

最節約復元順序集合の極値問題について
- On extremal problems of MPR-posets -

東海大・理・情報数理 宮川 幹平 (Kampei Miyakawa)
東海大・理・情報数理 成嶋 弘 (Hiroshi Narushima)

We first review several definitions and theorems used latter. The set $\{1, 2, \dots, n\}$ of n elements is denoted by $[n]$. Let a_i ($i \in [2n]$) be any elements in Ω , and be sorted in ascending order as follows:

$$x_1 \leq x_2 \leq \dots \leq x_n \leq x_{n+1} \leq \dots \leq x_{2n}.$$

Then we call x_n and x_{n+1} the *median two points* of the numbers a_i ($i \in [2n]$), and denote $\langle x_n, x_{n+1} \rangle$ by

$$\text{med2}\langle a_1, a_2, \dots, a_{2n} \rangle \text{ or } \text{med2}\langle a_i : i \in [2n] \rangle.$$

We also call x_{n-1}, x_n, x_{n+1} and x_{n+2} the *median four points* of the numbers a_i ($i \in [2n]$), and denote $\langle x_{n-1}, x_n, x_{n+1}, x_{n+2} \rangle$ by

$$\text{med4}\langle a_1, a_2, \dots, a_{2n} \rangle \text{ or } \text{med4}\langle a_i : i \in [2n] \rangle.$$

Let $I_i = [a_i, b_i]$ ($i \in [m]$) be any family of closed intervals in Ω . Then we denote the median two points $\text{med2}\langle a_i : i \in [m], b_i : i \in [m] \rangle$ of all the endpoints a_i and b_i of I_i ($i \in [m]$) by

$$\text{med2}\langle I_1, I_2, \dots, I_m \rangle \text{ or } \text{med2}\langle I_i : i \in [m] \rangle.$$

We also denote the median four points $\text{med4}\langle a_i : i \in [m], b_i : i \in [m] \rangle$ of all the endpoints a_i and b_i of I_i ($i \in [m]$) by

$$\text{med4}\langle I_1, I_2, \dots, I_m \rangle \text{ or } \text{med4}\langle I_i : i \in [m] \rangle.$$

Let $\text{med2}\langle I_i : i \in [m] \rangle = \langle x_m, x_{m+1} \rangle$. Then we call the closed interval $[x_m, x_{m+1}]$ in Ω the *median interval* of the closed intervals I_i ($i \in [m]$), which is the key concept in a series of our papers, and denote it by

$$\text{med}\langle I_1, I_2, \dots, I_m \rangle \text{ or } \text{med}\langle I_i : i \in [m] \rangle.$$

The following is Lemma 1 in [3] (lemma B in [5]). It is very useful to investigate characteristics of each MPR.

Lemma A. *Let a and b_i ($i \in [2m]$) be any elements in Ω . Then*

$$\text{med2}\langle a, a, b_i : i \in [2m] \rangle = \text{med2}\langle a, a, \text{med4}\langle b_i : i \in [2m] \rangle \rangle. \quad \square$$

From Theorem 1 in [1], we see that $\text{med}([\lambda(p(u)), \lambda(p(u))], I(v) : u \rightarrow v)$ is the MPR-set of node u under the condition that an element $\lambda(p(u))$ in $S_{p(u)}$ has been assigned to u 's parent $p(u)$. This subset of the MPR-set S_u is denoted by $S_u | x$. That is,

$$S_u | x = \text{med}([x, x], I(v) : u \rightarrow v),$$

where x is an element in $S_{p(u)}$. The following is Theorem 1 in [3].

Theorem B. *Let T be a rooted el-tree (T_s, r) . Then each MPR-set S_u for each internal node u of T is recursively decided by*

$$S_u = [\min(S_u | \min(S_{p(u)})), \max(S_u | \max(S_{p(u)}))]. \quad \square$$

We now show a sufficient condition for a $\sigma(r)$ -version MPR-poset to have both the greatest element and the least element.

Proposition 1. *Let T be an el-tree, and r be any element in V_O . If*

$$\sigma(r) \leq \min\{ \min(S_u) | \forall u \in V_H, S_u \text{ is a non-singleton} \}$$

or

$$\sigma(r) \geq \max\{ \max(S_u) | \forall u \in V_H, S_u \text{ is a non-singleton} \},$$

then $(\text{Rmp}(T), \leq_{\sigma(r)})$ has both the greatest element and the least element. $\square$

