

On mean-field approximation of particle systems with annihilation and spikes

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On a filtered probability space let us consider the following interactions of $N(\geq 2)$ Brownian particles each of which diffuses on the nonnegative half line $\mathbb{R}_+$ and is attracted towards the average position of all the particles. When a particle i attains the boundary 0, it is annihilated (default) and a new particle (also called i) spikes immediately in the middle of particles. More precisely, let us denote by $X_t := (X_t^1, \dots, X_t^N)$ the positions of these particles, where $X_t^i(\geq 0)$ is the position of particle i at time $t \geq 0$ for $i = 1, \dots, N$. With the average $\bar{X}_t := (X_t^1 + \dots + X_t^N) / N$ the dynamics of the system is determined by

$$\begin{aligned} X_t^i &= X_0^i + \int_0^t b(X_s^i, \bar{X}_s) ds + W_t^i + \int_0^t \bar{X}_{s-} \left(dM_s^i - \frac{1}{N} \sum_{j \neq i} dM_s^j \right); \quad t \geq 0, \\ M_t^i &:= \sum_{k=1}^{\infty} 1_{\{\tau_k^i \leq t\}}, \quad \tau_k^i := \inf \left\{ s > \tau_{k-1}^i : X_{s-}^i - \frac{\bar{X}_{s-}}{N} \sum_{j \neq i} (M_s^j - M_{s-}^j) \leq 0 \right\}, \end{aligned} \quad (1)$$

for $i = 1, \dots, N$, $k \in \mathbb{N}$, where $W_t := (W_t^1, \dots, W_t^N)$, $t \geq 0$ is an N -dimensional Brownian motion, M_t^i is the cumulative number of defaults by time $t \geq 0$, τ_k^i is the k -th default time with $\tau_0^i = 0$ of particle i . Here we assume that $b : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ is (globally) Lipschitz continuous, i.e., there exists a constant $\kappa > 0$ such that

$$|b(x_1, m_1) - b(x_2, m_2)| \leq \kappa(|x_1 - x_2| + |m_1 - m_2|) \quad (2)$$

for all $x_1, x_2, m_1, m_2 \in \mathbb{R}_+$, and we also impose the condition

$$\sum_{i=1}^N b(x^i, \bar{x}) \equiv 0 \quad (3)$$

for every $x := (x^1, \dots, x^N) \in \mathbb{R}_+^N$ and $\bar{x} := (x^1 + \dots + x^N) / N$ on the drift function $b(\cdot, \cdot)$.

Given a standard Brownian motion W , we shall consider a system $X := (X^1, \dots, X^N)$, $M := (M^1, \dots, M^N)$ described by (1) with (2)-(3) on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with filtration $\mathbb{F} := (\mathcal{F}_t, t \geq 0)$. In particular, we are concerned with (1) that there might be multiple defaults at the same time with positive probability, i.e.,

$$\mathbb{P}(\exists(i, j) \exists t \in [0, \infty) \text{ such that } X_t^i = X_t^j = 0) > 0.$$

We shall construct a solution to (1) with a specific boundary behavior of defaults until the time $\bar{\tau}_0 := \inf\{s > 0 : \max_{1 \leq i \leq N} X_s^i = 0\}$. Let us define the following map $\Phi(x) := (\Phi^1(x), \dots, \Phi^N(x)) : [0, \infty)^N \mapsto [0, \infty)^N$ and set-valued function $\Gamma : \mathbb{R}_+^N \rightarrow \{1, \dots, N\}$ defined by $\Gamma_0(x) := \{i \in \{1, \dots, N\} : x^i = 0\}$,

$$\Gamma_{k+1}(x) := \left\{ i \in \{1, \dots, N\} \setminus \bigcup_{\ell=1}^k \Gamma_{\ell}(x) : x^i - \frac{\bar{x}}{N} \cdot \left| \bigcup_{\ell=1}^k \Gamma_{\ell}(x) \right| \leq 0 \right\}; \quad k = 0, 1, 2, \dots, N-3$$

$$\Gamma(x) := \bigcup_{k=0}^{N-2} \Gamma_k(x), \quad \Phi^i(x) := x^i + \bar{x} \left(\left(1 + \frac{1}{N}\right) \cdot \mathbf{1}_{\{i \in \Gamma(x)\}} - \frac{1}{N} \cdot |\Gamma(x)| \right) \quad (4)$$

for $x = (x^1, \dots, x^N) \in \mathbb{R}_+^N$, $i = 1, \dots, N$ with $\bar{x} := (x^1 + \dots + x^N) / N \geq 0$. Note that $\Phi([0, \infty)^N \setminus \{0\}) \subseteq [0, \infty)^N \setminus \{0\}$ and $\Phi(0) = 0 = (0, \dots, 0)$.

