

## On the Orthogonal $L_1$ Linear Approximation of Points

Peter Yamamoto\*†, Keiko Imai‡ and Hiroshi Imai\*

\*Department of Computer Science and Communication Engineering  
Kyushu University, Fukuoka 812, Japan

†School of Computer Science, McGill University  
Montreal, PQ, Canada H3A 2K6

‡Department of Mathematical Engineering  
University of Tokyo, Tokyo 113, Japan

### Abstract

This paper presents algorithms for approximating a set of  $n$  points by a linear function, or a line, that minimizes the  $L_1$  norm of orthogonal distances. The algorithms find exact solutions based upon geometric properties of the problems as opposed to approximate solutions based upon existing numerical techniques. The algorithmic complexity of these problems appears not to have been investigated, although  $O(n^3)$  naive algorithms can be easily obtained based on some simple characteristics of optimal  $L_1$  solutions. In this paper, an  $O(n^{1.5} \log^2 n)$  algorithm is presented for the unweighted orthogonal problem, and an  $O(n^2)$  algorithm is presented for the weighted problem. An  $\Omega(n \log n)$  lower bound for the orthogonal  $L_1$  problem is shown under a certain model of computation. Also, the complexity of solving the orthogonal  $L_1$  problem is related to the construction of the  $k$ -belt of an arrangement of lines.

### 1. Introduction

Approximating a set of  $n$  points in the plane by a linear function, or a line, called the line-fitting problem, is a fundamental problem in scientific computing encountered in many fields, including statistics, econometrics, location theory, and signal-processing. Recently, the line-fitting problem and its variations have been considered from an algorithmic point of view in those areas. However, the algorithms presented are, for the most part, brute force and involve enumeration of all possible candidate solutions. Solutions to some of these problems, on the other hand, have inherent geometrical properties which have generated interest from the area of discrete and computational geometry.

The nature of the problem depends on three factors: the distance function used to measure the distance from a point to a line, the norm of the distance function used, and whether the distances are weighted by associating a weight to each point. The unweighted case corresponds to the situation where all weights are equal to one. Vertical,  $d_v$ , or orthogonal,  $d_o$ , distances are commonly used as the distance function (see Figure 1.1), although other measures, such as the rectangular distance, are also of interest.

The most frequently used norm is the  $L_2$  norm which may be efficiently solved by the least squares method for both vertical and horizontal distances. However, the  $L_2$  norm is not always the most appropriate criteria for a "best" fit. The two most popular alternatives are the  $L_1$  and the  $L_\infty$  (or Chebyshev) norms; however, their use in practice has been limited due to the lack of efficient algorithms (see the conclusion for a brief historical note). The full paper provides details and references concerning the applicability of the above approaches to particular problems.

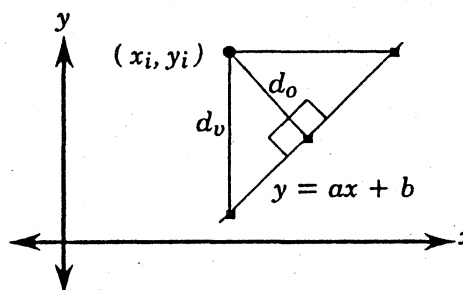


Figure 1.1. Vertical and orthogonal distances.

Megiddo [10] notes that by applying his multidimensional search technique to the vertical  $L_\infty$  problem, formulated as a linear programming problem, an exact solution may be found in linear time. Recently, Lee and Wu [9] provide optimal and efficient algorithms for the orthogonal  $L_\infty$  problem and some of its variations.

Optimal solutions to both the vertical and orthogonal  $L_1$  problems are known in statistics as robust estimators: the solutions are not easily influenced (relative to solutions to the  $L_2$  and  $L_\infty$  problems) by outliers, or noise, in the data (see Figure 1.2). For that reason, in applications in which the data is subject

to error, for example, signal processing, the  $L_1$  norms may be much more preferable than the more popular  $L_2$  norms. Also, from the viewpoint of location problems, the orthogonal  $L_1$  norm measures the total Euclidean distance of the points from the line.

