

A HOMFLY-PT type invariant for integral homology 3-spheres

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In this paper, we will introduce how to construct some invariants for integral homology 3-spheres, using skein algebras.

1 Skein algebras

Let Σ be a compact connected oriented surface with nonempty boundary. We fix base point sets J^- , J^+ and J of $\partial\Sigma$ such that $J^- \cap J^+ = \emptyset$ and $\sharp J^- = \sharp J^+$. We denote by $\mathcal{S}(\Sigma, J)$ the Kauffman bracket skein module, which is the quotient of the set of the free $\mathbb{Q}[[A+1]]$ -module with basis the set of the tangles in $\Sigma \times I$ with basis J by the relation shown in Fig. 2. Here, we denote by I the closed interval between 0 and 1. We denote by $\mathcal{A}(\Sigma, J^-, J^+)$ the HOMFLY-PT skein module which is the quotient of the set of the free $\mathbb{Q}[\rho][[h]]$ -module with basis the set of the tangles in $\Sigma \times I$ from J^- to J^+ by the relation shown in Fig. 1. See, for example, [7] p.21 Definition 3.2 and [10] p.9 Definition 3.2. We simply denote $\mathcal{S}(\Sigma) \stackrel{\text{def.}}{=} \mathcal{S}(\Sigma, \emptyset)$ and $\mathcal{A}(\Sigma) \stackrel{\text{def.}}{=} \mathcal{A}(\Sigma, \emptyset)$. Let e_1 and e_2 be embeddings from $\Sigma \times I$ into $\Sigma \times I$ defined by $e_1(x, t) = (x, \frac{t+1}{2})$ and $e_2(x, t) = (x, \frac{t}{2})$. For $x, x' \in \mathfrak{g}$ and $z \in \mathfrak{g}_M$, xx' , xy and yx is defined by $e_1(x) \cup e_2(x')$, $e_1(x) \cup e_2(y)$ and $e_1(y) \cup e_2(x)$ where $(\mathfrak{g}, \mathfrak{g}_M) = (\mathcal{S}(\Sigma), \mathcal{S}(\Sigma, J))$, $(\mathcal{A}(\Sigma), \mathcal{A}(\Sigma, J^-, J^+))$. We defined a Lie bracket $[x, x']$ and a Lie action $\sigma(x)(z)$ satisfying $\epsilon_{\mathfrak{g}}[x, x'] = xx' - x'x$ and $\epsilon_{\mathfrak{g}}\sigma(x)(z) = xz - zx$ where $\epsilon_{\mathcal{S}(\Sigma)} = -A + A^{-1}$ and $\epsilon_{\mathcal{A}(\Sigma)} = h$. For details, see [7] p.23 Definition 3.8. and [10] p.16 Definition 3.13. We can consider some skein algebras and there exists some Lie homomorphisms around skein algebras.

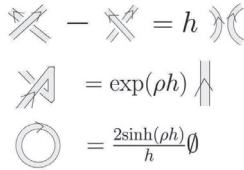


Fig. 1: $\mathcal{A}(\Sigma)$

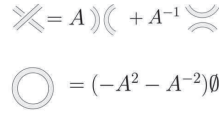


Fig. 2: $\mathcal{S}(\Sigma)$

We fix two base points $*_1, *_2 \in \partial\Sigma$. We simply denote $\mathcal{A}(\Sigma, *_1, *_2) \stackrel{\text{def.}}{=} \mathcal{A}(\Sigma, \{*_1\}, \{*_2\})$ and $\mathcal{S}(\Sigma, *_1, *_2) \stackrel{\text{def.}}{=} \mathcal{S}(\Sigma, \{*_1, *_2\})$.

1.1 $\psi_{\mathcal{A} \rightarrow \mathbb{Q}\hat{\pi}_1} : \mathcal{A}(\Sigma) \rightarrow S(\mathbb{Q}\hat{\pi}_1(\Sigma))$

