

SHARP ASYMPTOTICS FOR THE DAMPED WAVE EQUATIONS

室蘭工業大学・工学部 加藤正和 (Masakazu Kato)
Department of Engineering, Muroran Institute of Technology

神戸大学・海事科学部 上田好寛 (Yoshihiro Ueda)
Faculty of Maritime Sciences, Kobe University

1 Introduction

This note is concerned with large time behavior of the global solutions to the damped wave equations with a nonlinear convection term:

$$(1.1) \quad u_{tt} - u_{xx} + u_t + \alpha u_x + (f(u))_x = 0, \quad x \in \mathbb{R}, \quad t > 0,$$

$$(1.2) \quad u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x)$$

where $|\alpha| < 1$ and $f(u) = \frac{\beta}{2}u^2 + \frac{\gamma}{3!}u^3$. The subscripts t and x stand for the partial derivatives with respect to t and x , respectively. In Ueda and Kawashima [8], it was shown that solution of (1.1) and (1.2) tends to a nonlinear diffusion wave defined by

$$(1.3) \quad \chi(x, t) = \frac{1}{\sqrt{1+t}} \chi_* \left(\frac{x - \alpha(1+t)}{\sqrt{1+t}} \right), \quad x \in \mathbb{R}, \quad t \geq 0,$$

where

$$(1.4) \quad \chi_*(x) = \frac{\sqrt{\mu}}{\beta} \frac{(e^{\beta M/2\mu} - 1)e^{-\frac{x^2}{4\mu}}}{\sqrt{\pi} + (e^{\beta M/2\mu} - 1) \int_{x/\sqrt{4\mu}}^{\infty} e^{-y^2} dy},$$

$$(1.5) \quad M = \int_{\mathbb{R}} (u_0(x) + u_1(x)) dx, \quad \mu = 1 - \alpha^2.$$

By the Hopf-Cole transformation in Hopf [2] and Cole [1], we see that it is a solutions of the Burgers equation

$$(1.6) \quad \chi_t + \left(\alpha \chi + \frac{\beta}{2} \chi^2 \right)_x = \mu \chi_{xx}, \quad x \in \mathbb{R}, \quad t > 0,$$

satisfying

$$(1.7) \quad \int_{\mathbb{R}} \chi(x, 0) dx = M.$$

We set, for $1 \leq p \leq \infty$ and $s \geq 1$,

$$\begin{aligned} E_0^{(s,p)} &= \|u_0\|_{W^{s,p}} + \|u_0\|_{L^1} + \|u_1\|_{W^{s-1,p}} + \|u_1\|_{L^1}, \\ E_1^{(s,p)} &= \|u_0\|_{W^{s,p}} + \|u_0\|_{L_1^1} + \|u_1\|_{W^{s-1,p}} + \|u_1\|_{L_1^1}, \\ E_2^{(s,p)} &= E_1^{(s,p)} + E_1^{(2,1)}. \end{aligned}$$

Concerning the convergence rate of the nonlinear diffusion wave $\chi(x, t)$ to the original solution $u(x, t)$, we can infer the following result from the argument given in [8]: For any $\epsilon > 0$, if $u_0 \in W^{1,p} \cap L_1^1$ and $u_1 \in L^p \cap L_1^1$ and $E_0^{(1,p)}$ is small, then we have

$$(1.8) \quad \|\partial_x^l(u(\cdot, t) - \chi(\cdot, t))\|_{L^p} \leq C E_1^{(1,p)} (1+t)^{-\frac{1}{2}(2-\frac{1}{p}+l)+\epsilon}$$

for $l = 0, 1$. Here, $W^{s,p}$ denotes the space of functions $u = u(x)$ such that $\partial_x^l u$ are L^p -functions on $\mathbb{R}$ for $0 \leq l \leq s$, endowed with the norm $\|\cdot\|_{W^{s,p}}$, while $L_1^1(\mathbb{R})$ is subset of $L^1(\mathbb{R})$ whose elements satisfy $\|u\|_{L_1^1} \equiv \int_{\mathbb{R}} |u|(1+|x|)dx < \infty$.