Lemma 1 ([3]). *Given a standard Brownian motion W and the initial configuration $X_0 \in (0, \infty)^N$ one can construct the process (X, M) which is the unique, strong solution to (1) with (2), (3) on $[0, \bar{\tau}_0]$, such that if there is a default, i.e., $|\Gamma(X_{t-})| \geq 1$ at time t , then the post-default behavior is determined by the process with $X_t^i = \Phi^i(X_{t-})$ for $i = 1, \dots, N$.*

Now let us discuss the system (1) with (2)-(3) as a mean-field approximation for nonlinear equation of MCKEAN-VLASOV type. For the sake of concreteness, let us assume $b(x, m) = -a(x - m)$, $x, m \in [0, \infty)$ for some $a > 0$. By the theory of propagation of chaos (e.g., TANAKA (1984), SHIGA & TANAKA (1985) and SZNITMAN (1991)) as $N \rightarrow \infty$, the dynamics of the finite-dimensional marginal distribution of limiting representative process is expressed by

$$\mathcal{X}_t = \mathcal{X}_0 - a \int_0^t (\mathcal{X}_s - \mathbb{E}[\mathcal{X}_t]) ds + W_t + \int_0^t \mathbb{E}[\mathcal{X}_{s-}] d(\mathcal{M}_s - \mathbb{E}[\mathcal{M}_s]); \quad t \geq 0, \quad (5)$$

where W is the standard Brownian motion, $\mathcal{M}_t := \sum_{k=1}^{\infty} \mathbf{1}_{\{\tau^k \leq t\}}$, $\tau^k := \inf\{s > \tau^{k-1} : \mathcal{X}_{s-} \leq 0\}$, $k \geq 1$, $\tau^0 = 0$. Then taking expectations of both sides of (5), we obtain $\mathbb{E}[\mathcal{X}_t] = \mathbb{E}[\mathcal{X}_0]$, $t \geq 0$. When $\mathcal{X}_0 = x_0$ a.s. for some $x_0 > 0$, substituting this back into (5), we obtain

$$\mathcal{X}_t = \mathcal{X}_0 - a \int_0^t (\mathcal{X}_s - \mathcal{X}_0) ds + W_t + \mathcal{X}_0(\mathcal{M}_t - \mathbb{E}[\mathcal{M}_t]); \quad t \geq 0.$$

Transforming the state space from $[0, \infty)$ to $(-\infty, 1]$ by $\hat{\mathcal{X}}_t := (x_0 - \mathcal{X}_t) / x_0$, we see

$$\hat{\mathcal{X}}_t = - \int_0^t a \hat{\mathcal{X}}_s ds + \widehat{W}_t - \widehat{\mathcal{M}}_t + \mathbb{E}[\widehat{\mathcal{M}}_t]; \quad t \geq 0, \quad (6)$$

where we denote $\widehat{W} = W / x_0$, $\widehat{\mathcal{M}} = \mathcal{M}$.

This transformed process $\hat{\mathcal{X}}$ is similar to the nonlinear MCKEAN-VLASOV-type stochastic differential equation

$$\tilde{\mathcal{X}}_t = \tilde{\mathcal{X}}_0 + \int_0^t b(\tilde{\mathcal{X}}_s) ds + \widetilde{W}_t - \widetilde{\mathcal{M}}_t + \alpha \mathbb{E}[\widetilde{\mathcal{M}}_t]; \quad t \geq 0, \quad (7)$$