Let $\hat{\pi}_1(\Sigma)$ be the set of the conjugacy classes of $\pi_1(\Sigma)$. The quotient map is denoted by $|\cdot| : \mathbb{Q}\pi_1(\Sigma) \rightarrow \mathbb{Q}\hat{\pi}_1(\Sigma)$. Goldman defined the Lie bracket $[\cdot, \cdot] : \mathbb{Q}\hat{\pi}_1(\Sigma) \times \mathbb{Q}\hat{\pi}_1(\Sigma) \rightarrow \mathbb{Q}\hat{\pi}_1(\Sigma)$ in [1]. Kawazumi-Kuno defined the Lie action $\sigma : \mathbb{Q}\hat{\pi}_1(\Sigma) \times \mathbb{Q}\pi_1(\Sigma, *_1, *_2) \rightarrow \mathbb{Q}\pi_1(\Sigma, *_1, *_2)$ in [3]. Let $S(\mathbb{Q}\hat{\pi}_1(\Sigma)) \stackrel{\text{def.}}{=} \bigoplus_{i=0}^{\infty} S^i(\mathbb{Q}\hat{\pi}_1(\Sigma))$ be the symmetric tensor algebra of $\mathbb{Q}\hat{\pi}_1(\Sigma)$ over $\mathbb{Q}$. The Leibniz rule defines the Lie bracket $[\cdot, \cdot] : S(\mathbb{Q}\hat{\pi}_1(\Sigma)) \times S(\mathbb{Q}\hat{\pi}_1(\Sigma)) \rightarrow S(\mathbb{Q}\hat{\pi}_1(\Sigma))$. The Lie action $\sigma(\cdot)(\cdot) : S(\mathbb{Q}\hat{\pi}_1(\Sigma)) \times S(\mathbb{Q}\hat{\pi}_1(\Sigma)) \otimes \mathbb{Q}\pi_1(\Sigma, *_1, *_2) \rightarrow S(\mathbb{Q}\hat{\pi}_1(\Sigma)) \otimes \mathbb{Q}\pi_1(\Sigma, *_1, *_2)$ is defined by $\sigma(v_1 \cdots v_j)(V_1 \otimes \gamma) \stackrel{\text{def.}}{=} [v_1 \cdots v_j, V_1] \otimes \gamma + V_1 \cdot (\sum_{j'=1}^j v_1 \cdots v_{j'-1} \cdot v_{j'+1} \cdots v_j \otimes \sigma(v_{j'})(\gamma))$. Let L be an oriented framed link in $\Sigma \times I$ having connected components $l_1, \dots, l_j$. Then, we put $\psi_{\mathcal{A} \rightarrow \mathbb{Q}\hat{\pi}_1}(L) \stackrel{\text{def.}}{=} [l_1] \cdots [l_j] \in S^j(\mathbb{Q}\hat{\pi}_1(\Sigma))$. Let T be an oriented framed tangle from $*_1$ to $*_2$ having closed components $l_1, l_2, \dots, l_j$ and non-closed components r . Then, we also put $\psi_{\mathcal{A} \rightarrow \mathbb{Q}\hat{\pi}_1}(T) \stackrel{\text{def.}}{=} [l_1] \cdots [l_j] \otimes [r] \in S^j(\mathbb{Q}\hat{\pi}_1(\Sigma)) \otimes \mathbb{Q}\pi_1(\Sigma, *_1, *_2)$. The maps $\psi_{\mathcal{A} \rightarrow \mathbb{Q}\hat{\pi}_1}$ induces the $\mathbb{Q}$ -linear maps

$$\begin{aligned} \psi_{\mathcal{A} \rightarrow \mathbb{Q}\hat{\pi}_1} : \mathcal{A}(\Sigma) &\rightarrow S(\mathbb{Q}\hat{\pi}_1(\Sigma)) \\ \psi_{\mathcal{A} \rightarrow \mathbb{Q}\hat{\pi}_1} : \mathcal{A}(\Sigma, *_1, *_2) &\rightarrow S(\mathbb{Q}\hat{\pi}_1(\Sigma)) \otimes \mathbb{Q}\pi_1(\Sigma, *_1, *_2) \end{aligned}$$

satisfying

$$\begin{aligned} [(\psi_{\mathcal{A} \rightarrow \mathbb{Q}\hat{\pi}_1}(x), \psi_{\mathcal{A} \rightarrow \mathbb{Q}\hat{\pi}_1}(x'))] &= \psi_{\mathcal{A} \rightarrow \mathbb{Q}\hat{\pi}_1}([x, x']), \\ \sigma(\psi_{\mathcal{A} \rightarrow \mathbb{Q}\hat{\pi}_1}(x))(\psi_{\mathcal{A} \rightarrow \mathbb{Q}\hat{\pi}_1}(y)) &= \psi_{\mathcal{A} \rightarrow \mathbb{Q}\hat{\pi}_1}(\sigma(x)(y)) \end{aligned}$$

for any $x, x' \in \mathcal{A}(\Sigma)$ and $y \in \mathcal{A}(\Sigma, *_1, *_2)$.

