

Rational Curves on Hypersurfaces

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We study the family of rational curves on hypersurfaces $/k = \bar{k}$, $\text{ch} \geq 0$.

Family of Rational Curves

Let $e \geq 1$, $d \geq 1$, and $n \geq 3$.

Def. For $X = X_d \subset \mathbb{P}^n$: hypersurf of deg d , set

$$R_e(X) := \left\{ \begin{array}{l} \text{smooth rat curves } C \text{ of deg } e \text{ in } \mathbb{P}^n \\ \text{lying in } X \end{array} \right\} \subset \text{Hilb}^{e+1}(X/k).$$

Expected Dimension

For $\mu := (n+1-d)e + n - 4 = \chi(N_{C/X})$,

$$R_e(X) \neq \emptyset \Rightarrow \dim R_e(X) \geq \mu.$$

Starting Point (Lines, $e = 1$)

Thm A (W. Barth-A. Van de Ven '78/79 (C), J. Kollár '96 ($\text{ch} \geq 0$)).

(a) $R_1(X) = \emptyset$ for gen X if $\mu < 0$.

(b) $R_1(X)$: smooth of dim μ for gen X if $\mu \geq 0$.

(c) $R_1(X)$: connected for any X if $\mu \geq 1$, except when $X = X_2 \subset \mathbb{P}^3$.

What can we say about the family $R_e(X)$ for deg $e \geq 2$?

Thm B (J. Harris-M. Roth-J. Starr '04 (C)).

$e \geq 1$, $d < (n+1)/2 \Rightarrow R_e(X)$: integral, loc comp int sch of dim μ for gen X .

Main Thm (Furukawa '07 ($\text{ch} \geq 0$)).

(a) $d \geq \max(e-2, 1) \Rightarrow R_e(X) = \emptyset$ for gen X if $\mu < 0$.

(b) $\left. \begin{array}{l} 1 \leq e \leq 3, d \geq 1, \text{ or} \\ e \geq 4, d \geq 2e-3 \end{array} \right\} \Rightarrow R_e(X)$: smooth of dim μ for gen X if $\mu \geq 0$.

(c) $R_2(X)$: connected for gen X if $\mu \geq 1$, except when $X = X_3 \subset \mathbb{P}^3$.

Exceptional Case of (c)

Prop. $X = X_3 \subset \mathbb{P}^3$: smooth cubic
 $\Rightarrow R_2(X)$ has 27 connected comp.

$\therefore$ For the lines $L_1, \dots, L_{27} \subset X$,

$$R_2(X) \hookrightarrow \bigcup L_i^* \subset (\mathbb{P}^3)^\vee$$

$$C \mapsto (2\text{-pl spanned by } C),$$

where $L_i^* := \{H \subset \mathbb{P}^3 : 2\text{-pl}, L_i \subset H\}$. $\square$

Key of the Proof

Fix $f: \mathbb{P}^1 \xrightarrow{\sim} C \subset \mathbb{P}^n$: deg e .

$$I_C^\circ := \{X = X_d \subset \mathbb{P}^n : \text{smooth}, C \subset X\},$$

$$Z_C^\circ := \{X \in I_C^\circ \mid h^1(N_{C/X}) \neq 0\}.$$

Tools

- ◆ Castelnuovo-Mumford regularity of I_C^2 .
- ◆ Theory of catalecticant matrices.
- ◆ Bounds of a_i for $f^*N_{C/\mathbb{P}^n} \simeq \bigoplus \mathcal{O}_{\mathbb{P}^1}(a_i)$.

Prop. assumption of (b)

$$\Rightarrow \text{codim}(Z_C^\circ, I_C^\circ) \geq \mu + 1.$$

$\therefore 0 \rightarrow I_C^2 \rightarrow I_C \rightarrow N_{C/\mathbb{P}^n}^\vee \rightarrow 0$ induces

$$H^0(\mathbb{P}^n, I_C(d)) \xrightarrow{\delta_C} \text{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(f^*N_{C/\mathbb{P}^n}, f^*(\mathcal{O}_{\mathbb{P}^n}(d)))$$

$$\cup_b \quad \cup_\#$$

$$\hat{Z}_C = \delta_C^{-1}(Z) \twoheadrightarrow Z := \{\alpha \mid H^0(\alpha) : \text{not surj}\}.$$

(Prop) $\hookleftarrow \text{codim}(b) \hookleftarrow \text{codim}(\#)$. $\square$