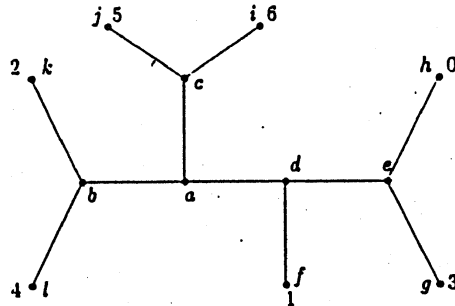


Figure 1: An el-tree T_1

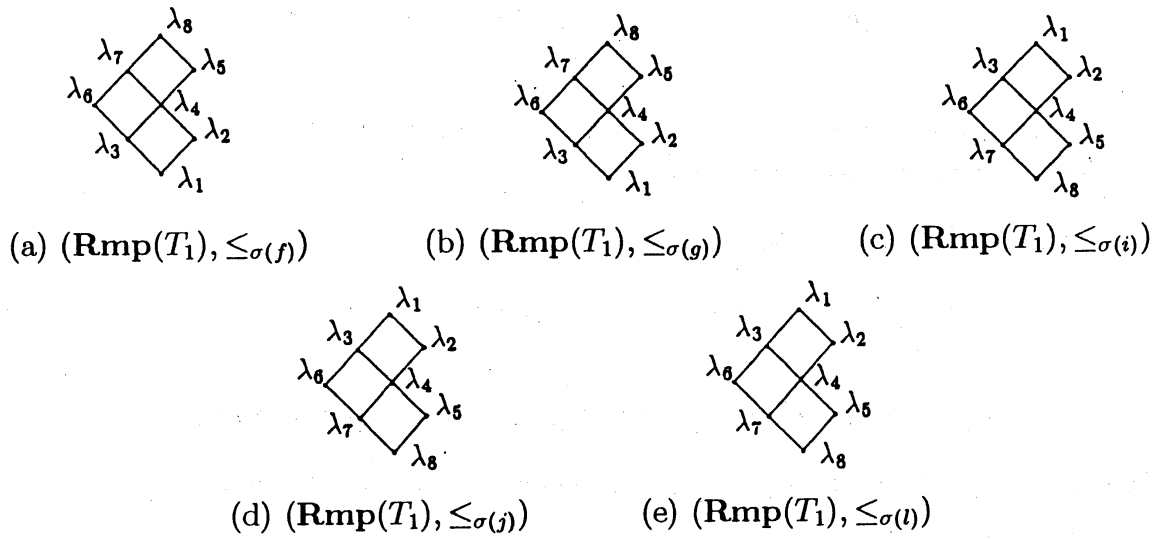
We here give some examples to illustrate proposition 1. Let T_1 be an el-tree shown in Fig 1. The set $\text{Rmp}(T_1)$ of MPRs is also given in Table 1. Then f, g, i, j and l in $V_O(T_1)$ satisfy the conditions in proposition 1. Therefore we see that $(\text{Rmp}(T_1), \leq_{\sigma(f)})$, $(\text{Rmp}(T_1), \leq_{\sigma(g)})$, $(\text{Rmp}(T_1), \leq_{\sigma(i)})$, $(\text{Rmp}(T_1), \leq_{\sigma(j)})$ and $(\text{Rmp}(T_1), \leq_{\sigma(l)})$ have both the greatest element and the least element (Fig 2 (a) ~ (e)).

We get easily the following remark from proposition 1.

Remark 1. *Let T be an el-tree rooted at r such that $\sigma(r) = \min(\sigma(V_O))$. Then $\sigma(r)$ -version MPR-poset, $(\text{Rmp}(T), \leq_{\sigma(r)})$ has both the greatest element and the least element.*

$\square$

λ^u	a	b	c	d	e	f	g	h	i	j	k	l
λ_1	2	2	5	1	1	1	3	0	6	5	2	4
λ_2	2	2	5	2	2	1	3	0	6	5	2	4
λ_3	3	3	5	1	1	1	3	0	6	5	2	4
λ_4	3	3	5	2	2	1	3	0	6	5	2	4
λ_5	3	3	5	3	3	1	3	0	6	5	2	4
λ_6	4	4	5	1	1	1	3	0	6	5	2	4
λ_7	4	4	5	2	2	1	3	0	6	5	2	4
λ_8	4	4	5	3	3	1	3	0	6	5	2	4

Table 1: $\mathbf{Rmp}(T_1)$ Figure 2: $\sigma(r)$ -version MPR-posets

We now have the main theorem in this paper, which answers for whether there exists the least element in a $\sigma(r)$ -version MPR-poset or not.

Let T be a rooted el-tree (T_s, r) . We define a reconstruction λ on T as follows. We define λ by $\lambda(u) = x$ in S_u satisfying $x \leq_{\sigma(r)} y$ for any y in S_u , that is, x is the least element of a subposet $(S_u, \leq_{\sigma(r)})$ in the poset $(\mathbf{Rmp}(T), \leq_{\sigma(r)})$. This reconstruction λ is particularly written as $\lambda_{\min}^{<\sigma(r)>}$.