1.2 $\mathcal{A}(\Sigma) = \bigoplus_{x \in H_1(\Sigma, \mathbb{Z})} \mathcal{A}_x(\Sigma)$

Let L_+ , L_- and L_0 be three framed links which are differ only in a closed ball as shown in the first, second and third terms of the first equation. The homology classes of L_+ , L_- and L_0 equal each other. Let L and L' be two framed links which are differ only in a closed ball as shown in the first and second terms of the second equation in Fig. 1. Then, the homology classes of L equals the one of L' . Hence, there exists a direct sum decomposition

$$\mathcal{A}(\Sigma) = \bigoplus_{x \in H_1(\Sigma, \mathbb{Z})} \mathcal{A}_x(\Sigma).$$

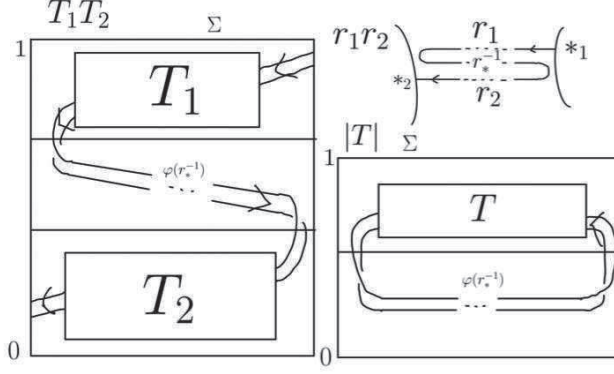
Then we have

$$\begin{aligned} \mathcal{A}_x(\Sigma) \cdot \mathcal{A}_y(\Sigma) &\subset \mathcal{A}_{x+y}(\Sigma) \\ [\mathcal{A}_x(\Sigma), \mathcal{A}_y(\Sigma)] &\subset \mathcal{A}_{x+y}(\Sigma) \end{aligned}$$

for any $x, y \in H_1(\Sigma, \mathbb{Z})$.

1.3 $\psi_{\mathcal{A} \rightarrow \mathcal{S}} : \mathcal{A}_0(\Sigma) \rightarrow \mathcal{S}(\Sigma)$

We put $\psi'_{\mathcal{A} \rightarrow \mathcal{S}}$ by the $\mathbb{Q}$ -module homomorphism from $\mathcal{A}(\Sigma, J^-, J^+)$ to $\mathcal{S}(\Sigma, J^- \cup J^+)$ defined by $T \mapsto (-A)^{w(T)}T$, $h \mapsto -A^2 + A^{-2}$ and $\exp(\rho h) = A^4$. Here $w(T) \stackrel{\text{def.}}{=}$

Fig. 3: T_1T_2 , $|T|$, r_1r_2

$\sharp\{\text{positive crossings of } T\} - \sharp\{\text{negative crossings of } T\}$ is the writhe number of T . By [10] Proposition 7.15, $\psi'_{\mathcal{A} \rightarrow \mathcal{S}}$ is well-defined. We remark that $\psi'_{\mathcal{A} \rightarrow \mathcal{S}}$ is not a Lie algebra homomorphism. The $\mathbb{Q}$ -module homomorphism $\psi_{\mathcal{A} \rightarrow \mathcal{S}}$ is defined by $\psi_{\mathcal{A} \rightarrow \mathcal{S}} \stackrel{\text{def.}}{=} \frac{1}{A+A^{-1}} \psi'_{\mathcal{A} \rightarrow \mathcal{S}}$. Then, $\psi_{\mathcal{A} \rightarrow \mathcal{S}} : \mathcal{A}_0(\Sigma) \rightarrow \mathcal{S}(\Sigma)$ is a Lie algebra homomorphism.

