

SEVERAL REVERSE INEQUALITIES OF OPERATORS

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ABSTRACT. In this report, we show reverse inequalities to Araki's inequality and investigate the equivalence among reverse inequalities of Araki, Cordes and Löwner-Heinz inequalities. Among others, we show that if A and B are positive operators on a Hilbert space H such that $0 < mI \leq A \leq MI$ for some scalars $m < M$, then

$$K(m, M, p) \|BAB\|^p \leq \|B^p A^p B^p\| \quad \text{for all } 0 < p < 1,$$

where $K(m, M, p)$ is a generalized Kantorovich constant by Furuta.

1. INTRODUCTION

Let A and B be positive operators on a Hilbert space H . The equivalence among Cordes and Löwner-Heinz inequalities was discussed by many authors. In [8], Furuta showed that the Cordes inequality

$$(1) \quad \|A^p B^p\| \leq \|AB\|^p \quad \text{for } 0 < p < 1$$

is equivalent to the Löwner-Heinz inequality (cf.[14])

$$(2) \quad A \geq B \geq 0 \quad \text{implies} \quad A^p \geq B^p \quad \text{for } 0 < p < 1$$

(cf. [5]). In [1], Araki showed a trace inequality which entails the following inequality:

$$(3) \quad \|B^p A^p B^p\| \leq \|BAB\|^p \quad \text{for } 0 < p < 1.$$

Moreover, it was shown in [6, 2] that the Cordes inequality (1) is equivalent to Araki's inequality (3).

On the other hand, Furuta [9] showed the following Kantorovich type inequalities: If A and B are positive operators with $0 < mI \leq A \leq MI$ for some scalars $m < M$, then

$$(4) \quad A \geq B \geq 0 \quad \text{implies} \quad K(m, M, p) A^p \geq B^p \quad \text{for } p > 1,$$

where a generalized Kantorovich constant $K(m, M, p)$ [3, 7, 11] is defined as

$$(5) \quad K(m, M, p) = \frac{mM^p - Mm^p}{(p-1)(M-m)} \left(\frac{p-1}{p} \frac{M^p - m^p}{mM^p - Mm^p} \right)^p \quad \text{for all real numbers } p.$$

We here cite Furuta's textbook [10] as a pertinent reference to Kantorovich inequalities.

Also, Yamazaki [16] showed the following difference type reverse inequalities of the Löwner-Heinz inequality: If A and B are positive operators with $0 < mI \leq B \leq MI$ for some scalars $m < M$, then

$$(6) \quad A \geq B \geq 0 \quad \text{implies} \quad C(m, M, p) + A^p \geq B^p \quad \text{for } p > 1,$$

where the constant $C(m, M, p)$ [12, 16] is defined as

$$(7) \quad C(m, M, p) = (p-1) \left(\frac{M^p - m^p}{p(M-m)} \right)^{\frac{p}{p-1}} + \frac{Mm^p - mM^p}{M-m} \quad \text{for all real numbers } p.$$

In this report, we show reverse inequalities to Araki's inequality (3) and the Cordes inequality (1): If A and B are positive operators with $0 < mI \leq A \leq MI$ for some scalars $m < M$, then the following inequalities hold

$$(8) \quad K(m, M, p) \|BAB\|^p \leq \|B^p A^p B^p\| \quad \text{for } 0 < p < 1,$$

$$(9) \quad K(m^2, M^2, p)^{1/2} \|AB\|^p \leq \|A^p B^p\| \quad \text{for } 0 < p < 1,$$

respectively. We moreover show that reverse inequalities (4), (8) and (9) are mutually equivalent.

2. MAIN RESULTS

First of all, we present our main theorem which is a reverse inequality to Araki's inequality (3).