We can get the following implicitly from proposition 1.

Remark 2. Let T be a rooted el-tree (T_s, r) . If

$$\sigma(r) \leq \min\{ \min(S_u) | \forall u \in V_H, S_u \text{ is a non-singleton} \},$$

then $\lambda_{\min}^{<\sigma(r)>} = \lambda_{\min}$. The dual case also holds. $\square$

Lemma 1. Let T be a rooted el-tree (T_s, r) . For each u in V_H , we have

$$\lambda_{\min}^{<\sigma(r)>}(u) = \begin{cases} \min(S_u) & (\sigma(r) \leq \min(S_u)) \\ \sigma(r) & (\min(S_u) < \sigma(r) < \max(S_u)) \\ \max(S_u) & (\sigma(r) \geq \max(S_u)) \end{cases} \quad \square$$

Theorem 1. Let T be a rooted el-tree (T_s, r) . Then the reconstruction $\lambda_{\min}^{<\sigma(r)>}$ is the least element of $(\mathbf{Rmp}(T), \leq_{\sigma(r)})$. $\square$

We here show some examples of the MPR $\lambda_{\min}^{<\sigma(r)>}$. Let $T_1 = (T_c, j)$ be the tree T_1 rooted at node j . Then for each u in V we can decide $\lambda_{\min}^{<\sigma(j)>}(u)$, which is shown in Fig 3. We also can see that $\lambda_{\min}^{<\sigma(j)>}$ is equal to λ_8 in $\mathbf{Rmp}(T_1)$, i.e, the least element of $(\mathbf{Rmp}(T_1), \leq_{\sigma(j)})$ (Fig 4).

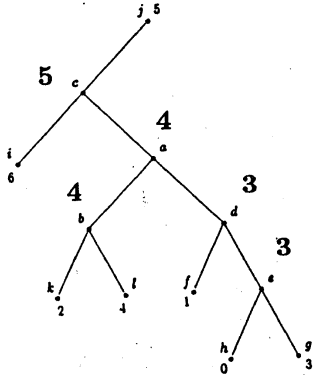


Figure 3: $\lambda_{\min}^{<\sigma(j)>}$ on $T_1 = (T_c, j)$

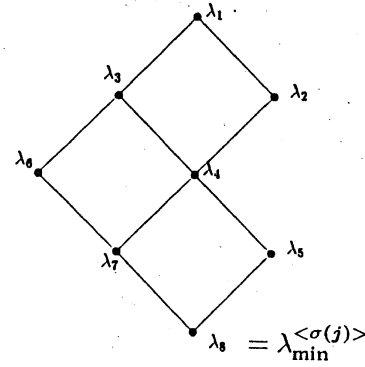


Figure 4: $(\mathbf{Rmp}(T_1), \leq_{\sigma(j)})$

It is known that $(\mathbf{Rmp}(T), \leq_{\sigma(r)})$ doesn't always have the greatest element. So, we show one of the requirements for a reconstruction λ in $\mathbf{Rmp}(T)$ to be a maximal element in the $\sigma(r)$ -version MPR-poset.

Lemma 2. Let T be a rooted el-tree (T_s, r) . For each u in V , we have $\max S_{p(u)} \leq \min I(u)$, $\max I(u) \leq \min S_{p(u)}$ or $S_{p(u)} \subseteq I(u)$ hold. $\square$

Let T be a rooted el-tree (T_s, r) . We define two reconstructions $\alpha^{<\sigma(r)>}, \beta^{<\sigma(r)>}$ on T as follows. We define $\alpha^{<\sigma(r)>}$ and $\beta^{<\sigma(r)>}$ by $\alpha^{<\sigma(r)>}(u) =$ the smallest element x under the usual ordering $\leq$ of maximal elements in the subposet $(S_u, \leq_{\sigma(r)})$ and $\beta^{<\sigma(r)>}(u) =$ the greatest element x under the usual ordering $\leq$ of maximal elements in the subposet $(S_u, \leq_{\sigma(r)})$.

Lemma 3. Let T be a rooted el-tree (T_s, r) . For each u in V_H , we have

$$\alpha^{<\sigma(r)>}(u) = \begin{cases} \min(S_u) & (\sigma(r) > \min(S_u)) \\ \max(S_u) & (\sigma(r) \leq \min(S_u)) \end{cases}$$

$$\beta^{<\sigma(r)>}(u) = \begin{cases} \min(S_u) & (\sigma(r) \geq \max(S_u)) \\ \max(S_u) & (\sigma(r) < \max(S_u)) \end{cases} \quad \square$$

Then, we get the following proposition.