This observation lead to a natural question whether it is possible to take $\epsilon = 0$ in (1.8) or not. The aim of note is to show that the optimal decay rate by studying the second asymptotic profile. Indeed, the second asymptotic profile of large time behavior of the solutions is given by

$$(1.9) \quad V(x, t) = -\kappa d V_* \left(\frac{(x - \alpha(1+t))}{\sqrt{1+t}} \right) (1+t)^{-1} \log(2+t), \quad t \geq 0, \quad x \in \mathbb{R},$$

where

$$(1.10) \quad V_*(x) = \frac{1}{\sqrt{4\pi}} \partial_x (\eta_*(x) e^{-\frac{x^2}{4}}),$$

$$(1.11) \quad \eta_*(x) = \exp \left(\frac{\beta}{2\mu} \int_{-\infty}^x \chi_*(y) dy \right),$$

$$(1.12) \quad \kappa = \frac{\alpha\beta^2}{4\mu} + \frac{\gamma}{3!}, \quad d = \int_{\mathbb{R}} \frac{1}{\eta_*(y)} \chi_*^3(y) dy.$$

Then we have the following result.

Theorem 1.1. *Let $s \geq 2$ and $1 \leq p \leq \infty$ and assume that $u_0 \in W^{s,p} \cap W^{2,1} \cap L_1^1$ and $u_1 \in W^{s-1,p} \cap W^{1,1} \cap L_1^1$. Let $u(x, t)$ be the global solution of the problem (1.1) and (1.2) constructed in Proposition 3.1. Then, If $E_0^{(s,p)} + E_0^{(2,1)}$ is small, then we have the following asymptotic relations:*

$$(1.13) \quad \|\partial_x^l(u(\cdot, t) - \chi(\cdot, t) - V(\cdot, t))\|_{L^p} \leq C E_2^{(s,p)} (1+t)^{-\frac{1}{2}(2-\frac{1}{p}+l)}.$$

for $0 \leq l \leq s-2$.

Using (1.10), (1.11) and (1.12), we can see that if $M \neq 0$, then $d \neq 0$, $V_*(x) \neq 0$. From (1.9), we have

$$(1.14) \quad \|\partial_x^l V(\cdot, t)\|_{L^p} = \kappa d \|\partial_x^l V_*(x)\|_{L^p} (1+t)^{-\frac{1}{2}(2-\frac{1}{p}+l)} \log(2+t).$$

Hence, we see from (1.13) and (1.14) that we can not take $\epsilon = 0$ in (1.8) unless $\kappa M \neq 0$. We remark that the estimate similar to (1.13) was obtained for Burgers equation such as the generalized Burgers equation in Kato [6] and KdV-Burgers in Hayashi and Naumkin [4] and Kaikina and Ruiz-Paredes [5], and Benjamin-Bona-Mathony-Burgers in Hayashi, Kaikina and Naumkin [3].

2 Basic estimates

To state the results, we introduce the modified heat kernel:

$$(2.1) \quad G_0(x, t) = \frac{1}{\sqrt{4\pi\mu t}} e^{-\frac{(x-\alpha t)^2}{4\mu t}}$$

which is the fundamental solution to the linear heat equation $w_t + \alpha w_x = \mu w_{xx}$. We show following two lemmas. The first one is concerned with $L^p - L^q$ estimate for the solution operator $G_0(t)*$. For the proof, see [8].

