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Discrepancy of Some Special Sequences

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Abstract

We obtained some results concerned with the discrepancy of the sequence $(\alpha n + \beta\log n)_{n=1}^{\infty}$, $\alpha \notin \mathbb{Q}$, $\beta \neq 0$.

First we give some definitions.

Definition 1. *Let (x_n), $n = 1, 2, \ldots,$ be a sequence of $\mathbb{R}$. Then the discrepancy of (x_n) is defined by*

$$D_N(x_n) = \sup_{0\le a<b\le 1}\left|\frac{1}{N}\sum_{n=1}^{N}\chi_{[a,\ b)}(x_n) - (b-a)\right|,$$

where $\chi_{[a,\ b)}(x)$ is the characteristic function mod 1 of $[a,\ b)$, that is, $\chi_{[a,\ b)}(x) = 1$ for $\{x\} = x - [x] \in [a,\ b)$ and $\chi_{[a,\ b)}(x) = 0$ otherwise.

Definition 2. *An irrational number α is said to be of constant type if there exists a constant $c > 0$ such that $\|\alpha h\| \geq c/h$ holds for all integers $h > 0$, where $\|x\| = \min\{\{x\}, 1-\{x\}\}$ for $x \in \mathbb{R}$.*

Definition 3. *An irrational number α is said to be of type η if η is the infimum of all real numbers τ for which there exists a positive constant $c = c(\tau,\ \alpha)$ such that $h^{\tau}\|\alpha h\| \geq c$ holds for all positive integers h.*

For an integer $s \geq 1$, let $U^s = \{(t_1,\ \ldots\ ,t_s) \in \mathbb{R}^s : 0 \leq t_i \leq 1 \text{ for } 1 \leq i \leq s\}$ be the s-dimensional unit cube. We set

$$\chi(\mathbf{x},\mathbf{y}) = \begin{cases} 1 & \text{if} \qquad \{x_i\} < y_i \quad (i = 1,\ \ldots\ ,\ s) \\ 0 & \text{otherwise} \end{cases},$$

for $\mathbf{x} = (x_1,\ \ldots\ ,x_s) \in \mathbb{R}^s$ and $\mathbf{y} = (y_1,\ \ldots\ ,y_s) \in U^s$.

Definition 4. *For $N \in \mathbb{N}$, the discrepancy of the sequence $(\mathbf{x}_n)$, $n = 1,\ 2,\ \ldots\ ,$ in $\mathbb{R}^s$ is defined by*

$$D_N(\mathbf{x}_n) = \sup_{\substack{\mathbf{y} \in U^s \\ \mathbf{y}=(y_1,\ \ldots\ ,y_s)}} \left| \frac{1}{N} \sum_{n=1}^{N} \chi(\mathbf{x}_n, \mathbf{y}) - y_1 \cdots y_s \right|.$$

where $\mathbf{y} = (y_1,\ \ldots\ ,y_s) \in U^s$.

Let $\boldsymbol{\alpha} = (\alpha_1,\ \ldots\ ,\alpha_s) \in \mathbb{R}^s$. Suppose that $1, \alpha_1, \ldots, \alpha_s$ are linearly independent over $\mathbb{Z}$.

Definition 5. *For a real number η, the vector $\boldsymbol{\alpha}$ is said to be of type η if η is the infimum of all real numbers τ for which there exists a positive constant $c = c(\tau, \alpha)$*

such that

$$r(\mathbf{h})^{\tau}\|\mathbf{h}\cdot\boldsymbol{\alpha}\| \geq c$$

holds for all lattice points $\mathbf{h} \neq \mathbf{0}$ *in* $\mathbb{R}^s$, *where* $r(\mathbf{h}) = \prod_{i=1}^{s}\max\{1, |h_i|\}$, $\mathbf{h} = (h_1, \ldots, h_s)$.

From Minkowski's linear form theorem, we have $\eta \geq 1$.