2 Formulas for Dehn twists

We denote by $\pi_1^+(\Sigma, *_1, *_2)$ the set of the regular homotopy classes of free immersed paths from $*_1$ to $*_2$. We fix a simple path $r_* \in \pi_1^+(\Sigma, *_1, *_2)$. A composite $\cdot : \pi_1^+(\Sigma, *_1, *_2) \times \pi_1^+(\Sigma, *_1, *_2) \rightarrow \pi_1^+(\Sigma, *_1, *_2)$ is defined by Fig. 3. We have $r \cdot r_* = r_* \cdot r = r$ for any $r \in \pi_1^+(\Sigma, *_1, *_2)$. For an immersed path $\varsigma : I \rightarrow \Sigma$, a framed oriented tangle $\varphi(\varsigma)$ is defined by $\varphi(\varsigma) : I \times I \rightarrow \Sigma \times I, (t, s) \mapsto (\varsigma(t), \frac{2-t+es}{3})$ where $\epsilon > 0$ is a number small enough. This φ induces a $\mathbb{Q}[\rho][[h]]$ -module homomorphism $\varphi : \mathcal{P}(\Sigma, *_1, *_2) \rightarrow \mathcal{A}(\Sigma, *_1, *_2)$ where $\mathcal{P}(\Sigma, *_1, *_2)$ is the quotient of the free $\mathbb{Q}[\rho][[h]]$ -module with basis $\pi_1^+(\Sigma, *_1, *_2)$ by the relation which is the second equation of Fig. 1. A composite $\mathcal{A}(\Sigma, *_1, *_2) \times \mathcal{A}(\Sigma, *_1, *_2) \rightarrow \mathcal{A}(\Sigma, *_1, *_2)$ is defined by Fig. 3. Then, the $\mathbb{Q}[\rho][[h]]$ -module homomorphism is a $\mathbb{Q}[\rho][[h]]$ -algebra homomorphism. We define the closure $\mathcal{A}(\Sigma, *_1, *_2) \rightarrow \mathcal{A}(\Sigma)$ by Fig. 3.

2.1 Filtrations

In this subsection, we will introduce some filtrations of skein algebras and skein modules. An augmentation map $\text{aug} : \mathcal{P}(\Sigma, *_1, *_2) \rightarrow \mathbb{Q}[\rho]$ is defined by $\text{aug}(r) = 1$ and $\text{aug}(h) = 0$ for $r \in \pi_1^+(\Sigma, *_1, *_2)$. Let $F^n \mathcal{P}(\Sigma, *_1, *_2)$ be a $\mathbb{Q}$ -linear subspace generated by $\sum_{2i+j \geq n} h^i (\ker \text{aug})^j$. For any $n \in \mathbb{Z}_{\geq 0}$, the filtrations $\{F^n \mathcal{A}(\Sigma, *_1, *_2)\}_{n \geq 0}$ and $\{F^n \mathcal{A}(\Sigma)\}_{n \geq 0}$ are defined by

$$F^n \mathcal{A}(\Sigma, *_1, *_2) \stackrel{\text{def.}}{=} \sum_{i_1+i_2+\dots \geq n} \varphi(F^{i_1} \mathcal{P}(\Sigma, *_1, *_2)) | \varphi(F^{i_2} \mathcal{P}(\Sigma, *_1, *_2)) | \dots,$$

$$F^n \mathcal{A}(\Sigma) \stackrel{\text{def.}}{=} \sum_{i_1+\dots \geq n} | \varphi(F^{i_1} \mathcal{P}(\Sigma, *_1, *_2)) | | \varphi(F^{i_2} \mathcal{P}(\Sigma, *_1, *_2)) | \dots.$$

These filtrations $\{F^n \mathcal{A}(\Sigma, *_1, *_2)\}_{n \geq 0}$ and $\{F^n \mathcal{A}(\Sigma)\}_{n \geq 0}$ induces the filtrations $\{F^n S(\mathbb{Q}\hat{\pi}_1(\Sigma))\}_{n \geq 0}$, $\{F^n S(\mathbb{Q}\hat{\pi}_1(\Sigma)) \otimes \mathbb{Q}\pi_1(\Sigma, *_1, *_2)\}_{n \geq 0}$, $\{F^n \mathbb{Q}\hat{\pi}_1(\Sigma)\}_{n \geq 0}$, $\{F^n \mathbb{Q}\pi_1(\Sigma, *_1, *_2)\}_{n \geq 0}$, $\{F^n \mathcal{S}(\Sigma)\}_{n \geq 0}$ and $\{F^n \mathcal{S}(\Sigma, *_1, *_2)\}_{n \geq 0}$ by