Lemma 2.1. *Let $1 \leq q \leq p \leq \infty$, and k and l be nonnegative integers. Then we have*

$$(2.2) \quad \|\partial_x^l \partial_t^k G_0(t) * \phi\|_{L^p} \leq C t^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p}+k+l)} \|\phi\|_{L^q}.$$

Also, if $\int \phi(x) dx = 0$, then we have

$$(2.3) \quad \|\partial_x^l \partial_t^k G_0(t) * \phi\|_{L^p} \leq C t^{-\frac{1}{2}(1-\frac{1}{p}+k+l)} (1+t)^{-\frac{1}{2}} \|\phi\|_{L_1^1}.$$

The second one is related to the diffusion wave $\chi(x, t)$. The explicit formula of $\chi(x, t)$ is given by (1.3). It is easy to see that

$$(2.4) \quad |\chi(x, t)| \leq C |M| (1+t)^{-\frac{1}{2}} e^{-(x-\alpha t)^2/(4\mu(1+t))}, \quad x \in \mathbb{R}, \quad t \geq 0.$$

Moreover, we get the following (see e.g. [7] and [8]).

Lemma 2.2. *Let k, l and m be nonnegative integers. If $|M| < 1$, then, for $1 \leq p \leq \infty$, the estimate*

$$(2.5) \quad \|(\partial_t + \alpha \partial_x)^m \partial_x^l \partial_t^k \chi(\cdot, t)\|_{L^p} \leq C |M| (1+t)^{-\frac{1}{2}(1-\frac{1}{p}+k+l+2m)}$$

holds.

For the latter sake, we introduce η defined by

$$(2.6) \quad \eta(x, t) \equiv \eta_* \left(\frac{x - \alpha(1+t)}{\sqrt{1+t}} \right) = \exp\left(\frac{\beta}{2\mu} \int_{-\infty}^x \chi(y, t) dy\right).$$

We easily have

$$(2.7) \quad \min\{1, e^{\frac{\beta M}{2\mu}}\} \leq \eta(x, t) \leq \max\{1, e^{\frac{\beta M}{2\mu}}\},$$

$$(2.8) \quad \min\{1, e^{-\frac{\beta M}{2\mu}}\} \leq \frac{1}{\eta(x, t)} \leq \max\{1, e^{-\frac{\beta M}{2\mu}}\}.$$

Moreover, we get the following corollary (see [6]).

Corollary 2.3. *Let l be a positive integer. If $|M| \leq 1$, then we have*

$$(2.9) \quad \|\partial_x^l \eta(\cdot, t)\|_{L^p} + \|\partial_x^l \frac{1}{\eta}(\cdot, t)\|_{L^p} \leq C|M|(1+t)^{-\frac{1}{2}(l-\frac{1}{p})}.$$

Next, we deal with the following linearized equation which corresponds to (3.6), (3.7) below:

$$(2.10) \quad z_t + (\alpha z + \beta \chi z)_x = \mu z_{xx}, \quad x \in \mathbb{R}, \quad t > \tau,$$

$$(2.11) \quad z(x, \tau) = z_0(x).$$

The explicit representation formula (2.12), and the decay estimate (2.14) and (2.15) below play a crucial role in our analysis. For the proof of Lemma 2.4, Lemma 2.5 and Lemma 2.6, see [6].

Lemma 2.4. *If we set*

$$(2.12) \quad U[w](x, t, \tau) = \int_{\mathbb{R}} \partial_x(\eta(x, t) G_0(x - y, t - \tau)) \frac{1}{\eta(y, \tau)} \int_{-\infty}^y w(\xi) d\xi dy, \\ x \in \mathbb{R}, \quad 0 \leq \tau < t,$$

then the solutions for (2.10) and (2.11) is given by

$$(2.13) \quad z(x, t) = U[z_0](x, t, \tau), \quad x \in \mathbb{R}, \quad t > \tau.$$

Lemma 2.5. *Let $1 \leq p \leq \infty$ and l be a nonnegative integer. Assume that $|M| \leq 1$, $z_0 \in L_1^1(\mathbb{R})$ and $\int_{\mathbb{R}} z_0(x) dx = 0$. Then, the estimate*

$$(2.14) \quad \|\partial_x^l U[z_0](\cdot, t, 0)\|_{L^p} \leq C t^{-\frac{1}{2}(2-\frac{1}{p}+l)} \|z_0\|_{L_1^1}, \quad t > 0$$

holds.

