

GEOMETRIC DESCRIPTION OF THE MODULI SPACE OF PARABOLIC CONNECTIONS ON $\mathbb{P}^1 \setminus \{t_1, \dots, t_5\}$ AND THE UNIVERSAL FAMILY

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Introduction

(Joint work with M.-H. Saito)

- n is a positive integer.
- $0, 1, t_1, \dots, t_{n-3}, \infty$ are n -distinct points on $\mathbb{P}^1$, and $D := 0 + 1 + t_1 + \dots + t_{n-3} + \infty$ is an effective divisor.
- $\nu_1, \dots, \nu_n$ are complex numbers satisfying some conditions.

Moduli space of $\mathfrak{sl}_2$ -connections ($\dim = 2(n-3)$)

$$M := \{(E, \nabla, \varphi)\} / \cong$$

1. E is a rank 2 vector bundle on $\mathbb{P}^1$,
2. $\nabla: E \rightarrow E \otimes \Omega_{\mathbb{P}^1}^1(D)$ is a connection,
3. $\varphi: \bigwedge^2 E \cong \mathcal{O}_{\mathbb{P}^1}(-1)$ is a horizontal isomorphism,
4. the residue $\text{res}_t(\nabla)$ of the connection ∇ at t_i has eigenvalues $\nu_i^\pm$,

$$1 \leq i \leq n \quad (\nu_i^\pm := \pm \nu_i, \nu_i^+ := \nu_n, \text{ and } \nu_n^- := 1 - \nu_n).$$

Moduli space of $\mathfrak{sl}_2$ -Higgs bundles ($\dim = 2(n-3)$)

$$M_H := \{(E, \Phi, \varphi)\} / \cong$$

1. E : a rank 2 vector bundle on $\mathbb{P}^1$,
2. $\Phi: E \rightarrow E \otimes \Omega_{\mathbb{P}^1}^1(D)$ is a Higgs field,
3. $\varphi: \bigwedge^2 E \cong \mathcal{O}_{\mathbb{P}^1}(-1)$ is an isomorphism and $\text{tr}(\Phi) = 0$,
4. the residue $\text{res}_t(\nabla)$ of the connection ∇ at t_i has eigenvalues $\nu_i^\pm$,

$$1 \leq i \leq n \quad (\nu_i^\pm := \pm \nu_i)$$

Stratifications

$$M = M^0 \cup \dots \cup M^{(n-3)/2}, \quad M_H = M_H^0 \cup \dots \cup M_H^{(n-3)/2}$$

where $M^k := \{(E, \nabla, \varphi) \mid E \cong \mathcal{O}(k) \oplus \mathcal{O}(-k-1)\} / \cong$ and $M_H^k := \{(E, \Phi, \varphi) \mid E \cong \mathcal{O}(k) \oplus \mathcal{O}(-k-1)\} / \cong$.

Problem

Give explicit descriptions of the moduli spaces.

Canonical coordinate of the moduli spaces

Apparent singularities and dual parameters for M_H .

$$(E, \Phi, \varphi, \sigma) \mapsto (G, C_\sigma, \sigma_G)$$

- $\sigma \in H^0(\mathbb{P}^1, E)$,
- C_σ is the spectral curve of (E, Φ) ,
- G is the torsion free sheaf of rank 1 on C_σ associated to E (Beauville-Narasimhan-Ramanan correspondence), and
- $\sigma_G \in H^0(C_\sigma, G) \cong H^0(\mathbb{P}^1, E)$ associated to σ .

For (G, C_σ, σ_G) ,

$$0 \longrightarrow \mathcal{O}_{C_\sigma} \xrightarrow{\sigma_G} G \longrightarrow T \longrightarrow 0$$

where T is a torsion sheaf on C_σ of length $n-3$.

$(E, \Phi, \varphi, \sigma) \mapsto \text{Supp}(T) = \{(q_1, p_1), \dots, (q_{n-3}, p_{n-3})\} \in \text{Sym}^{n-3}(\mathbb{L})$

where $\mathbb{L}$ is the total space of $\Omega_{\mathbb{P}^1}^1(D)$.