The unweighted vertical  $L_1$  problem may be solved by general numerical approximation methods, such as the method of descent, or it may be formulated as a linear programming problem in higher dimensions [3]. [8] notes an  $O(n^3)$  naive algorithm for finding an exact solution to the weighted vertical  $L_1$  problem based on characteristic properties of optimal solutions. In the case of the unweighted orthogonal  $L_1$  problem, [13] presents a numerical algorithm which corresponds to a concave quadratic programming algorithm. [11] presents an  $O(n^3)$  naive algorithm for finding an exact solution to the weighted orthogonal  $L_1$  problem based on characteristic properties of optimal solutions.

The complexity of the vertical and orthogonal  $L_1$  problems appears not to have been investigated from an algorithmic point of view before our work in [8]. The problems differ only by the divisor  $\sqrt{a^2 + 1}$  which converts the vertical distance into the orthogonal distance; however, the divisor is an important factor in determining the complexity of the problems. [8] presents a linear time algorithm for the unweighted vertical  $L_1$  problem; however, the results in this paper indicate that algorithms for the orthogonal  $L_1$  problems lie in a different complexity class.

This paper is concerned with the unweighted and weighted orthogonal  $L_1$  linear approximation problems in the plane. The weighted problem may be stated as follows.

**Problem 1.1. The Orthogonal  $L_1$  Problem.** Given a set,  $S$ , of  $n$  points,  $p_i \equiv (x_i, y_i)$  ( $i = 1, \dots, n$ ), in the  $(x, y)$ -plane, with corresponding weights,  $w_i$ , find a pair of values  $(a^*, b^*)$ , for the parameters  $a$  and  $b$ , which solves the following mini-sum problem:

$$\min_{a,b} \sum_{i=1}^n w_i \frac{|y_i - (ax_i + b)|}{\sqrt{a^2 + 1}}.$$

The unweighted problem corresponds to setting all the weights equal to one.

The algorithms presented in this paper find exact solutions to both the weighted and unweighted orthogonal  $L_1$  problems, as opposed to approximate solutions derived by numerical approximation techniques. The algorithms represent a significant improvement over the previous  $O(n^3)$  naive algorithms. The complexities of the orthogonal  $L_1$  problems are related to the construction of the  $k$ -belts of an arrangement of lines. The unweighted orthogonal  $L_1$  problem is solved in  $O(n^{1.5} \log^2 n)$  time based upon an algorithm for constructing  $k$ -belts. The weighted orthogonal  $L_1$  problem is solved in  $O(n^2)$  time using the topological sweep algorithm of [5].

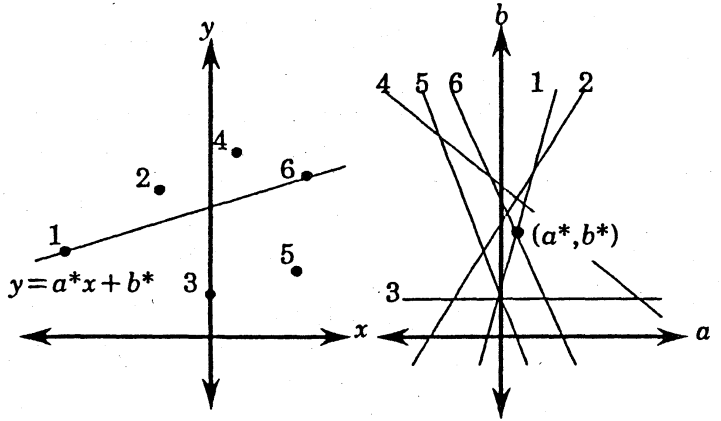


Figure 1.2. Effect of outliers on optimal solutions.

## 2. Preliminaries

This section defines the basic notation and concepts used in this paper. All of the algorithms make use of a point-line, or dual, transformation which is defined below. Some general notation is also introduced.

The transformation from the original, or  $(x, y)$ -, plane into the dual plane may be performed as follows (see Figure 2.1). The point  $(x', y')$  and the line  $y = a'x + b'$  in the  $(x, y)$ -plane are mapped to the line  $b = -x'a + y'$  and the point  $(a', b')$ , respectively, in the  $(a, b)$ -plane.