Lemma 2.6. *Let $1 \leq p \leq \infty$ and l be a nonnegative integer. Assume that $|M| \leq 1$, $z_0 \in W^{l,p}(\mathbb{R}) \cap L^1(\mathbb{R})$. Then the estimate*

$$(2.15) \quad \|\partial_x^l U[z_0](\cdot, t, 0)\|_{L^p} \leq C(1+t)^{-\frac{1}{2}(1-\frac{1}{p}+l)} (\|z_0\|_{L^1} + \|z_0\|_{W^{l,p}})$$

holds.

From Lemma 2.5 and Lemma 2.6, we get the following uniform estimate.

Corollary 2.7. *Let $1 \leq p \leq \infty$ and l be a nonnegative integer. Assume that $|M| < 1$, $z_0 \in L_1^1(\mathbb{R}) \cap W^{l,p}$ and $\int_{\mathbb{R}} z_0(x)dx = 0$. Then, the estimate*

$$(2.16) \quad \|\partial_x^l U[z_0](\cdot, t, 0)\|_{L^p} \leq C(1+t)^{-(2-\frac{1}{p}+l)/2} (\|z_0\|_{L_1^1} + \|z_0\|_{W^{l,p}}), \quad t > 0$$

holds.

In order to prove Proposition 3.5, we prepare the following lemma.

Lemma 2.8. *Let $1 \leq p \leq \infty$ and l be a nonnegative integer. Suppose $|M| \leq 1$. Then the estimates*

$$(2.17) \quad \|\partial_x^l U[\partial_x w](x, t, \tau)\|_{L^p} \leq C(t-\tau)^{-(2-\frac{1}{p}+l)/2} \left\| \frac{1}{\eta} w(\cdot, \tau) \right\|_{L^1},$$

$$(2.18) \quad \|\partial_x^l U[\partial_x w](x, t, \tau)\|_{L^p} \leq C(t-\tau)^{-1/2} \sum_{m=0}^l (1+t)^{-(l-m)/2} \|\partial_x^m \left(\frac{1}{\eta} w \right)(\cdot, \tau)\|_{L^p}$$

hold.

PROOF. From (2.12), we have

$$(2.19) \quad \begin{aligned} & \partial_x^l U[\partial_x w](x, t, \tau) \\ &= \sum_{n=0}^{l+1-n} {}_l C_n \partial_x^{l+1-n} \eta(x, t) \int_{\mathbb{R}} \partial_x^n G_0(x-y, t-\tau) \frac{1}{\eta(y, \tau)} w(y, \tau) dy. \end{aligned}$$

From Corollary 2.3, it follows that

$$(2.20) \quad \|\partial_x^l U[\partial_x w(\tau)](x, t, \tau)\|_{L^p} \leq C \sum_{n=0}^{l+1} (1+t)^{-(l+1-n)/2} \|\partial_x^n I(\cdot, t, \tau)\|_{L^p},$$

where we put

$$(2.21) \quad I(x, t, \tau) = \int_{\mathbb{R}} G_0(x-y, t-\tau) \frac{1}{\eta(y, \tau)} w(y, \tau) dy.$$

First, we shall prove (2.17). From Lemma 2.1, we have

$$(2.22) \quad \|\partial_x^n I(\cdot, t, \tau)\|_{L^p} \leq C(t-\tau)^{-(1-\frac{1}{p}+n)/2} \left\| \frac{1}{\eta} w(\cdot, \tau) \right\|_{L^1}.$$

Therefore, by (2.20) and (2.22), we obtain (2.17).

Next, we shall prove (2.18). From Lemma 2.1, we have

$$(2.23) \quad \|I(\cdot, t, \tau)\|_{L^p} \leq C\|w(\cdot, \tau)\|_{L^p}.$$

In the following, let $1 \leq n \leq l + 1$. From (2.2) and (2.9), it follows that

$$(2.24) \quad \|\partial_x^n I(\cdot, t)\|_{L^p} \leq C(t - \tau)^{-\frac{1}{2}} \|\partial_x^{n-1}(\frac{1}{\eta}w)(\cdot, \tau)\|_{L^p}.$$

Therefore, by (2.20), (2.23) and (2.24), we obtain (2.18). This completes the proof. $\square$

3 Proof of Theorem 1.1

We prepare the following two propositions concerning the decay rate and the asymptotic profile of the solution to the problem (1.1) and (1.2). All proposition stated below were proved mainly in [8].