$$\begin{aligned} F^n \mathcal{S}(\Sigma) &\stackrel{\text{def.}}{=} \psi_{\mathcal{A} \rightarrow \mathcal{S}}(F^n \mathcal{A}(\Sigma)), F^n \mathcal{S}(\Sigma, *_1, *_2) \stackrel{\text{def.}}{=} \psi_{\mathcal{A} \rightarrow \mathcal{S}}(F^n \mathcal{A}(\Sigma, *_1, *_2)), \\ F^n S(\mathbb{Q}\hat{\pi}_1(\Sigma)) &\stackrel{\text{def.}}{=} \psi_{\mathcal{A} \rightarrow \mathbb{Q}\hat{\pi}_1}(F^n \mathcal{A}(\Sigma)), F^n (S(\mathbb{Q}\hat{\pi}_1(\Sigma)) \otimes \mathbb{Q}\pi_1(\Sigma, *_1, *_2)) \stackrel{\text{def.}}{=} \psi_{\mathcal{A} \rightarrow \mathbb{Q}\hat{\pi}_1}(F^n \mathcal{A}(\Sigma, *_1, *_2)), \\ F^n \mathbb{Q}\hat{\pi}_1(\Sigma) &\stackrel{\text{def.}}{=} \mathbb{Q}\hat{\pi}_1(\Sigma) \cap S(\mathbb{Q}\hat{\pi}_1(\Sigma)), \\ F^n \mathbb{Q}\pi_1(\Sigma, *_1, *_2) &\stackrel{\text{def.}}{=} \mathbb{Q}\pi_1(\Sigma, *_1, *_2) \cap F^n (S(\mathbb{Q}\hat{\pi}_1(\Sigma)) \otimes \mathbb{Q}\pi_1(\Sigma, *_1, *_2)). \end{aligned}$$

These filtrations satisfy

$$[F^i \mathfrak{g}, F^j \mathfrak{g}] \subset F^{i+j-2} \mathfrak{g}, \sigma(F^i \mathfrak{g})(F^j \mathfrak{g}_M) \subset F^{i+j-2} \mathfrak{g}_M$$

for $(\mathfrak{g}, \mathfrak{g}_M) = (\mathcal{A}(\Sigma), \mathcal{A}(\Sigma, *_1, *_2)), (\mathcal{S}(\Sigma), \mathcal{S}(\Sigma, *_1, *_2)), (S(\mathbb{Q}\hat{\pi}_1(\Sigma)), S(\mathbb{Q}\hat{\pi}_1(\Sigma)) \otimes \mathbb{Q}\pi_1(\Sigma, *_1, *_2)), (\mathbb{Q}\hat{\pi}_1(\Sigma), \mathbb{Q}\pi_1(\Sigma, *_1, *_2))$. For $\mathfrak{g} = \mathcal{P}, \mathcal{A}, \mathcal{S}, \mathbb{Q}\hat{\pi}_1, \mathbb{Q}\pi_1$, Their completion is denoted by

$$\begin{aligned} \widehat{\mathfrak{g}}(\Sigma, *_1, *_2) &\stackrel{\text{def.}}{=} \varprojlim_{i \rightarrow \infty} \mathfrak{g}(\Sigma, *_1, *_2) / F^i \mathfrak{g}(\Sigma, *_1, *_2), \\ \widehat{\mathfrak{g}}(\Sigma) &\stackrel{\text{def.}}{=} \varprojlim_{i \rightarrow \infty} \mathfrak{g}(\Sigma) / F^i \mathfrak{g}(\Sigma). \end{aligned}$$

2.2 A formula for Dehn twists on $\mathbb{Q}\hat{\pi}_1(\Sigma)$

Kawazumi-Kuno and Massuyeau-Turaev give a formula for the action of Dehn twists on $\widehat{\mathbb{Q}\pi_1}(\Sigma, *_1, *_2)$. For a simple closed curve c , they put $L_{\mathbb{Q}\hat{\pi}_1}(c) \stackrel{\text{def.}}{=} \frac{1}{2} |(\log(r_c))^2|$ where r_c is an element of $\pi_1(\Sigma)$ such that $|r_c| = c$.