Theorem 1. *If A and B are positive operators on H such that $0 < mI \leq A \leq MI$ for some scalars $m < M$, then for each $\alpha > 0$*

$$(10) \quad \|BAB\|^p \leq \alpha \|B^p A^p B^p\| + \beta(m^p, M^p, \frac{1}{p}, \alpha) \|B\|^{2p} \quad \text{for all } 0 < p < 1,$$

or equivalently

$$(11) \quad \|B^p A^p B^p\|^{\frac{1}{p}} \leq \alpha \|BAB\| + \beta(m, M, p, \alpha) \|B\|^2 \quad \text{for all } p > 1,$$

where

$$(12) \quad \beta(m, M, p, \alpha) = \begin{cases} \frac{p-1}{p} \left(\frac{M^p - m^p}{p(M-m)} \right)^{\frac{1}{p-1}} + \frac{\alpha(Mm^p - mM^p)}{M^p - m^p} & \text{if } \frac{M^p - m^p}{pM^{p-1}(M-m)} \leq \alpha \leq \frac{M^p - m^p}{pm^{p-1}(M-m)}, \\ (1-\alpha)M & \text{if } 0 < \alpha \leq \frac{M^p - m^p}{pM^{p-1}(M-m)}, \\ (1-\alpha)m & \text{if } \alpha \geq \frac{M^p - m^p}{pm^{p-1}(M-m)}. \end{cases}$$

If we choose α satisfying $\beta(m, M, p, \alpha) = 0$ in Theorem 1, then we have the following ratio type reverse inequalities.

Corollary 2. *If A and B are positive operators on H such that $0 < mI \leq A \leq MI$ for some scalars $m < M$, then*

$$(13) \quad K(m, M, p) \|BAB\|^p \leq \|B^p A^p B^p\| \quad \text{for } 0 < p < 1,$$

or equivalently

$$(14) \quad \|BAB\|^p \leq K(m, M, p) \|B^p A^p B^p\| \quad \text{for } p > 1,$$

where $K(m, M, p)$ is defined as (5) in §1.

If we put $\alpha = 1$ in Theorem 1, then we have the following difference type reverse inequalities.

Corollary 3. *If A and B are positive operators on H such that $0 < mI \leq A \leq MI$ for some scalars $m < M$, then*

$$(15) \quad \|BAB\|^p - \|B^p A^p B^p\| \leq -C(m, M, p) \|B\|^{2p} \quad \text{for } 0 < p < 1,$$

or equivalently

$$(16) \quad \|B^p A^p B^p\|^{\frac{1}{p}} - \|BAB\| \leq -C(m^p, M^p, \frac{1}{p}) \|B\|^2 \quad \text{for } p > 1,$$

where $C(m, M, p)$ is defined as (7) in §1.

As special cases of Corollary 2 and Corollary 3, we have the following corollary.

Corollary 4. *If A and B are positive operators on H such that $0 < mI \leq A \leq MI$ for some scalars $m < M$, then*

$$(17) \quad \|B^2 A^2 B^2\| \leq \frac{(M+m)^2}{4Mm} \|BAB\|^2.$$

$$(18) \quad \|B^2 A^2 B^2\|^{\frac{1}{2}} - \|BAB\| \leq \frac{(M-m)^2}{4(M+m)} \|B\|^2.$$

$$(19) \quad \frac{2\sqrt[4]{Mm}}{\sqrt{M} + \sqrt{m}} \|BAB\|^{\frac{1}{2}} \leq \|B^{\frac{1}{2}} A^{\frac{1}{2}} B^{\frac{1}{2}}\|.$$

$$(20) \quad \|BAB\|^{\frac{1}{2}} - \|B^{\frac{1}{2}} A^{\frac{1}{2}} B^{\frac{1}{2}}\| \leq \frac{(\sqrt{M} - \sqrt{m})^2}{4(\sqrt{M} + \sqrt{m})} \|B\|.$$

Since $\|X^*X\| = \|X\|^2$ for an operator X , we obtain the following reverse inequality to the Cordes inequality by Corollary 2.