Proposition 3.1. *Let $s \geq 2$, $1 \leq p \leq \infty$. Assume that $u_0 \in W^{s,p} \cap L^1$, $u_1 \in W^{s-1,p} \cap L^1$ and $E_0^{(s,p)}$ is small. Then the initial value problem for (1.1) and (1.2) has a unique global solution $u(x, t)$ with*

$$u(x, t) \in \begin{cases} C([0, \infty); W^{s,p} \cap L^1), & 1 \leq p < \infty, \\ L^\infty((0, \infty); W^{s,\infty}) \cap C([0, \infty); L^1), & p = \infty. \end{cases}$$

Moreover, the solution satisfies

$$(3.1) \quad \|u(\cdot, t)\|_{L^1} \leq CE_0^{(s,p)},$$

$$(3.2) \quad \|\partial_x^l \partial_t^k u(\cdot, t)\|_{L^p} \leq CE_0^{(s,p)} (1+t)^{-\frac{1}{2}(1-\frac{1}{p}+k+l)}$$

for $k = 0, 1, 2$ and $0 \leq k + l \leq s$.

Proposition 3.2. *Let $s \geq 2$, $1 \leq p \leq \infty$ and assume that $u_0 \in W^{s,p} \cap L_1^1$ and $u_1 \in W^{s-1,p} \cap L_1^1$. Let $u(x, t)$ be the global solution of the problem for (1.1) and (1.2) constructed in Proposition 3.1. For any $\epsilon > 0$, if $E_0^{(s,p)}$ is small, then we have*

$$(3.3) \quad \|u(\cdot, t) - \chi(\cdot, t)\|_{L^1} \leq CE_1^{(s,p)} (1+t)^{-\frac{1}{2}+\epsilon},$$

$$(3.4) \quad \|\partial_x^l \partial_t^k (u - \chi)(\cdot, t)\|_{L^p} \leq CE_1^{(s,p)} (1+t)^{-\frac{1}{2}(2-\frac{1}{p}+k+l)+\epsilon}$$

for $k = 0, 1, 2$ and $0 \leq k + l \leq s$.

Corollary 3.3. *Assume the same condition as Proposition 3.2. For any $\epsilon > 0$, if $E_0^{(s,p)}$ is small, then we have*

$$(3.5) \quad \|\partial_x^l (\partial_t + \alpha \partial_x)(u - \chi)(\cdot, t)\|_{L^p} \leq CE_1^{(s,p)} (1+t)^{-\frac{1}{2}(4-\frac{1}{p}+l)+\epsilon}$$

for $0 \leq l \leq s - 2$.

PROOF. We get from (1.1) and (1.6)

$$\begin{aligned} (\partial_t + \alpha \partial_x)(u - \chi) &= (-\partial_t^2 + \partial_x^2)(u - \chi) - \frac{\beta}{2}(u^2 - \chi^2)_x - \left(\frac{\gamma}{3!}u^3\right)_x \\ &\quad - (\partial_t - \alpha \partial_x)(\partial_t + \alpha \partial_x)\chi. \end{aligned}$$

Hence we derive (3.5) from Proposition 3.2, Proposition 3.1, Lemma 2.2 and (1.6). This completes the proof. $\square$

In order to prove our result, we introduce the following auxiliary problem:

$$(3.6) \quad v_t + (\alpha v + \beta \chi v)_x - \mu v_{xx} = -\kappa(\chi^3)_x, \quad x \in \mathbb{R}, \quad t > 0,$$

$$(3.7) \quad v(x, 0) = 0.$$

Here κ is defined by (1.12). We show the asymptotics of the solution of the problem (3.6) and (3.7). For the proof of Proposition 3.4, see [6].

