

L-functions of holomorphic cusp forms on $U(2, 1)$

Atsushi Murase and Takashi Sugano

村瀬 篤 (京都産業大学・理)

菅野 孝史 (広島大学・理)

0. Introduction

In this note, we report several results on the standard L-functions of holomorphic cusp forms on the unitary group H of hermitian forms of signature $(2, 1)$. The L-functions we investigate here are associated to the 6-dimensional representation of the L-group ${}^L H$ of H that is induced from the standard representation of ${}^L H^0 = GL_3(\mathbb{C})$. Such L-functions has been studied by several mathematicians; for example, see Shintani [9], Gelbart and Piatetski-Shapiro[3], Kudla [5], Gelbart and Rogawski [4].

In §1 and §2, we recall basic facts about holomorphic cusp forms on the unitary groups. In §3, we recall the definition of the Hecke algebra of H and introduce the local Euler factor at each rational prime. After defining the global L-function $L(F; s)$ for a Hecke eigenform F and its gamma factor in §4, we state one of the main results of the paper in §5: the holomorphy and functional equation of $L(F; s)$ (here we have to impose a certain technical assumption on F). In the final section, we give a partial result on the critical values of $L(F; s)$. The method of proof is based on a certain integral expression of the L-functions studied in our previous paper [6]. Details will appear elsewhere.

1. Unitary groups

Let $K = \mathbb{Q}(\sqrt{d_K})$ be an imaginary quadratic field of discriminant $d_K < 0$ and O_K the ring of integers in K . Denote by σ the non-trivial automorphism of K .

Let $\{1, \theta\}$ be a $\mathbb{Z}$ -basis of O_K with $\text{Im } \theta > 0$ and put $\kappa = \theta - \theta^\sigma (= \sqrt{d_K})$. Let $H =$

$U(T)$ be the unitary group of a skew hermitian matrix $T = \begin{bmatrix} & -1 \\ -\kappa & \\ 1 & \end{bmatrix}$: $H_Q = \{h \in$

$GL_3(K) \mid {}^t h^\sigma T h = T\}$. Note that the signature of a hermitian matrix κT is $(1, 2)$.

Let $N_Q = \{n(w, v) = \begin{bmatrix} 1 & \kappa w^\sigma v + \frac{\kappa}{2} w w^\sigma \\ & 1 & w \\ & & 1 \end{bmatrix} \mid w \in K, v \in Q\}$ and $M_Q = \left\{ \begin{bmatrix} t & & \\ & \mu & \\ & & (t^\sigma)^{-1} \end{bmatrix} \mid \right.$

$t \in K^\times, \mu \in K^1\}$, where $K^1 = \{\mu \in K^\times \mid \mu \mu^\sigma = 1\}$. Then $P = NM$ is a maximal parabolic subgroup of H .

Let $D = \{Z = \begin{bmatrix} z \\ w \end{bmatrix} \in \mathbb{C}^2 \mid \frac{1}{\kappa} (z - \bar{z}) - |w|^2 > 0\}$ be a symmetric domain and

put $Z^\sim = \begin{bmatrix} z \\ w \\ 1 \end{bmatrix} \in \mathbb{C}^3$ for $Z = \begin{bmatrix} z \\ w \end{bmatrix} \in D$. Then the action of $H_\infty = H(\mathbb{R})$ on D : (h, Z)

$\mapsto h \langle Z \rangle$ is given by $h \cdot Z^\sim = (h \langle Z \rangle)^\sim \cdot J_H(h, Z)$, where $J_H(h, Z) \in \mathbb{C}^\times$. Denote by

U_∞ the isotropy subgroup of $Z_\theta = \begin{bmatrix} \theta \\ 0 \end{bmatrix} \in D$ in H_∞ .

For a rational prime p , we write $K_p = K \otimes_Q \mathbb{Q}_p$ and $O_{K,p} = O_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Put $H_p = H(\mathbb{Q}_p)$, $U_p = H_p \cap GL_3(O_{K,p})$ and $U_p^* = \{h \in U_p \mid (h-1)T^{-1} \in M_3(O_{K,p})\}$. Then U_p^* is a normal subgroup of U_p and $[U_p : U_p^*]$ is equal to 1 if $p \nmid d_K$ and 2 if $p \mid d_K$. Note that the Iwasawa decomposition $H_p = N_p M_p U_p^*$ holds. We normalize the Haar measure dh on H_p by