Theorem 5. *If A and B are positive operators on H such that $0 < mI \leq A \leq MI$ for some scalars $m < M$, then*

$$(21) \quad K(m^2, M^2, p)^{\frac{1}{2}} \|AB\|^p \leq \|A^p B^p\| \quad \text{for all } 0 < p < 1,$$

or equivalently

$$(22) \quad \|A^p B^p\| \leq K(m^2, M^2, p)^{\frac{1}{2}} \|AB\|^p \quad \text{for all } p > 1.$$

In particular,

$$(23) \quad \sqrt{\frac{2\sqrt{Mm}}{M+m}} \|AB\|^{\frac{1}{2}} \leq \|A^{\frac{1}{2}} B^{\frac{1}{2}}\|.$$

and

$$(24) \quad \|A^2 B^2\| \leq \frac{M^2 + m^2}{2Mm} \|AB\|^2$$

The equivalence among the reverse inequalities of Araki, Cordes and Löwner-Heinz inequalities is now given as follows.

Theorem 6. For a given $p > 1$, the following are mutually equivalent: For all $A, B \geq 0$ and $0 < mI \leq A \leq MI$

- (A) $A \geq B \geq 0$ implies $K(m, M, p)A^p \geq B^p$.
- (B) $\|A^p B^p\| \leq K(m^2, M^2, p)^{1/2} \|AB\|^p$.
- (C) $\|B^p A^p B^p\| \leq K(m, M, p) \|BAB\|^p$.
- (B') $K(m^2, M^2, 1/p)^{1/2} \|AB\|^p \leq \|A^p B^p\|$.
- (C') $K(m, M, 1/p) \|BAB\|^p \leq \|B^p A^p B^p\|$.

3. LEMMAS

We start with the following three lemmas before we give proofs of the results in §2.

Let A be a positive operator on a Hilbert space H and x a unit vector in H . Then it follows from Hölder-McCarthy inequality that

$$(25) \quad (Ax, x) \leq (A^p x, x)^{\frac{1}{p}} \quad \text{for all } p > 1.$$

By using the Mond-Pečarić method [12, 13], we have the following reverse inequality of (25) [15, 4]:

Lemma 7. If A is a positive operator on H such that $0 < mI \leq A \leq MI$ for some scalars $0 < m < M$, then for each $\alpha > 0$

$$(26) \quad (A^p x, x)^{\frac{1}{p}} \leq \alpha(Ax, x) + \beta(m, M, p, \alpha) \quad \text{for all } p > 1$$

holds for every unit vector $x \in H$, where $\beta(m, M, p, \alpha)$ is defined as (12) in Theorem 1.

Proof. For the sake of reader's convenience, we give a proof. Put $\beta = \beta(m, M, p, \alpha)$ and $f(t) = (at+b)^{\frac{1}{p}} - \alpha t$ for $a = \frac{M^p - m^p}{M - m}$ and $b = \frac{Mm^p - mM^p}{M - m}$, then we have $f'(t) = \frac{a}{p}(at+b)^{\frac{1}{p}-1} - \alpha$. It follows that the equation $f'(t) = 0$ has exactly one solution $t_0 = \frac{1}{a}(\frac{\alpha p}{a})^{\frac{p}{1-p}} - \frac{b}{a}$. If $m \leq t_0 \leq M$, then we have $\beta = \max_{m \leq t \leq M} f(t) = f(t_0)$ since $f''(t) = \frac{a^2(1-p)}{p^2}(at+b)^{\frac{1}{p}-2} < 0$ and the condition $m \leq t_0 \leq M$ is equivalent to the condition

$$\frac{M^p - m^p}{pM^{p-1}(M - m)} \leq \alpha \leq \frac{M^p - m^p}{pm^{p-1}(M - m)}.$$

If $M \leq t_0$, then $f(t)$ is increasing on $[m, M]$ and hence we have $\beta = \max_{m \leq t \leq M} f(t) = f(t_0) = (1 - \alpha)M$ for $t_0 = M$. Similarly, we have $\beta = \max_{m \leq t \leq M} f(t) = f(t_0) = (1 - \alpha)m$ for $t_0 = m$ if $t_0 \leq m$. Hence it follows that

$$(at+b)^{\frac{1}{p}} - \alpha t \leq \beta \quad \text{for all } t \in [m, M].$$

Since t^p is convex for $p > 1$, it follows that $t^p \leq at + b$ for $t \in [m, M]$. By the spectral theorem, we have $A^p \leq aA + b$ and hence $(A^p x, x) \leq a(Ax, x) + b$ for every unit vector $x \in H$. Therefore we have

$$\begin{aligned} (A^p x, x)^{\frac{1}{p}} - \alpha(Ax, x) &\leq (a(Ax, x) + b)^{\frac{1}{p}} - \alpha(Ax, x) \\ &\leq \max_{m \leq t \leq M} f(t) = \beta(m, M, p, \alpha). \end{aligned}$$

□

By Lemma 7, we have the following estimates of both the difference and the ratio in the inequality (25).