Definition 6. *The vector* $\boldsymbol{\alpha}$ *is said to be of constant type if there exists a positive constant* c *such that*

$$r(\mathbf{h})\|\mathbf{h}\cdot\boldsymbol{\alpha}\| \geq c$$

holds for all lattice points $\mathbf{h} \neq \mathbf{0}$ *in* $\mathbb{R}^s$.

1 Theorems and Examples

Tichy and Turnwald[4] proved the following:

Theorem 1. *For any* $\epsilon > 0$

$$D_N(\omega) \ll_{\alpha,\beta} N^{-\frac{1}{\eta+1}+\epsilon},$$

provided α *is an irrational number of finite type* $\eta \geq 1$.

In 1999, Ohkubo[3] improved Theorem 1 as follows :

Theorem 2. *If α is an irrational number of finite type $\eta \geq 1$, then for any $\epsilon > 0$*

$$D_N(\omega) \ll_\beta N^{-\frac{1}{\eta+1/2}+\epsilon},$$

and also if α is an irrational number of constant type, then

$$D_N(\omega) \ll_\beta N^{-\frac{2}{3}} \log N.$$

We found out an another proof of Theorem 2 and an extension to the multidimensional case as follows (see [1]):

Theorem 3. *Let ϵ be an arbitrary positive number, $\boldsymbol{\alpha} = (\alpha_1, \ \ldots \ , \alpha_s) \in \mathbb{R}^s$ and $\boldsymbol{\beta} = (\beta_1, \ \ldots \ , \beta_s) \in \mathbb{R}^s$ with $\boldsymbol{\beta} \neq \mathbf{0}$. If $1, \ \alpha_1, \ \ldots \ , \alpha_s$ are linearly independent over $\mathbb{Z}$ and $\boldsymbol{\alpha}$ is of finite type η , then we have*

$$D_N(n\boldsymbol{\alpha} + (\log n)\boldsymbol{\beta}) \ll_\beta N^{-\frac{1}{s(\eta-1)+3/2}+\epsilon}.$$

Theorem 4. *Let $\boldsymbol{\alpha} = (\alpha_1, \ \ldots \ , \alpha_s) \in \mathbb{R}^s$ and $\boldsymbol{\beta} = (\beta_1, \ \ldots \ , \beta_s) \in \mathbb{R}^s$ with $\boldsymbol{\beta} \neq \mathbf{0}$. If $1, \ \alpha_1, \ \ldots \ , \alpha_s$ are linearly independent over $\mathbb{Z}$ and $\boldsymbol{\alpha}$ is of constant type, then we have*

$$D_N(n\boldsymbol{\alpha} + (\log n)\boldsymbol{\beta}) \ll_\beta N^{-\frac{2}{3}}(\log N)^s.$$

Recently, we also generalized Theorem 2.

Theorem 5. *Let $f(x)$ be twice differentiable for $x \geq 1$. Suppose also that there exists an irrational number α of finite type η such that either*

$$f'(x) > \alpha, \ f''(x) < 0 \quad \text{or} \quad f'(x) < \alpha, \ f''(x) > 0 \quad \text{for } x \geq 1$$

and $f'(x) = \alpha + O(|f''(x)|^{1/2})$. *Then for any* $\epsilon > 0$

$$D_N(f(n)) \ll N^{-\frac{1}{\eta+1/2}+\epsilon}.$$

Proof. Let h be a positive integer. Applying [5, p.74, Lemma 4.7], we get

$$\left|\sum_{n=1}^{N} e^{2\pi i h f(n)}\right| \ll \sum_{A-1/2<\nu<B+1/2} \left|\int_1^N e^{2\pi i\{hf(x)-\nu x\}}dx\right| + \log(B-A+2),$$

where $A = hf'(N)$ and $B = hf'(1)$. We set $g(x) = h\{f(x) - \alpha x\}$. Using integration by parts, we have

$$\begin{aligned}\int_1^N e^{2\pi i\{hf(x)-\nu x\}}dx &= \int_1^N e^{2\pi i\{(h\alpha-\nu)x+g(x)\}}dx = \int_1^N e^{2\pi i(h\alpha-\nu)x}e^{2\pi i g(x)}dx \\ &= \left[\frac{e^{2\pi i(h\alpha-\nu)x}}{2\pi i(h\alpha-\nu)}e^{2\pi i g(x)}\right]_1^N - \frac{1}{h\alpha-\nu}\int_1^N g'(x)e^{2\pi i\{(h\alpha-\nu)+g(x)\}}dx.\end{aligned}$$