Theorem 2.1 ([2] [3][5]). *For a Dehn twist t_c along c , we obtain*

$$t_c(\cdot) = \exp(\sigma(L_{\mathbb{Q}\hat{\pi}_1}(c)))(\cdot) \stackrel{\text{def.}}{=} \sum_{i=0}^{\infty} \frac{1}{i!} (\sigma(L_{\mathbb{Q}\hat{\pi}_1}(c)))^i(\cdot) : \widehat{\mathbb{Q}\pi_1}(\Sigma, *_1, *_2) \rightarrow \widehat{\mathbb{Q}\pi_1}(\Sigma, *_1, *_2).$$

2.3 A formula for Dehn twists on $\mathcal{A}(\Sigma)$

We denote by φ_n the $\mathbb{Q}$ -module homomorphism from $(\hat{\mathcal{P}}(\Sigma, *_1, *_2))^{\hat{\otimes} n}$ to $\hat{\mathcal{A}}(\Sigma, *_1, *_2)$ defined by $\varphi_n(r_1 \otimes \cdots \otimes r_n) = |\varphi(r_1)| \cdots |\varphi(r_n)|$. By induction, we define $\lambda^{[n]}(X_1, \cdots, X_n)$ by $\lambda^{[1]}(X_1) = \frac{1}{2X_1} (\log(X_1))^2$ and

$$\lambda^{[n+1]}(X_1, \cdots, X_{n+1}) \stackrel{\text{def.}}{=} \frac{\lambda^{[n]}(X_1, \cdots, X_n) - \lambda^{[n]}(X_2, \cdots, X_{n+1})}{X_1 - X_{n+1}}.$$

For a simple closed curve c , let r_c be an element of $\pi_1^+(\Sigma, *_1, *_2)$ such that $|\varphi(r_c)|$ is an element of $\mathcal{A}(\Sigma)$ represented by a knot presented by a diagram c . We put

$$L_{\mathcal{A}}(c) \stackrel{\text{def.}}{=} \left(\frac{h/2}{\text{arcsinh}(h/2)} \right)^2 \left(\sum_{n=1}^{\infty} \frac{h^{n-1} \exp(n\rho h)}{n} \varphi_n(r_{1,n} \cdots r_{n,n} \lambda^{[n]}(r_{1,n}, \cdots, r_{n,n})) - \frac{1}{3} \rho^3 h^2 \right),$$

where $r_{i,n} \stackrel{\text{def.}}{=} r_*^{\otimes(i-1)} \otimes \exp(-\rho h) r_c \otimes r_*^{\otimes n-i}$. Here, for $F(X_1, X_2, \dots, X_n) = \sum a_{i_1, i_2, \dots, i_n} (X_1 - 1)^{i_1} (X_2 - 1)^{i_2} \dots (X_n - 1)^{i_n} \in \mathbb{Q}[[X_1 - 1, X_2 - 1, \dots, X_n - 1]]$ and $x_1, x_2, \dots, x_n \in r_*^{\otimes n} + (\ker \text{aug})^{\otimes n}$, we define

$$F(x_1, x_2, \dots, x_n) = \sum a_{i_1, i_2, \dots, i_n} (x_1 - r_*^{\otimes n})^{i_1} (x_2 - r_*^{\otimes n})^{i_2} \dots (x_n - r_*^{\otimes n})^{i_n}.$$

Here $r_*^{\otimes n}$ is a unit of $\hat{\mathcal{P}}(\Sigma, *_1, *_2)^{\otimes n}$.

Theorem 2.2 ([10] p.29 Theorem 5.2). *The element $L_{\mathcal{A}}(c)$ is well-defined.*

Theorem 2.3 ([10] p.28 Theorem 5.1). *For a Dehn twist t_c along c , we obtain*

$$t_c(\cdot) = \exp(\sigma(L_{\mathcal{A}}(c)))(\cdot) \stackrel{\text{def.}}{=} \sum_{i=0}^{\infty} \frac{1}{i!} (\sigma(L_{\mathcal{A}}(c)))^i(\cdot) : \hat{\mathcal{A}}(\Sigma, *_1, *_2) \rightarrow \hat{\mathcal{A}}(\Sigma, *_1, *_2).$$

2.4 A formula for Dehn twists on $\mathcal{S}(\Sigma)$

For a simple closed curve c , we denote by

$$L_{\mathcal{S}}(c) \stackrel{\text{def.}}{=} \frac{-A + A^{-1}}{4 \log(-A)} (\text{arccosh}(\frac{-c}{2}))^2 - (-A + A^{-1}) \log(-A).$$

Here we also denote by c an element of $\mathcal{S}(\Sigma)$ represented by a knot presented by a diagram c .