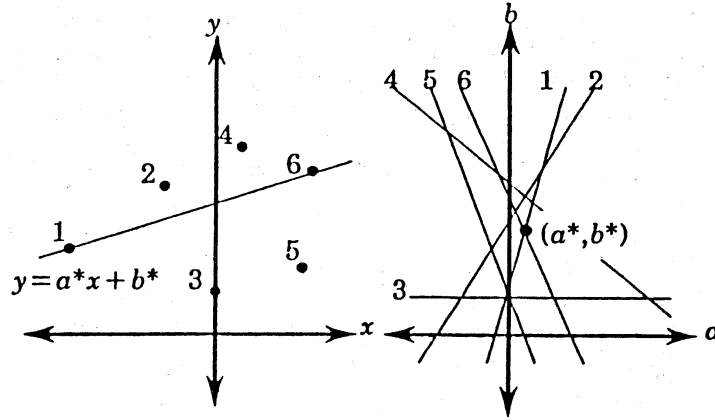


Figure 2.1. Primal-dual transformation.

The following notation is used for describing the relative position of a point with respect to a line. A point  $(x', y')$  lies above, on, or below, the line  $y = a'x + b'$  if  $y' - (a'x' + b')$  is, respectively, greater than, equal to, or less than, zero. If the line is defined by  $x = a'$  (a vertical line), then  $(x', y')$  lies to the right of, on, or to the left of, the line  $x = a'$  if  $x' - a'$  is, respectively, greater than, equal to, or less than, zero. The definitions also hold for a point  $(a', b')$  and a line  $(b = x'a + y')$  in the  $(a, b)$ -plane.

The duality preserves the vertical distance (hence, also the above-below) and incidence relations between points and lines. A vertical line in one plane is conceptually mapped to a point at infinity in the other plane.

The value of the orthogonal  $L_1$  norm will be referred to as

$$D_o(a, b) \equiv \sum_{i=1}^n w_i \frac{|y_i - (ax_i + b)|}{\sqrt{a^2 + 1}}.$$

The absolute value signs may be eliminated by considering the value of the function for fixed values of the parameters  $a$  and  $b$ . For fixed  $a$  and  $b$ , define sets  $I_A(a, b)$  and  $I_B(a, b)$  of indices by:

$$I_A(a, b) = \{i \mid y_i > ax_i + b \quad (i = 1, \dots, n)\}$$

and

$$I_B(a, b) = \{i \mid y_i < ax_i + b \quad (i = 1, \dots, n)\}.$$

Note that the above notation may be interpreted as the set of indices of data points  $(x_i, y_i)$  above and below a line defined by  $(a, b)$  in the  $(x, y)$ -plane, or as the corresponding set of indices of data lines defined by  $(-x_i, y_i)$  above and below a point  $(a, b)$  in the  $(a, b)$ -plane.

$D_o(a, b)$  may then be written as follows.

$$\begin{aligned} D_o(a, b) &= \sum_{i=1}^n w_i \frac{|y_i - (ax_i + b)|}{\sqrt{a^2 + 1}} \\ &= \sum_{i \in I_A(a, b)} w_i \frac{y_i - (ax_i + b)}{\sqrt{a^2 + 1}} - \sum_{i \in I_B(a, b)} w_i \frac{y_i - (ax_i + b)}{\sqrt{a^2 + 1}} \\ &= \frac{a(X_B - X_A) + b(W_B - W_A) + Y_A - Y_B}{\sqrt{a^2 + 1}} \end{aligned}$$

where

$$\begin{aligned} W_A &= \sum_{i \in I_A} w_i, & X_A &= \sum_{i \in I_A} w_i x_i, & Y_A &= \sum_{i \in I_A} w_i y_i, \\ W_B &= \sum_{i \in I_B} w_i, & X_B &= \sum_{i \in I_B} w_i x_i, & Y_B &= \sum_{i \in I_B} w_i y_i. \end{aligned}$$

Hence, given the values of  $W_A, X_A, Y_A, W_B, X_B$  and  $Y_B$ , the value of  $D_o(a, b)$  can be computed in constant time.

Suppose the values of the above variables are computed for some fixed parameter values  $a'$  and  $b'$ . If the next computation is for new values,  $a''$  and  $b''$ , such that the above-below relationships of the data points and the lines defined by  $y = a'x + b'$  and  $y = a''x + b''$  do not change, then the values of those variables also do not change. Hence, the objective function for the parameters  $a''$  and  $b''$  can be computed in constant time. The above representation scheme is used in all the algorithms for maintaining the contribution of the data points to the objective function.

Morris and Norback [11] present two characteristic properties of an optimal orthogonal  $L_1$  problem solution.