Lemma 8. *If A is a positive operator on H such that $0 < mI \leq A \leq MI$ for some scalars $0 < m < M$, then for each $p > 1$*

$$(27) \quad (A^p x, x)^{\frac{1}{p}} \leq K(m, M, p)^{\frac{1}{p}} (Ax, x)$$

and

$$(28) \quad (A^p x, x)^{\frac{1}{p}} - (Ax, x) \leq -C(m^p, M^p, \frac{1}{p})$$

hold for every unit vector $x \in H$, where $K(m, M, p)$ is defined as (5) in §1 and $C(m, M, p)$ is defined as (7) in §1.

Proof. If we choose α satisfying $\beta(m, M, p, \alpha) = 0$ in Lemma 7, then we have $\alpha = K(m, M, p)^{\frac{1}{p}}$. If we put $\alpha = 1$ in Lemma 7, then we have $\beta(m, M, p, 1) = -C(m^p, M^p, \frac{1}{p})$. $\square$

We remark that $K(m, M, 2)$ coincides with the Kantorovich constant $\frac{(M+m)^2}{4Mm}$ if $p = 2$.

We summarize some important properties of a generalized Kantorovich constant [3, 11].

Lemma 9. *Let $m < M$ be given. Then a generalized Kantorovich constant $K(m, M, p)$ has the following properties.*

- (i) $K(m, M, p) = K(M, m, p)$ for all $p \in \mathbb{R}$.
- (ii) $K(m, M, p) = K(m, M, 1-p)$ for all $p \in \mathbb{R}$.
- (iii) $K(m, M, 0) = K(m, M, 1) = 1$ for all $p \in \mathbb{R}$.
- (iv) $K(m, M, p)$ is increasing for $p > \frac{1}{2}$ and decreasing for $p < \frac{1}{2}$.
- (v) $K(m^r, M^r, \frac{p}{r})^{\frac{1}{p}} = K(m^p, M^p, \frac{r}{p})^{-\frac{1}{r}}$ for $pr \neq 0$.

4. PROOF OF RESULTS

Based on Lemmas in the preceding section, we give proofs of the results mentioned in the second section.

Proof of Theorem 1.

For every unit vector $x \in H$, it follows that

$$\begin{aligned} & ((BAB)^p x, x) \\ & \leq (BABx, x)^p \quad \text{by Hölder-McCarthy inequality and } 0 < p < 1 \\ & = \left((A^p)^{\frac{1}{p}} \frac{Bx}{\|Bx\|}, \frac{Bx}{\|Bx\|} \right)^p \|Bx\|^{2p} \\ & \leq \left(\alpha (A^p \frac{Bx}{\|Bx\|}, \frac{Bx}{\|Bx\|}) + \beta(m^p, M^p, \frac{1}{p}, \alpha) \right) \|Bx\|^{2p} \quad \text{by Lemma 7} \\ & = \alpha (A^p Bx, Bx) \|Bx\|^{2p-2} + \beta(m^p, M^p, \frac{1}{p}, \alpha) \|Bx\|^{2p} \\ & = \alpha \left(B^p A^p B^p \frac{B^{1-p}x}{\|B^{1-p}x\|}, \frac{B^{1-p}x}{\|B^{1-p}x\|} \right) \|Bx\|^{2p-2} \|B^{1-p}x\|^2 + \beta(m^p, M^p, \frac{1}{p}, \alpha) \|Bx\|^{2p} \end{aligned}$$

and

$$\begin{aligned}\|Bx\|^{2p-2}\|B^{1-p}x\|^2 &= (B^2x, x)^{p-1}(B^{2-2p}x, x) \\ &\leq (B^2x, x)^{p-1}(B^2x, x)^{1-p} = 1 \quad \text{by } 0 < 1-p < 1.\end{aligned}$$

By combining two inequalities above, we have

$$\begin{aligned}\|BAB\|^p &= \|(BAB)^p\| \\ &\leq \alpha\|B^pA^pB^p\| + \beta(m^p, M^p, \frac{1}{p}, \alpha)\|B\|^{2p}\end{aligned}$$

and hence we have the desired inequality (10).