$$\int_{H_p} f(h) dh = \int_{N_p} dn \int_{M_p} dm \int_{U_p^*} du^* f(nmu^*) \delta(m)^{-1},$$

where f is any integrable function on H_p , $\delta(m) = d(mnm^{-1})/dn$ and the Haar measures dn, dm and du^* on N_p, M_p and U_p^* are normalized so that $\text{vol}(N_p \cap U_p^*) = \text{vol}(M_p \cap U_p^*) = \text{vol}(U_p^*) = 1$. We normalize the Haar measure dh_∞ on H_∞ by

$$\int_{H_\infty} f(h_\infty) dh_\infty = \int_D \tilde{f}(Z) d\mu(Z),$$

where $f(h_\infty \langle Z_\theta \rangle) = \int_{U_\infty} f(h_\infty u_\infty) du_\infty$ for $h_\infty \in H_\infty$ and $d\mu(Z) = \left(\frac{1}{\kappa} (z - \bar{z}) - |w|^2\right)^{-3} d\operatorname{Re}(z) d\operatorname{Im}(z) d\operatorname{Re}(w) d\operatorname{Im}(w)$. The Haar measure dh on H_A is defined to be the product measures $\prod_{v \leq \infty} dh_v$. We set $U_{A_f}^* = \prod_{p < \infty} U_p^*$ and $U_A^* = U_\infty \cdot U_{A_f}^*$.

2. Automorphic forms

Let ℓ be a positive integer with $\ell \equiv 0 \pmod{|O_K^\times|}$. Let $S_\ell(U_A^*)$ be the space of holomorphic cusp forms on H of weight ℓ defined as follows:

$$S_\ell(U_A^*) = \{ F : H_Q \backslash H_A / U_{A_f}^* \rightarrow \mathbb{C} \mid$$

- (i) $F(hu_\infty) = F(h) \cdot J_H(u_\infty, Z_\theta)^{-\ell}$ for $u_\infty \in U_\infty$.
- (ii) The function $h_\infty \rightarrow F(h_\infty h_f) \cdot J_H(h_\infty, Z_\theta)^\ell$ gives rise to a holomorphic function of $h_\infty \langle Z_\theta \rangle \in D$ for any $h_f \in H_f$.
- (iii) F is bounded on H_A .

It is known that $F \in S_\ell(U_A^*)$ satisfies the cuspidal condition

$$\int_{N_Q \backslash N_A} F(nh) dn = 0$$

for any $h \in H_A$. The Petersson inner product of $S_\ell(U_A^*)$ is defined by

$$\langle F, F' \rangle = \int_{H_Q \backslash H_A} F(h) \overline{F'(h)} dh \quad F, F' \in S_\ell(U_A^*).$$

3. Hecke algebra

For a rational prime p , let H_p be the algebra of compactly supported bi U_p^* -invariant functions on H_p . The object of this section is to recall Satake's parametrization of $\operatorname{Hom}_{\mathbb{C}}(H_p, \mathbb{C})$ (cf. [7]).

First we consider the case where $\left(\frac{K/Q}{p}\right) \neq 1$ and hence K_p is a field. Put $K_p^1 = \{ \mu \in K_p^\times \mid \mu \mu^\sigma = 1 \}$ and $K_p^1(\kappa) = \{ \mu \in K_p^1 \mid \frac{\mu-1}{\kappa} \in O_{K,p} \}$. It is easy to see that

U_p/U_p^* is isomorphic to $K_p^1/K_p^1(\kappa)$, which is trivial if $\left(\frac{K/Q}{p}\right) = -1$ and the cyclic group of order 2 if $\left(\frac{K/Q}{p}\right) = 0$ (in this case, $K_p^1/K_p^1(\kappa)$ consists of 1 and π/π^σ where π is a prime element of K_p). Let χ_1 be an unramified character of $K_p^\times$ and χ_0 a character of $K_p^1/K_p^1(\kappa)$. For a pair $\chi = (\chi_0, \chi_1)$, we define a function ϕ_χ on H_p by

$$\phi_\chi\left(n \begin{bmatrix} t & & \\ & \mu & \\ & & (t^\sigma)^{-1} \end{bmatrix} u^*\right) = \chi_0(\mu) \chi_1(t) |t t^\sigma|_p$$

for $n \in N_p$, $t \in K_p^\times$, $\mu \in K_p^1$ and $u^* \in U_p^*$. Here $|\cdot|_p$ denotes the normalized valuation of $\mathbb{Q}_p^\times$.