studied by DELARUE, INGLIS, RUBENTHALER & TANRÉ (2015 a,b). Here $\tilde{\mathcal{X}}_0 < 1$, $\alpha \in (0, 1)$, $b : (-\infty, 1] \rightarrow \mathbb{R}$ is assumed to be Lipschitz continuous with at most linear growth. $\widetilde{W}$ is the standard Brownian motion, $\widetilde{\mathcal{M}} = \sum_{k=1}^{\infty} \mathbf{1}_{\{\tilde{\tau}^k \leq \cdot\}}$ with $\tilde{\tau}^k := \inf\{s > \tilde{\tau}^{k-1} : \tilde{\mathcal{X}}_{s-} \geq 1\}$, $k \geq 1$, $\tilde{\tau}^0 = 0$. When we specify $\tilde{\mathcal{X}}_0 = 0$, $b(x) = -ax$, $x \in \mathbb{R}_+$, and $\alpha = 1$, the solution $(\hat{\mathcal{X}}, \widehat{\mathcal{M}})$ to (7) reduces to the solution $(\tilde{\mathcal{X}}, \widetilde{\mathcal{M}})$ to (6), however, the previous study of (7) does not guarantee the uniqueness of solution to (7) in the case $\alpha = 1$.

Proposition 1 ([3]). *Assume $\mathbb{E}[\mathcal{X}_0] \geq 1$ and $b(x, m) = -a(x - m)$, $x, m \in [0, \infty)$ for some $a > 0$. There exists a unique strong solution to (5) on $[0, T]$. Moreover, for every $T > 0$, there exists a constant c_T such that every solution to (5) satisfies $(d/dt)\mathbb{E}[\mathcal{M}_t] \leq c_T$ for $0 \leq t \leq T$.*

The proof is based on a fixed point argument. For example, when $a = 0$, we may reformulate the solution $(\hat{\mathcal{X}}, \hat{\mathcal{M}})$ in (6) as

$$\hat{Z}_t = \hat{\mathcal{X}}_t + \hat{\mathcal{M}}_t = \widehat{W}_t + \mathbb{E}[\widehat{\mathcal{M}}_t], \quad \widehat{\mathcal{M}}_t = \lfloor \sup_{0 \leq s \leq t} (\widehat{Z}_s)^+ \rfloor; \quad t \geq 0, \quad (8)$$

where $\lfloor x \rfloor$ is the integer part. Given a candidate solution e_t for $\mathbb{E}[\widehat{\mathcal{M}}_t]$, $t \geq 0$, we shall consider

$$\widehat{Z}_t^e := \widehat{W}_t + e_t, \quad \widehat{\mathcal{M}}_t^e := \lfloor \sup_{0 \leq s \leq t} (\widehat{Z}_s^e)^+ \rfloor; \quad t \geq 0, \quad (9)$$

where the superscripts e of $\widehat{Z}^e$ and $\widehat{\mathcal{M}}^e$ represent the dependence on e . Then uniqueness of the solution to (6) is reduced to uniqueness of the fixed point $e^* = \mathfrak{M}(e^*)$ of the map $\mathfrak{M} : C(\mathbb{R}_+, \mathbb{R}_+) \rightarrow C(\mathbb{R}_+, \mathbb{R}_+)$ defined by

$$\mathfrak{M}_t(e) := \mathbb{E}[\lfloor \sup_{0 \leq s \leq t} (\widehat{Z}_s^e)^+ \rfloor] = \mathbb{E}[\widehat{\mathcal{M}}_t^e]; \quad t \geq 0. \quad (10)$$

To solve the equation (10) let us define recursively $e^{(0)} \equiv 0$, $e^{(n+1)} := \mathfrak{M}(e^{(n)})$ for $n \in \mathbb{N}_0$. Then one can verify $e^{(n)} \leq e^{(n+1)}$ for $n \in \mathbb{N}_0$. Let us also define

$$\mathcal{L} := \{e \in C^1([0, \infty)) : \dot{e} \geq 0, e_0 = 0, e_t \leq \ell(t) := t/x_0, t \geq 0\}.$$

Then one can show that if $e^{(n)} \in \mathcal{L}$, then $e^{(n+1)} \in \mathcal{L}$. (For example, if $a = 0$ and $x_0 \geq 1$, then $\widehat{Z} = \widehat{W} + e = (W/x_0) + e$ for every $e \in \mathcal{L}$, and hence by an application of the renewal theory

$$\mathfrak{M}_t(e) = \sum_{k=1}^{\infty} \mathbb{P}(\sup_{0 \leq s \leq t} (\widehat{W}_s + e_s)^+ \geq k) \leq \sum_{k=1}^{\infty} \mathbb{P}(\sup_{0 \leq s \leq t} (W_s + s)^+ \geq kx_0) \leq \frac{t}{x_0}$$

for $t \geq 0$.) By utilizing this monotone property of the map $\mathfrak{M}_t$ and the first passage time distribution for diffusions, we verify the contraction property and then find a unique fixed point in the class of continuously differentiable, nonnegative functions bounded by a linear line with slope $1/x_0$. Note that in some numerical evaluation we observe the slow convergence of PICARD iteration even for the case $x_0 < 1$.

