < j \leq 4}, \sum_i \int_0^t (A_{\pi/2}^{(1)} \tilde{h})_t^i dw_t^i \right)_{(i,j) \neq (1,2)},$$

Clearly $g_{\varepsilon,1}^{(1)}(w) = g_{\varepsilon}^{(1)}(w)$. Then $g_{\varepsilon,t}^{(1)}(w)$ satisfies the following S.D.E. :

$$dg_{\varepsilon,t}^{(1)}(w) = \sum_{\alpha=1}^4 L_{\alpha}(\varepsilon, t, g_{\varepsilon,t}^{(1)}(w)) \circ dw_t^{\alpha}$$

where $L_{\alpha}(\varepsilon, t, \xi)$, $\alpha = 1, \dots, 4$, $\xi = (\xi_1, \dots, \xi_{11}) = (x, x^1) \in R^{11}$, are given by

$$L_1(\varepsilon, t, \xi) = \frac{\partial}{\partial x_1} - \frac{1}{2} \left(x_2 \cdot \frac{\partial}{\partial x_{(12)}} + (\varepsilon x_3 + 2\tilde{h}_t^3) \cdot \frac{\partial}{\partial x_{(13)}} + (\varepsilon x_4 + 2\tilde{h}_t^4) \cdot \frac{\partial}{\partial x_{(14)}} \right),$$

$$L_2(\varepsilon, t, \xi) = \frac{\partial}{\partial x_2} + \frac{1}{2} \left(x_1 \cdot \frac{\partial}{\partial x_{(12)}} - (\varepsilon x_3 + 2\tilde{h}_t^3) \cdot \frac{\partial}{\partial x_{(23)}} - (\varepsilon x_4 + 2\tilde{h}_t^4) \cdot \frac{\partial}{\partial x_{(24)}} \right),$$

$$L_3(\varepsilon, t, \xi) = \frac{\partial}{\partial x_3} + \frac{1}{2} \left(\varepsilon x_1 \cdot \frac{\partial}{\partial x_{(13)}} + \varepsilon x_2 \cdot \frac{\partial}{\partial x_{(23)}} - (\varepsilon x_4 + 2\hbar_t^4) \cdot \frac{\partial}{\partial x_{(34)}} \right) - \sqrt{4\pi u} \sin 2\pi t \frac{\partial}{\partial x^1}$$

and

$$L_4(\varepsilon, t, \xi) = \frac{\partial}{\partial x_4} + \frac{1}{2} \left(\varepsilon x_1 \cdot \frac{\partial}{\partial x_{(14)}} + \varepsilon x_2 \cdot \frac{\partial}{\partial x_{(24)}} + (\varepsilon x_3 + 2\hbar_t^3) \cdot \frac{\partial}{\partial x_{(34)}} \right) + \sqrt{4\pi u} \cos 2\pi t \frac{\partial}{\partial x^1}$$

Let Y_t^ε be the 11×11 matrix given by

$$dY_t^\varepsilon = \partial L_\alpha(\varepsilon, t, g_{\varepsilon, t}^{(1)}) Y_t^\varepsilon \circ dw_t^\alpha$$

where $\partial L_\alpha(\varepsilon, t, \xi)$ is the 11×11 matrix given by $(\partial L_\alpha(\varepsilon, t, \xi))_{ij} = \frac{\partial}{\partial \xi_j} L_\alpha^i(\varepsilon, t, \xi)$. Then we have

$$\langle Dg_{\varepsilon}^{(1)}(w), Dg_{\varepsilon}^{(1)}(w) \rangle_H = Y_1^\varepsilon \sum_{\alpha=1}^4 \int_0^1 (Y_t^\varepsilon)^{-1} L_\alpha(\varepsilon, t, g_{\varepsilon, t}^{(1)}(w)) \otimes (Y_t^\varepsilon)^{-1} L_\alpha(\varepsilon, t, g_{\varepsilon, t}^{(1)}(w)) dt {}^t Y_1^\varepsilon.$$

By a slight computation, we know that $\det Y_1^\varepsilon = 1$. Therefore we will only evaluate the integral part which will be denoted by $\sigma(\varepsilon, w)$.

Let $l = {}^t(l_i, l_{jk}, l^1)_{\substack{i=1, \dots, 4 \\ 1 \leq j < k \leq 4}} \in \mathbb{R}^{11}$. Then we can easily

compute that

$$\begin{aligned} & {}^t l \sigma(\varepsilon, w) l \\ &= \int_0^1 \left(\{ l_1 - l_{12} \cdot w_t^2 - l_{13}(\varepsilon w_t^3 + \sqrt{u/\pi} \sin 2\pi t) \right. \\ & \quad \left. - l_{14}(\varepsilon w_t^4 + \sqrt{u/\pi}(1 - \cos 2\pi t)) \}^2 \right. \\ & \quad + \{ l_2 + l_{12} \cdot w_t^1 - l_{23}(\varepsilon w_t^3 + \sqrt{u/\pi} \sin 2\pi t) \\ & \quad \left. - l_{24}(\varepsilon w_t^4 + \sqrt{u/\pi}(1 - \cos 2\pi t)) \}^2 \right. \\ & \quad + \{ l_3 + l_{13} \cdot w_t^1 + l_{23} \cdot w_t^2 \\ & \quad \left. - l_{34}(\varepsilon w_t^4 + \sqrt{u/\pi}(1 - \cos 2\pi t)) - l^1 \cdot \sqrt{4\pi u} \sin 2\pi t \}^2 \right. \\ & \quad \left. + \{ l_4 + l_{14} \cdot w_t^1 + l_{24} \cdot w_t^2 \right. \end{aligned}$$

$$+ l_{34}(\varepsilon w_t^3 + \sqrt{u/\pi} \sin 2\pi t)) + l^1 \cdot \sqrt{4\pi u} \cos 2\pi t \}^2 \Big) dt \quad .$$

Now we will prove that for any T large enough,

$$(5.16) \quad P\left(\inf_{|l|=1} t_l \sigma(\varepsilon, w) \mid l \mid < \frac{1}{T} \right) \leq c_1 e^{-c_2 T^{c_3}}$$

for some positive constants c_1 , c_2 and c_3 all of which are independent of ε . We know easily that

$$P\left(\sup_{|l|=1} t_l \sigma(\varepsilon, w) \mid l \mid \geq T \right) \leq c_4 e^{-c_5 T}$$

for some positive constants c_4 and c_5 which are independent both of ε and l . Thus it is enough to estimate

$$P\left(t_l \sigma(\varepsilon, w) \mid l \mid < \frac{1}{T} \right)$$

uniformly in l (cf. S.Kusuoka-D.W.Stroock [12], Appendix).

Appealing to J.Norris [18] or N.Ikeda-S.Watanabe [10], however, it is easy to check that

$$P\left(t_l \sigma(\varepsilon, w) \mid l \mid < \frac{1}{T} \right) \leq c_6 e^{-c_7 T^{c_8}}$$

where c_6 , c_7 and c_8 are positive constants all of which are independent of l . Thus (5.16) is concluded.

Then it is easy to see that

$$\overline{\lim}_{\varepsilon \downarrow 0} E[|\det \langle g_{\varepsilon}^{(1)}(w), g_{\varepsilon}^{(1)}(w) \rangle_H|^{-p}] < \infty$$

for all $p > 0$, and this completes the proof. //

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