

LABELED CONFIGURATION SPACES AND GROUP-COMPLETION

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1. STATEMENT OF THE RESULTS

In [7] we assigned to any pointed space Y and any topological abelian monoid M the configuration space $C^M(Y)$ of finite subsets of Y with labels in M . As a set $C^M(Y)$ consists of those pairs (S, σ) , where S is a finite subset of the complement of the basepoint in Y and σ is a map $S \rightarrow M$. But (S, σ) is identified with (S', σ') if $S \subset S'$, $\sigma'|_S = \sigma$, and $\sigma'(x) = 0$ when $x \notin S$. The topology of $C^M(Y)$ depends not only on the topology of Y and M but also on the partial monoid structure of M .

Take $\mathbf{R}^\infty \ltimes X = \mathbf{R}^\infty \times X / \mathbf{R}^\infty \times *$ as Y and let $C^M(\mathbf{R}^\infty, X)$ be the subspace of $C^M(\mathbf{R}^\infty \ltimes X)$ consisting of those (S, σ) such that S can be embedded into $\mathbf{R}^\infty$ by the projection $\mathbf{R}^\infty \ltimes X \rightarrow \mathbf{R}^\infty$. In other words,

$$C^M(\mathbf{R}^\infty, X) = C^{X \wedge M}(\mathbf{R}^\infty),$$

where $X \wedge M$ is endowed with the partial monoid structure such that the sum of non-zero elements $(x_1, a_1), \dots, (x_k, a_k)$ exists in $X \wedge M$ if and only if $x_1 = \dots = x_k$ and the sum $a_1 + \dots + a_k$ exists in M .

Let us write $E^M(X) = \Omega C^M(\mathbf{R}^\infty, \Sigma X)$. Then the results of [7] imply the following.

- (1) The inclusion $C^M(\mathbf{R}^\infty, X) \rightarrow C^M(\mathbf{R}^\infty \ltimes X)$ is a homotopy equivalence if X is a euclidean neighborhood retract.
- (2) The natural map $C^M(\mathbf{R}^\infty, X) \rightarrow E^M(X)$ is a group-completion, that is, induces an isomorphism of Pontrjagin ring

$$H_*(C^M(\mathbf{R}^\infty, X))[\pi^{-1}] \cong H_*(E^M(X))$$

where $\pi = \pi_0 C^M(\mathbf{R}^\infty, X) \subset H_*(C^M(\mathbf{R}^\infty, X))$.

- (3) $E^M(X)$ is an infinite loop space, and the correspondence $X \rightarrow \pi_* E^M(X)$ defines a generalized homology theory.

Among examples, we have

- (1) If $M = \mathbf{N}$ is the set of positive integers then $C^M(\mathbf{R}^\infty, X)$ is equivalent to the free abelian monoid generated by X modulo the relation $*$ = 0. In this case we have $\pi_* E^M(X) = H_*(X)$ by the Dold-Thom theorem [2].

More generally, if M is a topological abelian group M then

$$\pi_{\bullet}E^M(X) = \bigoplus_i H_{\bullet+i}(X, \pi_i M)$$

is the homology theory defined by the generalized Eilenberg-Mac Lane spectrum $\bigvee K(\pi_i M, i)$.

- (2) If M is the subset $\{1\}$ in the additive group $\mathbf{Z}$ then $\pi_{\bullet}E^M(X) = \pi_{\bullet}^S X$ is the stable homotopy of X . This is a consequence of the Barratt-Priddy-Quillen theorem.
- (3) Let $M = \text{Gr}(\mathbf{R}^{\infty})$ be the Grassmannian of finite dimensional subspaces of $\mathbf{R}^{\infty}$, regarded as a partial monoid such that $V_1 + \cdots + V_k$ exists if and only if $V_i \perp V_j$ holds for $i \neq j$. Then $\pi_{\bullet}E^M(X) = ko_{\bullet}(X)$ is the connective homology theory associated to the real K -theory $KO_{\bullet}$. (See [6].)