For the stationary distribution of the solution $\mathcal{X}$ to (5) we have the following proposition.

Proposition 2 ([3]). *When $a > 0$, the stationary distribution of*

$$\mathcal{X}_t = \mathcal{X}_0 - a \int_0^t (\mathcal{X}_s - x_0) ds + W_t + x_0(\mathcal{M}_t - \mathbb{E}[\mathcal{M}_t]); \quad t \geq 0$$

has the density

$$p_a(x) := 2c_0 \left(\int_0^{x \wedge x_0} e^{ay^2 + 2x_0(c_0+a)y} dy \right) e^{-ax^2 - 2x_0(c_0+a)x}; \quad x \geq 0,$$

where $c_0 := \lim_{t \rightarrow \infty} d\mathbb{E}[\mathcal{M}_t]/dt$ is a unique solution to

$$\frac{c_0}{a} \int_{\sqrt{2/ac_0 - \sqrt{2a}x_0}}^{\sqrt{2/ac_0}x_0} e^{x^2/2} \left(\int_x^\infty e^{-y^2/2} dy \right) dx = 1.$$

When $a = 0$, we have $c_0 = 1/x_0^2$ and $p_0(x) = (1 - e^{-2x/x_0})/x_0$ if $0 < x < x_0$ and $p_0(x) = e^{-2x/x_0}(e^2 - 1)/x_0$ if $x > x_0$.

It follows from Proposition 1 that the propagation-of-chaos result holds for the reformulated solution $(Z, \mathcal{M})$ from the original X in (1). Thus we have the following.

Proposition 3 ([3]). *Let us assume that X_0^i , $i \in \mathbb{N}$ are independently, identically distributed with a finite mean. Under the same assumption as in Proposition 1, for every $k \geq 1$, $\ell \geq 1$, $t_1, \dots, t_\ell$, as $N \rightarrow \infty$ the vector $(X_{t_j}^i, M_{t_j}^i)$, $1 \leq i \leq k$, $1 \leq j \leq \ell$ defined from (1) converges towards the finite dimensional marginals at times $t_1, \dots, t_\ell$ of k independent copies of $(\mathcal{X}, \mathcal{M})$ in (5).*

Research supported in part by the National Science Foundation under grants NSF-DMS-13-13373 and DMS-16-15229. Part of research is joint work with ROMUALD ELIE and MATHIEU LAURIERE.

Bibliography

- [1] F. DELARUE, J. INGLIS, S. RUBENTHALER, AND E. TANRÉ. (2015a) Global solvability of a networked integrate-and-fire model of McKean–Vlasov type, *Ann. Appl. Probab.*, 25, pp. 2096–2133.
- [2] F. DELARUE, J. INGLIS, S. RUBENTHALER, AND E. TANRÉ. (2015b) Particle systems with a singular mean-field self-excitation. Application to neuronal networks, *Stochastic Process. Appl.*, 125, pp. 2451–2492.
- [3] R. ELIE, T. ICHIBA, AND M. LAURIERE. (2017) Large Banking Systems with Default and Recovery, *Preprint*.
- [4] T. SHIGA & H. TANAKA. (1985) Central limit theorem for a system of Markovian particles with mean field interactions, *Probab. Theory Related Fields* 69 (3) (1985) 439–459.
- [5] A.-S. SZNITMAN. (1991) Topics in propagation of chaos, in: *École d’Été de Probabilités de Saint-Flour XIX –1989*, Springer, pp. 165–251.
- [6] H. TANAKA. (1984) Limit theorems for certain diffusion processes with interaction, in *Stochastic Analysis* (Katata/Kyoto, 1982), North-Holland Math. Library 32, 469–488.

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