Proposition 3.4. *Assume that $|M| \leq 1$. Then the estimate*

$$(3.8) \quad \|\partial_x^l(v(\cdot, t) - V(\cdot, t))\|_{L^p(\mathbb{R})} \leq C |M|^3 (1+t)^{-(2-\frac{1}{p}+l)/2}$$

holds.

To prove Theorem 1.1, it is sufficient to show Proposition 3.5 below by virtue of Proposition 3.4.

Proposition 3.5. *Let $s \geq 2$, $1 \leq p \leq \infty$ and assume that $u_0 \in W^{s,p} \cap W^{2,1} \cap L_1^1$ and $u_1 \in W^{s-1,p} \cap W^{1,1} \cap L_1^1$. Let $u(x, t)$ be the global solution of the problem (1.1) and (1.2) constructed in Proposition 3.2. If $E_0^{(s,p)} + E_0^{(2,1)}$ is small, then we have the following asymptotic relations:*

$$(3.9) \quad \|\partial_x^l(u(\cdot, t) - \chi(\cdot, t) - v(\cdot, t))\|_{L^p} \leq C E_2^{(s,p)} (1+t)^{-\frac{1}{2}(2-\frac{1}{p}+l)}$$

for $0 \leq l \leq s-2$.

PROOF. From (2.6), we obtain

$$\kappa \chi^3 = \left(\frac{\alpha \beta^2}{4\mu} + \frac{\gamma}{3!} \right) \chi^3 = 2\alpha(\beta \chi \chi_x - \mu \chi_{xx}) + \frac{\gamma}{3!} \chi^3 + \eta \left(\frac{1}{\eta} g_1(\chi) \right)_x,$$

where

$$(3.10) \quad g_1(\chi) = 2\alpha \mu \chi_x - \frac{\alpha \beta}{2} \chi^2.$$

We put

$$(3.11) \quad w(x, t) = u(x, t) + u_t(x, t) - \chi(x, t) - v(x, t).$$

Then $w(x, t)$ satisfies

$$(3.12) \quad w_t + (\alpha w + \beta \chi w)_x - \mu w_{xx} = (g(u, \chi))_x, \quad x \in \mathbb{R}, \quad t > 0,$$

$$(3.13) \quad w(x, 0) = w_0(x),$$

where we have set $w_0(x) = u_0(x) + u_1(x) - \chi_0(x)$ and

$$(3.14) \quad \begin{aligned} g(u, \chi) &= \eta \left(\frac{1}{\eta} g_1(\chi) \right)_x + g_2(u, \chi) \\ g_2(u, \chi) &= \alpha(\partial_t + \alpha \partial_x)(u - \chi) + \beta \chi(u_t + \alpha \chi_x) - \mu(u_t + \alpha \chi_x)_x \\ &\quad - \left\{ \frac{\beta}{2}(u - \chi)^2 + \frac{\gamma}{3!}(u^3 - \chi^3) \right\}. \end{aligned}$$

Since $u_0(x), u_1(x), \chi(x, 0) \in W^{s-1,p} \cap W^{1,1} \cap L_1^1$, we have $w_0(x) \in W^{s-1,p} \cap W^{1,1} \cap L_1^1$. Besides by (1.5) and (1.7),

$$(3.16) \quad \int_{\mathbb{R}} w_0(x) dx = 0.$$

To prove (3.9), it is sufficient to show the decay estimate (3.17) below by Proposition 3.1, Lemma 2.2, Proposition 3.4 and (1.14).

$$(3.17) \quad \|\partial_x^l w(\cdot, t)\|_{L^p} \leq C E_2^{(s,p)} t^{-\frac{1}{2}(2-\frac{1}{p}+l)}.$$