We next consider the case $\left(\frac{K/Q}{p}\right) = 1$. Once and for all we fix an isomorphism of K_p onto $\mathbb{Q}_p \oplus \mathbb{Q}_p$. Then $H_p = \{h = (h_1, h_2) \in GL_3(\mathbb{Q}_p) \times GL_3(\mathbb{Q}_p) \mid {}^th_2 T_1 h_1 = T_1\}$, where T_1 is the first component of $T \in GL_3(K_p) = GL_3(\mathbb{Q}_p) \times GL_3(\mathbb{Q}_p)$. In what follows we identify H_p with $GL_3(\mathbb{Q}_p)$ via $h \rightarrow h_1$ and identify H_p with $H_p(GL_3(\mathbb{Q}_p), GL_3(\mathbb{Z}_p))$. For a triplet $\chi = (\chi_1, \chi_2, \chi_3)$ of unramified characters of $\mathbb{Q}_p^\times$, put

$$\phi_\chi\left(\begin{bmatrix} t_1 & * & * \\ & t_2 & * \\ & & t_3 \end{bmatrix} u\right) = \prod_{j=1}^3 |t_j|_p^{2-j} \chi_j(t_j)$$

for $t_j \in \mathbb{Q}_p^\times$ ($1 \leq j \leq 3$) and $u \in GL_3(\mathbb{Z}_p)$.

In both cases, we put

$$\chi^\wedge(\varphi) = \int_{H_p} \phi_\chi(h) \varphi(h^{-1}) dh \quad \varphi \in H_p.$$

Then $\varphi \rightarrow \chi^\wedge(\varphi)$ defines an algebra homomorphism of H_p to $\mathbb{C}$. Moreover every algebra homomorphism of H_p to $\mathbb{C}$ is of the form $\chi^\wedge$ for some χ .

Let $\Lambda \in \text{Hom}_{\mathbb{C}}(H_p, \mathbb{C})$ and ω be an unramified character of $K_p^\times$. Choose χ so that $\Lambda = \chi^\wedge$. Denote by ω' the first component of ω under the identification

$K_p^\times = Q_p^\times \times Q_p^\times$ if $\left(\frac{K/Q}{p}\right) = 1$. We define a local L-factor $L_p(\Lambda \otimes \omega; s)$ attached to Λ and ω to be

$$L_p(\Lambda \otimes \omega; s)^{-1} = \begin{cases} (1 - (\chi_1 \omega')(p)p^{-s}) (1 - (\chi_1 \omega')^{-1}(p)p^{-s}) (1 - (\chi_2 \omega')(p)p^{-s}) \\ \times (1 - (\chi_2 \omega')^{-1}(p)p^{-s}) (1 - (\chi_3 \omega')(p)p^{-s}) (1 - (\chi_3 \omega')^{-1}(p)p^{-s}) & \text{if } \left(\frac{K/Q}{p}\right) = 1 \\ (1 - (\chi_1 \omega)(p)p^{-2s}) (1 - (\chi_1 \omega)^{-1}(p)p^{-2s}) (1 - \omega(p)p^{-2s}) & \text{if } \left(\frac{K/Q}{p}\right) = -1 \\ (1 - (\chi_1 \omega)(\pi)p^{-s}) (1 - (\chi_1 \omega)^{-1}(\pi)p^{-s}) (1 - \omega(\pi)\chi_o(\pi/\pi^\sigma)p^{-s}) & \text{if } \left(\frac{K/Q}{p}\right) = 0. \end{cases}$$

4. Automorphic L-functions

Fix a Hecke character ω of $K^\times \backslash K_A^\times$ that is unramified everywhere (namely ω is trivial on $\prod_{p < \infty} O_{K,p}^\times$) and satisfies $\omega(x_\infty) = \left(\frac{x_\infty}{|x_\infty|}\right)^\ell$ for $x_\infty \in K_\infty^\times = \mathbb{C}^\times$. Let $F \in S_\ell(U_A^*)$ be a Hecke eigenform corresponding to $\Lambda_p \in \text{Hom}_{\mathbb{C}}(H_p, \mathbb{C})$ for each p . That is to say, we have $(F * \phi_p)(h) := \int_{H_p} F(hx) \phi_p(x^{-1}) dx = \Lambda_p(\phi_p) F$ for every p and every $\phi_p \in H_p$ (cf. 3). The global L-function attached to F and ω is defined by $L(F \otimes \omega; s) = \prod_{p < \infty} L_p(\Lambda_p \otimes \omega_p; s)$, where ω_p is the local component of ω at p . The gamma factor for $L(F \otimes \omega; s)$ is given by

$$L_\infty(F \otimes \omega; s) = (2\pi)^{-3s} |d_K|^{\frac{3}{2}s} \Gamma(s + \frac{\ell}{2}) \Gamma(s + \frac{\ell}{2} - 1)^2.$$

We put $\xi(F \otimes \omega; s) = L_\infty(F \otimes \omega; s) L(F \otimes \omega; s)$.