**Lemma 2.1.** *There is an approximate line to the point set  $S$  which minimizes the orthogonal  $L_1$  norm and which passes through two points of  $S$ .*

Lemma 2.1 suggests an  $O(n^3)$  brute force approach to solving the problem. For each of the possible  $\binom{n}{2}$  pairs of points, evaluate the function for a line which passes through the pair.

**Lemma 2.2.** *The sum of the weights on the optimal approximation line is greater than the difference of the sums of weights on either side of the line.*

Both Lemma 2.1 and Lemma 2.2 also hold in  $d$ -dimensions and can be solved by an  $O(n^d)$ -time and  $O(n)$ -space algorithm. The details will appear in a later paper.

Morris and Norback suggest using both properties to find candidate solutions, pairs of data points in the  $(x, y)$ -plane (or pairs of lines in the  $(a, b)$ -plane) which satisfy both properties. They suggest the brute force approach of inspecting each of the possible  $\binom{n}{2}$  pairs to see if the line defined by the pair satisfies the second property; if so, then compute the  $L_1$  norm with respect to the candidate line defined by the pair of points. Clearly, that algorithm is not a great improvement over computing the  $L_1$  norm for every possible pair. However, the approach does raise the question of how many candidate pairs there are and what is the complexity of finding them. That question can be related to the number of  $k$ -sets as described below.

Given a set  $S$  of  $n$  points, a  $k$ -set of  $S$  is a subset  $S'$  such that  $S'$  contains  $k$  points and there exists a line,  $l_k$ , which separates  $S'$  from  $S - S'$ . The number of  $k$ -sets of a set of  $n$  points has been shown to have complexities of  $\Omega(n \log k)$  and  $O(nk^{1/2})$  [7]. The computation of  $k$ -sets is studied in [6]. In that paper, they introduce the notion of  $k$ -belts in the dual plane.

For each point  $p$  in the dual plane, let  $b(p)$ ,  $o(p)$ , and  $a(p)$  denote the number of lines which lie below  $p$ , on  $p$ , and above  $p$ , respectively. Clearly  $b(p) + o(p) + a(p) = n$ , for all  $p$ . The  $k$ -belt, for  $0 \leq k \leq \lfloor n/2 \rfloor$ , of the arrangement of lines in the dual plane is defined as the set of points  $p$  in the dual plane such that  $b(p) + o(p) \geq k$  and  $a(p) + o(p) \geq k$ . The 0-belt is the whole plane, and for  $k \geq 1$ , the  $k$ -belt is bounded above and below by an unbounded monotone polygonal chain. For  $n$  odd and  $k = \lfloor n/2 \rfloor$ , the two boundaries of the  $\lfloor n/2 \rfloor$ -belt, referred to as the median-belt, coincide. The bold line in Figure 2.2(b) represents the median-belt for the seven lines shown in Figure 2.2(a).

In the unweighted problem, Lemma 2.2 is just a cardinality problem: the difference in the number of points above and below the line must not exceed the number of points on the line. The number of candidate lines is related to the number of  $k$ -sets, or "median"-sets since  $k = \lfloor n/2 \rfloor$ , of a set of  $n$  points in the  $(x, y)$ -plane. Similarly, the number of candidate lines in the weighted problem is related to the number of "weighted"  $k$ -sets, or *weighted median-sets*, since the line must satisfy the second property which balances the weights equally on each side of the line.

### 3. The Unweighted Problem

In the unweighted orthogonal  $L_1$  problem, the dual transformations of the candidate lines correspond to the vertices of the  $\lfloor n/2 \rfloor$ -belt, or median-belt (see Figure 2.2). The algorithm presented below evaluates the function at each of those vertices in order to determine an optimal value (the existence is guaranteed by Lemma 2.2). The function evaluations can be performed in the same amount of time as it takes to construct the median-belt.