For nonnegative integer m and $1 \leq q \leq \infty$, we get

$$(3.18) \quad \begin{aligned} \|\partial_x^m \left(\frac{1}{\eta} g_1(\chi) \right)(\cdot, t)\|_{L^q} &\leq C \sum_{k=0}^m (1+t)^{-\frac{m-k}{2}} \|\partial_x^k g_1(\cdot, t)\|_{L^q} \\ &\leq C |M| (1+t)^{-\frac{1}{2}(2-\frac{1}{q}+m)}. \end{aligned}$$

We shall show that for $0 \leq m \leq s-2$

$$(3.19) \quad \left\| \frac{1}{\eta} g_2(\cdot, t) \right\|_{L^1} \leq C E_1^{(2,1)} (1+t)^{-\frac{3}{2}+\epsilon},$$

$$(3.20) \quad \left\| \partial_x^m \left(\frac{1}{\eta} g_2 \right)(\cdot, t) \right\|_{L^p} \leq C E_1^{(s,p)} (1+t)^{-\frac{1}{2}(4-\frac{1}{p}+m)+\epsilon}.$$

We shall prove only (3.19), since we can prove (3.20) in a similar way. From Lemma 2.2, Proposition 3.1, Proposition 3.2 and Corollary 3.3, we have

$$(3.21) \quad \begin{aligned} \|\chi(\cdot, t)(u_t + \alpha \chi_x)(\cdot, t)\|_{L^1} &\leq \|\chi(\cdot, t)\|_{L^\infty} (\|\partial_t(u - \chi)(\cdot, t)\|_{L^1} + \|(\partial_t + \alpha \partial_x)\chi(\cdot, t)\|_{L^1}) \\ &\leq C E_1^{(2,1)} (1+t)^{-\frac{3}{2}+\epsilon}, \end{aligned}$$

$$(3.22) \quad \begin{aligned} \|(u_t + \alpha \chi_x)_x(\cdot, t)\|_{L^1} &\leq \|\partial_x \partial_t(u - \chi)(\cdot, t)\|_{L^1} + \|\partial_x(\partial_t + \alpha \partial_x)\chi(\cdot, t)\|_{L^1} \\ &\leq C E_1^{(2,1)} (1+t)^{-\frac{3}{2}+\epsilon}, \end{aligned}$$

$$\begin{aligned}
\|(u - \chi)^2(\cdot, t)\|_{L^1} &\leq \|(u - \chi)(\cdot, t)\|_{L^\infty} \|(u - \chi)(\cdot, t)\|_{L^1} \\
(3.23) \quad &\leq CE_1^{(2,1)}(1+t)^{-\frac{3}{2}+\epsilon},
\end{aligned}$$

$$\begin{aligned}
\|(u^3 - \chi^3)(\cdot, t)\|_{L^1} &\leq C(\|u^2(\cdot, t)\|_{L^\infty} + \|\chi^2(\cdot, t)\|_{L^\infty}) \|(u - \chi)(\cdot, t)\|_{L^1} \\
(3.24) \quad &\leq CE_1^{(2,1)}(1+t)^{-\frac{3}{2}+\epsilon},
\end{aligned}$$

Summing up these estimate, we obtain (3.19) from (3.15) and Corollary 3.3. Applying the Duhamel principle for the problem (3.12) and (3.13), we have from Lemma 2.4

$$(3.25) \quad w(x, t) = U[w_0](x, t, 0) + \int_0^t U[\partial_x g(u, \chi)(\tau)](x, t, \tau) d\tau, \quad x \in \mathbb{R}, \quad t > 0.$$