In this note we give an alternative construction of group-completion by using the combinatorial structure of $C^M(\mathbf{R}^{\infty}, X)$. More precisely, we will see that the partial monoid structure of $C^M(\mathbf{R}^{\infty}, X)$ enables us to define an analogue of the classifying space (for topological monoids) which gives rise to a group-completion that, unlike $E^M(X) = \Omega C^M(\mathbf{R}^{\infty}, \Sigma X)$, depends only on $C^M(\mathbf{R}^{\infty}, X)$.

For each $k \geq 0$, let $BC^M(\mathbf{R}^{\infty}, X)_k$ be the subspace of $C^M(\mathbf{R}^{\infty}, X)^k$ consisting of those k -tuples $((S_1, \sigma_1), \dots, (S_k, \sigma_k))$ such that for every $J \subset \{1, \dots, k\}$ and $v \in \bigcup_{j \in J} S_j$ the sum $\sum_{j \in \Lambda(v)} \sigma_j(v)$ exists in $X \wedge M$, where $\Lambda(v) = \{j \mid v \in S_j\}$. Such a k -tuple will be called admissible. With respect to the evident face and degeneracy operators $BC^M(\mathbf{R}^{\infty}, X)_{\bullet}$ is a simplicial space, whose realization is denoted by $BC^M(\mathbf{R}^{\infty}, X)$.

Similarly, let $EC^M(\mathbf{R}^{\infty}, X)$ be the realization of the simplicial space $EC^M(\mathbf{R}^{\infty}, X)_{\bullet}$ such that

$$EC^M(\mathbf{R}^{\infty}, X)_k \subset C^M(\mathbf{R}^{\infty}, X)^k \times C^M(\mathbf{R}^{\infty}, X)$$

is the set of admissible $(k+1)$ -tuples, and that the projection $EC^M(\mathbf{R}^{\infty}, X)_k \rightarrow BC^M(\mathbf{R}^{\infty}, X)_k$ is compatible with face and degeneracy maps. Then the fiber of the induced map $EC^M(\mathbf{R}^{\infty}, X) \rightarrow BC^M(\mathbf{R}^{\infty}, X)$ at the basepoint is $C^M(\mathbf{R}^{\infty}, X)$. As $EC^M(\mathbf{R}^{\infty}, X)$ is contractible, we obtain a natural map

$$C^M(\mathbf{R}^{\infty}, X) \rightarrow \Omega BC^M(\mathbf{R}^{\infty}, X).$$

The main result of this note is the following two theorems.

Theorem 1. *For any topological partial monoid M , the natural map*

$$C^M(\mathbf{R}^{\infty}, X) \rightarrow \Omega BC^M(\mathbf{R}^{\infty}, X)$$

is a group-completion.

Theorem 2. *Let M be a subset of a topological abelian group A and let $\pm M = M \cup -M \subset A$. Then the natural map*

$$C^M(\mathbf{R}^\infty, X) \rightarrow C^{\pm M}(\mathbf{R}^\infty, X),$$

induced by the inclusion $M \subset \pm M$, is a group-completion.

In particular, if $M = \{1\} \in \mathbf{Z}$ then $C^{\pm M}(\mathbf{R}^\infty, X)$ is nothing but the space of positive and negative particles $C^\pm(\mathbf{R}^\infty, X)$ introduced by McDuff [3]. Thus we have

Corollary 3 (Caruso [1]). *For any pointed space X the space $C^\pm(\mathbf{R}^\infty, X)$ is weakly equivalent to $\Omega^\infty \Sigma^\infty X$.*

2. PROOFS

Theorem 1 follows from Proposition 1.5 of [5], because the correspondence $\mathbf{k} \mapsto BC^M(\mathbf{R}^\infty, X)_\mathbf{k}$ is a Γ -space such that the maps

$$BC^M(\mathbf{R}^\infty, X)_\mathbf{k} \rightarrow BC^M(\mathbf{R}^\infty, X)^\mathbf{k}$$

induced by the projections $p_s: \mathbf{k} \rightarrow \mathbf{1}$ are homotopy equivalences.