Hence,

$$\int_1^N e^{2\pi i\{hf(x)-\nu x\}}dx \ll \frac{1}{|h\alpha-\nu|} + \frac{1}{|h\alpha-\nu|}\left|\int_1^N g'(x)e^{2\pi i\{(h\alpha-\nu)x+g(x)\}}dx\right|.$$

We suppose that

$$f'(x) > \alpha \text{ and } f''(x) < 0 \text{ for } x \geq 1.$$

From [6, p.226, Lemma 10.5] and the hypothesis, it follows that

$$\left|\int_1^N g'(x)e^{2\pi i\{(h\alpha-\nu)x+g(x)\}}dx\right| \ll h^{1/2} \max_{1\leq x\leq N}\left\{\frac{f'(x)-\alpha}{|f''(x)|^{1/2}}\right\} + 1 \ll h^{1/2}.$$

Hence we have

$$\begin{aligned}\left|\sum_{n=1}^{N} e^{2\pi i h f(n)}\right| \ll & h^{1/2} \sum_{A-1/2<\nu<B+1/2} \frac{1}{|h\alpha-\nu|} + \log(B-A+2) \\ \ll & h^{1/2}\left\{\frac{1}{\|h\alpha\|} + \int_{\|h\alpha\|}^{h\{f'(1)-\alpha\}+1/2} \frac{1}{x}dx\right\} \\ & + \log[h\{f'(1)-f'(N)\}+2] \\ \ll & h^{1/2}\left\{\frac{1}{\|h\alpha\|} + \log\left[h\{f'(1)-\alpha\}+2\right]\right\}.\end{aligned}$$

Applying Erdös-Turán inequality and $\sum_{h=1}^{m} \frac{1}{h^{1/2}\|h\alpha\|} \ll m^{\eta-1/2+\delta}$ (see [2, p.123, Lemma 3.3]), for any positive integer m, we obtain

$$\begin{aligned}D_N(f(n)) \ll & \frac{1}{m} + \frac{1}{N}\left\{\sum_{h=1}^{m} \frac{1}{h^{1/2}\|h\alpha\|} + \sum_{h=1}^{m} \frac{\log[h\{f'(1)-\alpha\}+2]}{h^{1/2}}\right\} \\ \ll & \frac{1}{m} + \frac{1}{N}\left(m^{\eta-1/2+\delta} + m^{1/2}\log m\right) \\ \ll & \frac{1}{m} + \frac{1}{N} m^{\eta-1/2+\delta},\end{aligned}$$

for any $\delta > 0$.

Choosing $m = \left[N^{\frac{1}{\eta+1/2}}\right]$, we have

$$D_N(f(n)) \ll N^{-\frac{1}{\eta+1/2}} + N^{-\frac{1}{\eta+1/2}+\frac{\delta}{\eta+1/2}} \ll N^{-\frac{1}{\eta+1/2}+\epsilon}.$$

In the case $f'(x) < \alpha$, $f''(x) > 0$ for $x \geq 1$, the proof runs along the same lines as above. □

Remark 1. *The following was shown by van der Corput: If $f(x)$, $x \geq 1$, is differentiable for sufficiently large x and $\lim_{x\to\infty} f'(x) = \alpha$(irrational), then the sequence $(f(n))$ is uniformly distributed* mod 1 (*see* [2, p.28, Theorem 3.3 and p.31, Exercises 3.5]). *If the function $f(x)$ in Theorem 5 also satisfies the condition $\lim_{x\to\infty} f''(x) = 0$, then $\lim_{x\to\infty} f'(x) = \alpha$. Therefore, Theorem 5 gives a quantitative aspect of van der Corput's result.*

Examples. $f(x) = \alpha x + \beta \log\log x$, or $f(x) = \alpha x + \beta \log x$.

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