Proposition 2. *Let T be a rooted el-tree (T_s, r) . Then, both $\alpha^{<\sigma(r)>}$ and $\beta^{<\sigma(r)>}$ are maximal elements of $(\mathbf{Rmp}(T), \leq_{\sigma(r)})$. $\square$*

We here show some examples of the MPR $\alpha^{<\sigma(r)>}$ and $\beta^{<\sigma(r)>}$. Let $T_1 = (T_b, k)$ be the tree T_1 rooted at node k . Then for each u in V we can decide $\alpha^{<\sigma(k)>}(u)$ and $\beta^{<\sigma(k)>}(u)$, which are shown in Fig 5(a) and (b) respectively. We also see that $\alpha^{<\sigma(k)>}$ and $\beta^{<\sigma(k)>}$ are equal to λ_6 and λ_8 in $\mathbf{Rmp}(T_1)$, respectively, which are maximal elements of $(\mathbf{Rmp}(T_1), \leq_{\sigma(k)})$ (Fig 6).

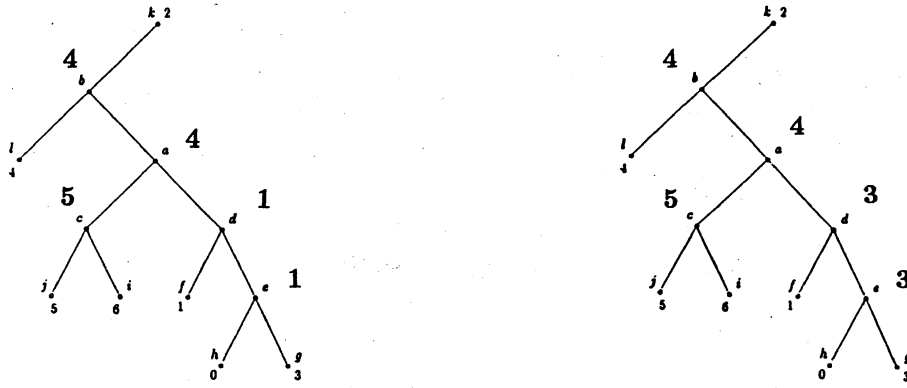


Figure 5(a): $\alpha^{<\sigma(k)>}$ on $T_1 = (T_b, k)$

(b): $\beta^{<\sigma(k)>}$ on $T_1 = (T_b, k)$

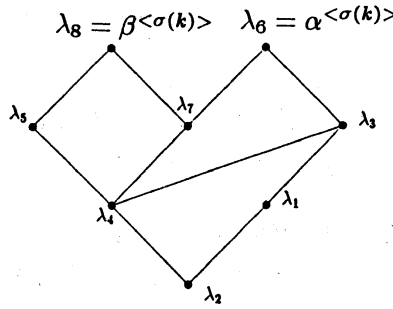
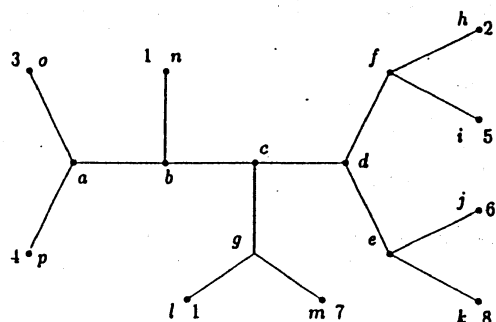
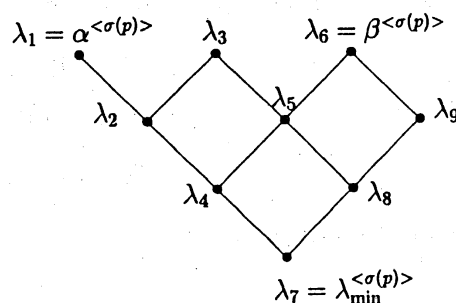


Figure 6: $(\mathbf{Rmp}(T_1), \leq_{\sigma(k)})$

Finally, we show interesting examples on the number of maximal elements of a $\sigma(r)$ -version MPR-poset. Let T_2 be an el-tree shown in Fig 7. When T_2 is rooted at p in $V_O(T_2)$, we see that $(\mathbf{Rmp}(T_2), \leq_{\sigma(p)})$ has three maximal elements λ_1, λ_3 and λ_6 shown in Fig 8. In other words, it shows that the number of maximal elements of a $\sigma(r)$ -version MPR-poset is not necessarily at most two.

Figure 7: An el-tree T_2 Figure 8: $(\mathbf{Rmp}(T_2), \leq_{\sigma(p)})$

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