To prove Theorem 2, let $C^{\pm M}(\mathbf{R}^\infty, X)_{C^M(\mathbf{R}^\infty, X)}$ be the realization of the simplicial space $E_\bullet$ such that $E_k \subset C^M(\mathbf{R}^\infty, X)^k \times C^{\pm M}(\mathbf{R}^\infty, X)$ is the subset of admissible $(k+1)$ -tuples. Let

$$\xi: C^{\pm M}(\mathbf{R}^\infty, X)_{C^M(\mathbf{R}^\infty, X)} \rightarrow BC^M(\mathbf{R}^\infty, X)$$

be the map induced by the projection $\xi_\bullet: E_\bullet \rightarrow BC^M(\mathbf{R}^\infty, X)_\bullet$. Then each ξ_k is a homology fibration since it is equivalent to the projection

$$C^M(\mathbf{R}^\infty, X)^k \times C^{\pm M}(\mathbf{R}^\infty, X) \rightarrow C^M(\mathbf{R}^\infty, X)^k.$$

As $C^M(\mathbf{R}^\infty, X)$ acts on $C^{\pm M}(\mathbf{R}^\infty, X)$ through homology equivalences, we see from [4, Proposition 4] that ξ is a homology fibration with fiber $C^{\pm M}(\mathbf{R}^\infty, X)$.

Assume that $C^{\pm M}(\mathbf{R}^\infty, X)_{C^M(\mathbf{R}^\infty, X)}$ is contractible. Then $C^{\pm M}(\mathbf{R}^\infty, X)$ is weakly equivalent to $\Omega BC^M(\mathbf{R}^\infty, X)$, and Theorem 2 follows from the commutative diagram

$$\begin{array}{ccccc} C^M(\mathbf{R}^\infty, X) & \longrightarrow & EC^M(\mathbf{R}^\infty, X) & \longrightarrow & BC^M(\mathbf{R}^\infty, X) \\ \downarrow & & \downarrow & & \parallel \\ C^{\pm M}(\mathbf{R}^\infty, X) & \longrightarrow & C^{\pm M}(\mathbf{R}^\infty, X)_{C^M(\mathbf{R}^\infty, X)} & \longrightarrow & BC^M(\mathbf{R}^\infty, X) \end{array}$$

together with Theorem 1.

Thus, to prove Theorem 2 we need only show

Lemma 4. *$C^{\pm M}(\mathbf{R}^\infty, X)_{C^M(\mathbf{R}^\infty, X)}$ is contractible.*

(My proof of this lemma is rather complicated, and is omitted here.)

REFERENCES

1. J. Caruso, *A simpler approximation to QX* , Trans. Amer. Math. Soc. **265** (1981), 163–167.
2. A. Dold and R. Thom, *Quasifaserungen und unendliche symmetrische produkte*, Ann. Math. **67** (1958), 239–281.
3. D. McDuff, *Configuration spaces of positive and negative particles*, Topology **14** (1975), 91–107.
4. D. McDuff and G. Segal, *Homology fibrations and the “group-completion” theorem*, Invent. Math. **31** (1976), 279–284.
5. G. Segal, *Categories and cohomology theories*, Topology **13** (1974), 293–312.
6. ———, *K-homology theory and algebraic K-theory*, *K-Theory and Operator Algebras* (A. Dold and B. Eckmann, eds.), Lecture Notes in Math., vol. 575, Springer-Verlag, 1977, pp. 113–127.
7. K. Shimakawa, *Configuration spaces with partially summable labels and homology theories*, Math. J. Okayama Univ. **43** (2001), (in press).