For $l \leq s - 2$, we have from (3.25), Corollary 2.7 and Lemma 2.8

$$\begin{aligned}
\|\partial_x^l w(\cdot, t)\|_{L^p} &\leq CE_2^{(s,p)}(1+t)^{-\frac{1}{2}(2-\frac{1}{p}+l)} \\
&+ C \sum_{m=0}^{l+1} \|\partial_x^{l+1-m} \eta(\cdot, t)\|_{L^\infty} \int_0^{t/2} \|\partial_x^{m+1} G_0(t-\tau) * (\frac{1}{\eta} g_1(\chi))(\cdot, \tau)\|_{L^p} d\tau \\
&+ C \int_0^{t/2} (t-\tau)^{-\frac{1}{2}(2-\frac{1}{p}+l)} \|\frac{1}{\eta} g_2(\cdot, \tau)\|_{L^1} d\tau \\
&+ C \sum_{m=0}^l \int_{t/2}^t (t-\tau)^{-\frac{1}{2}} (1+t)^{-(l-m)/2} \|\partial_x^m (\frac{1}{\eta} g)(\cdot, \tau)\|_{L^p} d\tau \\
(3.26) \quad &\equiv I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

First we evaluate I_2 . We have from Corollary 2.3 and Lemma 2.1

$$\begin{aligned}
I_2 &\leq C \sum_{m=0}^{l+1} (1+t)^{-\frac{l+1-m}{2}} \int_0^{t/2} (t-\tau)^{-\frac{1}{2}(2-\frac{1}{p}+m)} \|\frac{1}{\eta} g_1(\chi)(\cdot, \tau)\|_{L^1} d\tau \\
&\leq C |M| t^{-\frac{1}{2}(3-\frac{1}{p}+l)} \int_0^{t/2} (1+\tau)^{-\frac{1}{2}} d\tau \\
(3.27) \quad &\leq C |M| t^{-\frac{1}{2}(2-\frac{1}{p}+l)}.
\end{aligned}$$

Next we evaluate I_3 . From (3.19), we have

$$\begin{aligned}
I_3 &\leq CE_1^{(2,1)} t^{-\frac{1}{2}(2-\frac{1}{p}+l)} \int_0^{t/2} (1+\tau)^{-\frac{5}{4}} d\tau \\
(3.28) \quad &\leq CE_1^{(2,1)} t^{-\frac{1}{2}(2-\frac{1}{p}+l)}.
\end{aligned}$$

Finally we evaluate I_4 . From (3.20), it follows that

$$\begin{aligned}
I_4 &\leq CE_1^{(s,p)} \int_{t/2}^t (t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{1}{2}(3-\frac{1}{p}+l)} d\tau \\
(3.29) \quad &\leq CE_1^{(s,p)} (1+t)^{-\frac{1}{2}(2-\frac{1}{p}+l)}.
\end{aligned}$$

Summing up these estimates, we obtain (3.17). This completes the proof. $\square$

References

- [1] J.D. Cole: *On a quasi-linear parabolic equation occurring in aerodynamics*, *Quart. Appl. Math.* **IX** (1951), 225–236.
- [2] E. Hopf: *The partial differential equation $u_t + uu_x = \mu u_{xx}$* , *Comm. Pure Appl. Math.* **3** (1950), 201–230.
- [3] N. Hayashi, E.I. Kaikina and P.I. Naumkin: *Large time asymptotics for the BBM-Burgers equation*, *Ann. Henri Poincaré*. **8** (2007), no.3, 485–511.
- [4] N. Hayashi and P.I. Naumkin: *Asymptotics for the Korteweg-de Vries-Burger Equation*, *Acta Math. Sin Engl. Ser.* **22** (2006), no.5, 1441–1456.
- [5] E.I. Kaikina and H.F. Ruiz-Paredes: *Second term of asymptotics for KdVB equation with large initial data*, *Osaka J. Math.* **42** (2005), 407–420.
- [6] M. Kato: *Large time behavior of solutions to the generalized Burgers equations*, *Osaka J. Math.* **Vol.44**, No.4 (2007), 923–943.
- [7] A. Matsumura and K. Nishihara: *Global Solution of Nonlinear Differential Equations-Mathematical Analysis for Compressible Viscous Fluids*, Nippon-Hyoron-Sha, Tokyo, 2004, (in Japanese).
- [8] Y.Ueda, and S.Kawashima: *Large time behavior of solutions to a semilinear hyperbolic system with relaxation*, *J. Hyperbolic Differ Equ.* **4** (2007), 147–179.