Next, we show (10) $\implies$ (11). For $p > 1$, since $0 < \frac{1}{p} < 1$, it follows from (10) that

$$\|BAB\|^{\frac{1}{p}} \leq \alpha \|B^{\frac{1}{p}}A^{\frac{1}{p}}B^{\frac{1}{p}}\| + \beta(m^{\frac{1}{p}}, M^{\frac{1}{p}}, p, \alpha)\|B\|^{\frac{2}{p}}.$$

By replacing A by A^p and B by B^p in the above inequality respectively, we have

$$\|B^pA^pB^p\|^{\frac{1}{p}} \leq \alpha \|BAB\| + \beta(m, M, p, \alpha)\|B^p\|^{\frac{2}{p}},$$

and so we have the desired inequality (11). Similarly we can show (11) $\implies$ (10). Therefore (10) is equivalent to (11). $\square$

Proof of Corollary 2.

For $p > 1$, if we put $\beta(m, M, p, \alpha) = 0$ in Theorem 1, then it follows that

$$\frac{p-1}{p} \left(\frac{M^p - m^p}{p(M-m)} \right)^{\frac{1}{p-1}} + \alpha^{\frac{p}{p-1}} \frac{(Mm^p - mM^p)}{M^p - m^p} = 0$$

and hence

$$\alpha^{\frac{p}{p-1}} = -\frac{p-1}{p} \left(\frac{M^p - m^p}{p(M-m)} \right)^{\frac{1}{p-1}} \frac{M^p - m^p}{Mm^p - mM^p}.$$

Therefore, we have

$$\begin{aligned}\alpha^p &= \frac{M^p - m^p}{p(M-m)} \left(\frac{p-1}{p} \frac{M^p - m^p}{mM^p - Mm^p} \right)^{p-1} \\ &= K(m, M, p)\end{aligned}$$

and we obtain the desired inequality (14).

For $0 < p < 1$, since $1/p > 1$, it follows from (14) that

$$\|BAB\|^{\frac{1}{p}} \leq K(m, M, \frac{1}{p})\|B^{\frac{1}{p}}A^{\frac{1}{p}}B^{\frac{1}{p}}\|.$$

By replacing A and B by A^p and B^p respectively, then we have

$$\|B^pA^pB^p\|^{\frac{1}{p}} \leq K(m^p, M^p, \frac{1}{p})\|BAB\|.$$

Hence it follows from Lemma 9 that

$$\begin{aligned}\|B^pA^pB^p\| &\leq K(m^p, M^p, \frac{1}{p})^p \|BAB\|^p \\ &\leq K(m, M, p)^{-1} \|BAB\|^p,\end{aligned}$$

and we have the desired inequality (13). Similarly we have the implication (13) $\implies$ (14). $\square$

Proof of Corollary 3.

If we put $\alpha = 1$ in Theorem 1, then it follows that

$$\begin{aligned}\beta(m^p, M^p, \frac{1}{p}, 1) &= \frac{\frac{1}{p} - 1}{\frac{1}{p}} \left(\frac{M - m}{\frac{1}{p}(M^p - m^p)} \right)^{\frac{1}{\frac{1}{p} - 1}} + \frac{M^p m - m^p M}{M - m} \\ &= (1 - p) \left(\frac{p(M - m)}{M^p - m^p} \right)^{\frac{p}{1-p}} + \frac{M^p m - m^p M}{M - m} \\ &= -C(m, M, p).\end{aligned}$$

Similarly it follows that $\beta(m, M, p, 1) = -C(m^p, M^p, \frac{1}{p})$. Hence we have the equivalence (15) $\iff$ (16) $\square$

Proof of Corollary 4.