Apparent singularities and dual parameters

$\{q_1, \dots, q_{n-3}\}$: apparent singularities of $(E, \Phi, \varphi, \sigma)$

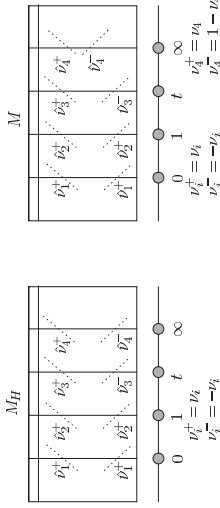
$\{p_1, \dots, p_{n-3}\}$: dual parameters of $(E, \Phi, \varphi, \sigma)$

Apparent singularities and dual parameters for M .

We can define the apparent singularities for M . On the other hand, we can define the dual parameters for only M^0 , which is a Zariski open set.

$$n = 4$$

In this case, $M = M^0$ and $M_H = M_H^0$.



$$n = 5 \text{ (Higgs case)}$$

In this case, $M = M^0 \cup M^1$ and $M_H = M_H^0 \cup M_H^1$.

Moduli space of $\mathfrak{sl}_2$ -Higgs bundles with a cyclic vector

$$\widetilde{M}_H := \{(E, \Phi, \varphi, [\sigma])\} / \cong$$

1. $(E, \Phi, \varphi) \in M_H$,
2. $[\sigma] \subset H^0(\mathbb{P}^1, E)$ is a 1-dimensional subspace generated by a nonzero section $\sigma \in H^0(\mathbb{P}^1, E)$.

• $\widetilde{M}_H$ is the blowing up of M_H along M_H^1 .

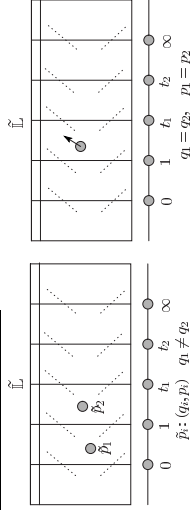
- By the apparent singularities and the dual parameters, we have $\widetilde{M}_H \rightarrow \text{Sym}^2(\mathbb{L})$.

Theorem

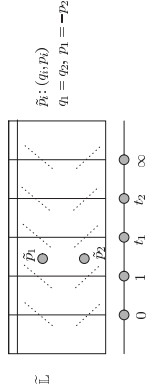
The map $\widetilde{M}_H \rightarrow \widetilde{\text{Hilb}}^2(\mathbb{L})$ is injective.

Here, $\widetilde{\text{Hilb}}^2(\mathbb{L})$ is a blowing up of $\text{Hilb}^{n-3}(\mathbb{L})$.

The locus of $\mathcal{O} \oplus \mathcal{O}(-1)$.



The locus of $\mathcal{O}(1) \oplus \mathcal{O}(-2)$.

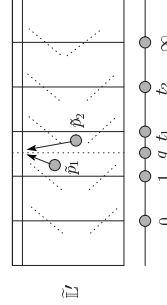


$$n = 5 \text{ (Connection case)}$$

By the apparent singularities and the dual parameters, we obtain that $M^0 \longrightarrow \text{Hilb}^2(\mathbb{L})$ is injective. We extend the map to the moduli space $\widetilde{M}$ of $\mathfrak{sl}_2$ -connection with a cyclic vector.

Strategy

1. Construct a jumping family Φ_{jump} of Higgs bundles, that is, Φ_{jump} is type $\mathcal{O} \oplus \mathcal{O}(-1)$ for $q_1 \neq q_2$ and $\lim_{q_2 \rightarrow q_1} \Phi_{\text{jump}}$ is type $\mathcal{O}(1) \oplus \mathcal{O}(-2)$.
2. Construct a connection ∇_0 such that
 - $\lim_{q_2 \rightarrow q_1} \nabla_0$ is type $\mathcal{O}(1) \oplus \mathcal{O}(-2)$.
 - $\nabla_0 + \Phi_{\text{jump}}$ satisfies the conditions of the residue matrices.
3. Follow up the apparent singularities and the dual parameters of $\nabla_0 + \Phi_{\text{jump}}$ as $q_2 \rightarrow q_1$.



References

- [1] A. Komyo, M.-H. Saito, *Geometric description of the moduli space of parabolic connections on $\mathbb{P}^1 \setminus \{t_1, \dots, t_5\}$ and the universal family*, in preparation.
- [2] M.-H. Saito, S. Seki, *Apparent singularities and canonical coordinates of moduli spaces of parabolic connections and parabolic Higgs bundles*, in preparation.