5. Functional equation

Let $G = U(S)$ be the unitary group of $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and embed G into H via

$\iota_0 \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} a & b \\ & 1 \\ c & d \end{bmatrix}$. Then, for $F \in S_\ell(U_A^*)$, the pullback $\iota_0^* F(g) = F(\iota_0(g))$ by ι_0 is

an automorphic form on G . Our first main result is as follows:

Theorem 1 *Let $F \in S_\ell(U_A^*)$ be a Hecke eigenform. Assume that $\ell > 4$ and that $\iota_0^* F$ is not identically equal to zero. Then $\xi(F \otimes \omega; s)$ can be continued to an entire function of s on $\mathbb{C}$ and satisfies the functional equation*

$$\xi(F \otimes \omega; s) = \xi(F \otimes \omega; 1-s).$$

6. Special values of automorphic L-functions

In this section, we assume that the class number of K is one. Note that the Hecke character ω is uniquely determined in this case. Let $\overline{\mathbb{Q}}$ be the algebraic closure of K in $\mathbb{C}$. For $F \in S_\ell(U_A^*)$, we put $F^{\text{dm}}(Z) = F(h_Z) J_H(h_Z, Z_\theta)^\ell$ where h_Z is any element of H_∞ such that $h_Z \langle Z_\theta \rangle = Z \in D$. Then F^{dm} is a holomorphic function on D that satisfies $F^{\text{dm}}(\gamma \langle Z \rangle) = J_H(\gamma, Z)^\ell F^{\text{dm}}(Z)$ for $\gamma \in \Gamma^* = \{ \gamma \in H_Z \mid (\gamma - 1)T^{-1} \in M_3(O_K) \}$. It follows that F^{dm} admits the Fourier-Jacobi expansion

$$F^{\text{dm}} \left(\begin{bmatrix} z \\ w \end{bmatrix} \right) = \sum_{r=1}^{\infty} g_r(w) e[rz], \text{ where we put } e[x] = \exp(2\pi i x) \text{ for } x \in \mathbb{C}. \text{ We say that } F$$

is $\overline{\mathbb{Q}}$ -rational if $g_r(v) e[\frac{rk}{2} vv^\sigma] \in \overline{\mathbb{Q}}$ for any $v \in K$ and any $r \geq 1$. By virtue of

Shimura ([8]), the space $S_\ell(U_A^*)_{\overline{\mathbb{Q}}}$ of $\overline{\mathbb{Q}}$ -rational forms of weight ℓ on U_A^*

spans $S_\ell(U_A^*)$. Let $L(\omega; s)$ be the Hecke L-function of K attached to ω . The

second main result of this note is stated as follows:

Theorem 2 *Assume that the class number of K is one and that $\ell > 4$. Let $F \in S_\ell(U_A^*)$ be a $\overline{\mathbb{Q}}$ -rational Hecke eigenform with $\iota_0^* F \neq 0$. Then there exists a $\overline{\mathbb{Q}}$ -rational Hecke eigenform $F' \in S_\ell(U_A^*)$ with the same eigenvalues as F such that*

$$\xi(F \otimes \omega; \frac{\ell}{2} - 1) = c \cdot \pi^{\frac{3}{2}\ell} L(\omega; \frac{\ell}{2}) \langle F', F' \rangle$$

with a non-zero constant $c \in \overline{\mathbf{Q}}^\times$.

Remark. The set of the critical points (in the sense of [1]) of $\xi(F \otimes \omega; s)$ is $\{k \mid 2 - \frac{\ell}{2} \leq k \leq \frac{\ell}{2} - 1\}$.

In view of Garrett's results on Petersson inner products of arithmetic Siegel modular forms ([2]), the following conjecture seems to be plausible.

Conjecture Let $F, F' \in S_\ell(U_A^*)$ be $\overline{\mathbf{Q}}$ -rational Hecke eigenforms with the same Hecke eigenvalues. Then we have $\frac{\langle F, F \rangle}{\langle F', F' \rangle} \in \overline{\mathbf{Q}}$.

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