Theorem 2.4 ([7] p.26 Theorem 4.1). *For a Dehn twist t_c along c , we obtain*

$$t_c(\cdot) = \exp(\sigma(L_{\mathcal{S}}(c)))(\cdot) \stackrel{\text{def.}}{=} \sum_{i=0}^{\infty} \frac{1}{i!} (\sigma(L_{\mathcal{S}}(c)))^i(\cdot) : \hat{\mathcal{S}}(\Sigma, *_1, *_2) \rightarrow \hat{\mathcal{S}}(\Sigma, *_1, *_2).$$

3 Embeddings of the Torelli group into $(F^3 \hat{\mathfrak{g}}(\Sigma_{g,1}), \text{bch})$ for $\mathfrak{g} = \mathbb{Q}\hat{\pi}_1, \mathcal{S}, \mathcal{A}$

We call the kernel of the action the mapping class group of $\Sigma_{g,1}$ on the homology class of $\Sigma_{g,1}$ the Torelli group. The Torelli group $\mathcal{I}(\Sigma_{g,1})$ is generated by $t_{c_1} t_{c_2}^{-1}$ for a pair of disjoint simple closed curves (c_1, c_2) bounding a surface, which is called by a bounding pair. We can consider $F^3 \hat{\mathfrak{g}}(\Sigma_{g,1})$ as the group where the composition is the Baker-Campbell-Hausdorff series, which is simply denote by $\text{bch}(\cdot, \cdot)$. We can consider $F^3 \hat{\mathfrak{g}}(\Sigma_{g,1})$ as the group whose composition is the Baker-Campbell-Hausdorff series, which is simply denote by $\text{bch}(\cdot, \cdot)$.

Theorem 3.1 ($\mathfrak{g} = \mathbb{Q}\hat{\pi}_1$: [2] [3], $\mathfrak{g} = \mathcal{S}$: [8] p.15 Theorem 3.13, p.16 Corollary 3.15, $\mathfrak{g} = \mathcal{A}$: [10] p.54 Theorem 7.13, Corollary 7.14.). *The group homomorphism $\zeta_{\mathfrak{g}}$ from $\mathcal{I}(\Sigma_{g,1})$ to $(F^3 \hat{\mathfrak{g}}(\Sigma_{g,1}), \text{bch})$ defined by $\zeta(t_{c_1} t_{c_2}^{-1}) = L_{\mathfrak{g}}(c_1) - L_{\mathfrak{g}}(c_2)$ for any bounding pair (c_1, c_2) is well-defined and injective. Furthermore, for $\xi \in \mathcal{I}(\Sigma_{g,1})$, we have $\xi(\cdot) = \exp(\sigma(\zeta_{\mathfrak{g}}(\xi)))(\cdot) : \hat{\mathfrak{g}}(\Sigma, *_1, *_2) \rightarrow \hat{\mathfrak{g}}(\Sigma, *_1, *_2)$.*

Let $\{\mathcal{I}^{(n)}(\Sigma_{g,1})\}_{n \geq 1}$ be the lower central series, defined by $\mathcal{I}^{(1)}(\Sigma_{g,1}) \stackrel{\text{def.}}{=} \mathcal{I}(\Sigma_{g,1})$ and $\mathcal{I}^{(n+1)}(\Sigma_{g,1}) \stackrel{\text{def.}}{=} [\mathcal{I}(\Sigma_{g,1}), \mathcal{I}^{(n)}(\Sigma_{g,1})]$. Using the embedding $\zeta_{\mathfrak{g}} : \mathcal{I}(\Sigma_{g,1}) \rightarrow F^3 \mathfrak{g}(\Sigma_{g,1})$, we can define a filtration $\{\mathcal{M}_{\mathfrak{g}}^{(n)}\}_{n \geq 1}$ of $\mathcal{I}(\Sigma_{g,1})$ by $\mathcal{M}_{\mathfrak{g}}^{(n)}(\Sigma_{g,1}) \stackrel{\text{def.}}{=} \zeta_{\mathfrak{g}}^{-1}(F^{n+2} \widehat{\mathfrak{g}}(\Sigma_{g,1}))$ for $\mathfrak{g} = \mathbb{Q}\hat{\pi}_1, \mathcal{S}, \mathcal{A}$. We call $\{\mathcal{M}_{\mathbb{Q}\hat{\pi}_1}^{(n)}\}_{n \geq 1}$ the Johnson filtration. They satisfy

$$\mathcal{I}^{(n)}(\Sigma_{g,1}) \subset \mathcal{M}_{\mathcal{A}}^{(n)}(\Sigma_{g,1}) \subset \mathcal{M}_{\mathcal{S}}^{(n)}(\Sigma_{g,1}) \cap \mathcal{M}_{\mathbb{Q}\hat{\pi}_1}^{(n)}(\Sigma_{g,1}).$$

4 Invariants for integral homology 3-spheres

In this subsection, the symbol $\mathfrak{g}$ is a symbol $\mathcal{S}$ or a symbol $\mathcal{A}$. Using the embedding $\zeta_{\mathfrak{g}} : \mathcal{I} \rightarrow F^3 \widehat{\mathfrak{g}}(\Sigma_{g,1})$ we obtain an invariant for integral homology 3-spheres.