In Corollary 2 and 3, we have only to put $p = 2$ and $p = 1/2$. $\square$

Proof of Theorem 5

By Corollary 2, it follows that

$$K(m, M, p) \|A^{\frac{1}{2}} B\|^{2p} \leq \|A^{\frac{p}{2}} B^p\|^2.$$

By replacing A by A^2 , we have

$$K(m^2, M^2, p) \|AB\|^{2p} \leq \|A^p B^p\|^2.$$

Therefore we have (21). Similarly, we have the equivalence (21) $\iff$ (22). $\square$

Proof of Theorem 6

The proof is divided into three parts, namely the equivalence (A) $\implies$ (B) $\implies$ (C) $\implies$ (A), (B) $\iff$ (B') and (C) $\iff$ (C').

(A) $\implies$ (B). It follows that

$$\begin{aligned}(A) &\iff \|A^{-\frac{1}{2}} B^{\frac{1}{2}}\| \leq 1 \rightarrow \|A^{-\frac{p}{2}} B^{\frac{p}{2}}\|^2 \leq K(m, M, p) \\ &\iff \|A^{\frac{1}{2}} B^{\frac{1}{2}}\| \leq 1 \rightarrow \|A^{\frac{p}{2}} B^{\frac{p}{2}}\|^2 \leq K(M^{-1}, m^{-1}, p) = K(m, M, p) \\ &\iff \|AB\| \leq 1 \rightarrow \|A^p B^p\| \leq K(m^2, M^2, p).\end{aligned}$$

If we put $B_1 = B/\|AB\|$, then it follows from $\|AB_1\| = 1$ that

$$\|A^p B_1^p\| \leq K(m^2, M^2, p)^{\frac{1}{2}} \iff \|A^p B^p\| \leq K(m^2, M^2, p)^{\frac{1}{2}} \|AB\|^p.$$

(B) $\implies$ (C). If we replace A by $A^{\frac{1}{2}}$ in (A), then it follows that

$$\|A^{\frac{p}{2}} B^p\| \leq K(m, M, p)^{\frac{1}{2}} \|A^{\frac{1}{2}} B\|^p.$$

Square both sides, we have

$$\|B^p A^p B^p\| \leq K(m, M, p) \|BAB\|^p.$$

(C) $\implies$ (A). If we replace B by $B^{\frac{1}{2}}$ and A by A^{-1} in (C), then it follows that

$$\|B^{\frac{p}{2}} A^{-p} B^{\frac{p}{2}}\| \leq K(M^{-1}, m^{-1}, p) \|B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}}\|^p.$$

By rearranging it, we have

$$\|A^{-\frac{p}{2}} B^p A^{-\frac{p}{2}}\| \leq K(m, M, p) \|A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\|^p.$$

Since $A \geq B \geq 0$, it follows from $A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq 1$ that

$$\|A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}}\| \leq K(m, M, p)$$

and hence

$$B^p \leq K(m, M, p)A^p.$$

(B) $\iff$ (B'): If we replace A and B by $A^{\frac{1}{p}}$ and $B^{\frac{1}{p}}$ in (B) respectively, then it follows that

$$\begin{aligned} (B) &\iff \|AB\| \leq K(m^{\frac{2}{p}}, M^{\frac{2}{p}}, p)^{\frac{1}{2}} \|A^{\frac{1}{p}}B^{\frac{1}{p}}\|^p \\ &\iff \|AB\|^{\frac{1}{p}} \leq K(m^{\frac{2}{p}}, M^{\frac{2}{p}}, p)^{\frac{1}{2p}} \|A^{\frac{1}{p}}B^{\frac{1}{p}}\| \\ &\iff K(m^2, M^2, p)^{\frac{1}{2}} \|AB\|^{\frac{1}{p}} \leq \|A^{\frac{1}{p}}B^{\frac{1}{p}}\| \quad \text{by Lemma 9} \\ &\iff (B') \end{aligned}$$

Similarly we have (C) $\iff$ (C') and so the proof is complete. $\square$

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