Theorem 4.1 ($\mathfrak{g} = \mathcal{S}$: [9] p.1 Theorem 1.1, $\mathfrak{g} = \mathcal{A}$: [10] p.60 Theorem 9.1.). *Fix a Heegaard splitting of $S^3 = H_g^+ \cup_{\iota} H_g^-$ where H_g^+ and H_g^- are handle bodies of genus g and ι is a diffeomorphism from ∂H_g^+ to ∂H_g^- . We consider $\Sigma_{g,1}$ as the closure of ∂H_g^+ except for a closed disk. Let $e : \Sigma_{g,1} \rightarrow S^3$ be a tubular neighborhood inducing $e_* : \widehat{\mathfrak{g}}(\Sigma_{g,1}) \rightarrow R_{\mathfrak{g}}$ where $R_{\mathcal{S}} \stackrel{\text{def.}}{=} \mathbb{Q}[[A+1]]$ and $R_{\mathcal{A}} \stackrel{\text{def.}}{=} \mathbb{Q}[[\rho]][[h]]$. Then, the map $Z_{\mathfrak{g}} : \mathcal{I}(\Sigma_{g,1}) \rightarrow R_{\mathfrak{g}}$ defined by $Z_{\mathfrak{g}}(\xi) \stackrel{\text{def.}}{=} \sum_{i=1}^{\infty} \frac{1}{i! \epsilon_{\mathfrak{g}}} e_*((\zeta_{\mathfrak{g}}(\xi))^i)$ induces a map $z_{\mathfrak{g}} : \mathcal{H} \stackrel{\text{def.}}{=} \{\text{integral homology 3 spheres}\} \rightarrow R_{\mathfrak{g}}$, $H_g^+ \cup_{\iota \circ \xi} H_g^- \mapsto Z(\xi)$.*

We denote

$$z^{\text{sl}_2}(M) = 1 + z_1^{\text{sl}_2}(M)(q-1) + z_2^{\text{sl}_2}(M)(q-1)^2 + \dots$$

the invariant of an integral homology 3-sphere M defined by T. Ohtsuki [6]. We also denote

$$z^{\text{sl}_N}(M) = 1 + z_1^{\text{sl}_N}(M)(q-1) + z_2^{\text{sl}_N}(M)(q-1)^2 + \dots$$

using the sl_N -quantum group in [4].

The theorem will appear in our paper in preparation.

Theorem 4.2 ([11]). *We obtain $z^{\text{sl}_2}(\cdot) = z_{\mathcal{S}}(\cdot)_{|A^4=q} = z_{\mathcal{A}}(\cdot)_{|\exp(\rho h)=q, h=-q^{-\frac{1}{2}}+q^{-\frac{1}{2}}}$ and $z^{\text{sl}_N}(\cdot) = z_{\mathcal{A}}(\cdot)_{|\exp(\rho h)=q^{N/2}, h=-q^{\frac{1}{2}}+q^{-\frac{1}{2}}}$.*

Since $e_*(F^{2n-1} \mathfrak{g}) \subset (\epsilon_{\mathfrak{g}}^n)$, we have

$$z_{\mathfrak{g}}(H_g^+ \cup_{\iota \circ \xi} H_g^-) \in 1 + (\epsilon_{\mathfrak{g}}^n)$$

for $\xi \in \mathcal{M}_{\mathfrak{g}}^{(2n-1)}$. Hence, if $\xi \in \mathcal{M}_{\mathcal{S}}^{(2n-1)}$, we have

$$z^{\text{sl}_2}(H_g^+ \cup_{\iota \circ \xi} H_g^-) \in 1 + ((q-1)^n).$$

Furthermore, if $\xi \in \mathcal{M}_{\mathcal{A}}^{(2n-1)}$, we have

$$z^{\text{sl}_N}(H_g^+ \cup_{\iota \circ \xi} H_g^-) \in 1 + ((q-1)^n)$$

for any N .

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