The construction of the  $k$ -belt of an arrangement,  $H$ , of  $n$  lines can be performed in  $O(b_k(n) \log^2 n)$  time and  $O(n)$  space, where  $b_k(n)$  is the maximum number of vertices in the  $k$ -belt for all arrangements,  $H$ , of  $n$  lines [6]. In the unweighted orthogonal  $L_1$  problem,  $k = \lfloor n/2 \rfloor$

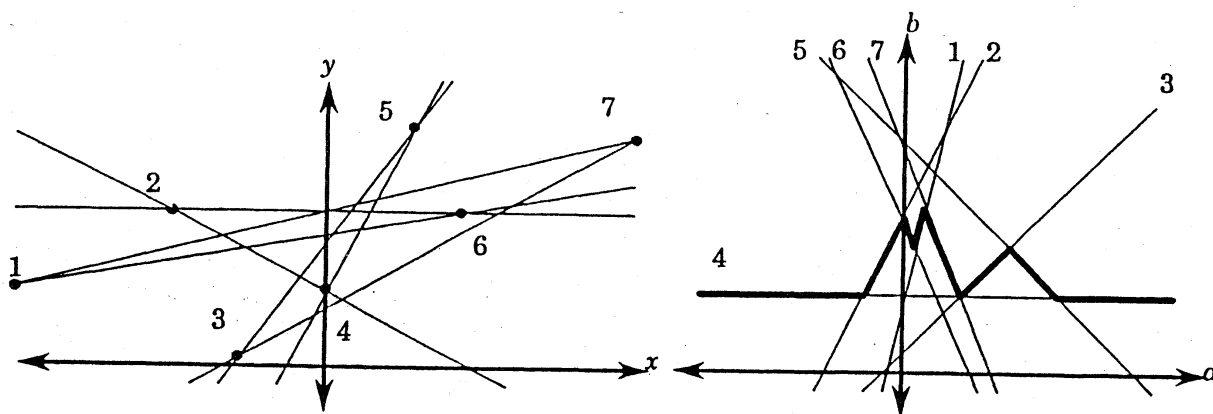


Figure 2.2.(a) Candidates in the  $(x, y)$ -plane. Figure 2.2.(b) Candidates in the  $(x, y)$ -plane.

and  $b_{\lceil n/2 \rceil}(n) = O(n^{1.5})$ .

The algorithm sweeps the arrangement by a vertical line  $L$  from left to right. At  $a = -\infty$ , the data line,  $l_m$ , with the median slope also has the median intersection with  $L$ . Let  $H_A$  be a set of data lines above  $l_m$  and similarly  $H_B$  a set of data lines below  $l_m$  with respect to the median intersection point. Hence, the sets  $I_A$  and  $I_B$ , as defined in Section 2, refer to the indices of the lines in  $H_A$  and  $H_B$ , respectively, and the variables used to maintain  $D_o(a, b)$  (see Section 2) can be initialized in linear time.

The algorithm then sweeps the arrangement to find the next vertex on the belt which is the intersection of  $l_m$  and another data line  $l_A$ , where without loss of generality the line  $l_A$  is assumed to be in the set  $H_A$ . The value of  $D_o(a, b)$  may be computed at the new vertex in constant time as follows. Subtract  $x_A$  from  $X_A$  and  $y_A$  from  $Y_A$ . Add  $x_m$  to  $X_A$  and  $y_m$  to  $Y_A$ . Those operations take constant time and  $D_o(a, b)$  may be computed in constant time with the updated values. The algorithm keeps a pointer to the swept vertex which minimizes  $D_o(a, b)$ ; clearly, such a pointer can be updated in constant time. Thus, each step of the plane sweep can be done without increasing the complexity of the  $k$ -belt construction algorithm. Since  $O(n^{1.5} \log^2 n)$  time is spent in constructing the median-belt, the following result is obtained.

**Theorem 3.1.** *The unweighted orthogonal  $L_1$  linear approximation problem for  $n$  points can be solved in  $O(n^{1.5} \log^2 n)$  time and  $O(n)$  space.*

The above algorithm, although efficient, is still an exhaustive search of all the candidate solutions. Note that the algorithm is for constructing the general  $k$ -belt; for at least one value of  $k$ ,  $k = 1$ , a more efficient algorithm,  $O(n \log n)$  time, can be obtained by considering the particular problem. Hence, the authors conjecture that a more efficient algorithm may be obtained by considering the particular properties of the median-belt.

#### 4. The Weighted Problem

In the dual plane, the candidate solutions to the weighted orthogonal  $L_1$  problem lie on the boundary of the weighted median-belt of the arrangement, the set of points which are the dual transformations of the weighted median lines in the  $(x, y)$ -plane. Hence, the complexity of finding a solution can be related to the number of vertices (due to Lemma 2.2),  $n_w$ , on the boundary of the weighted median-belt.

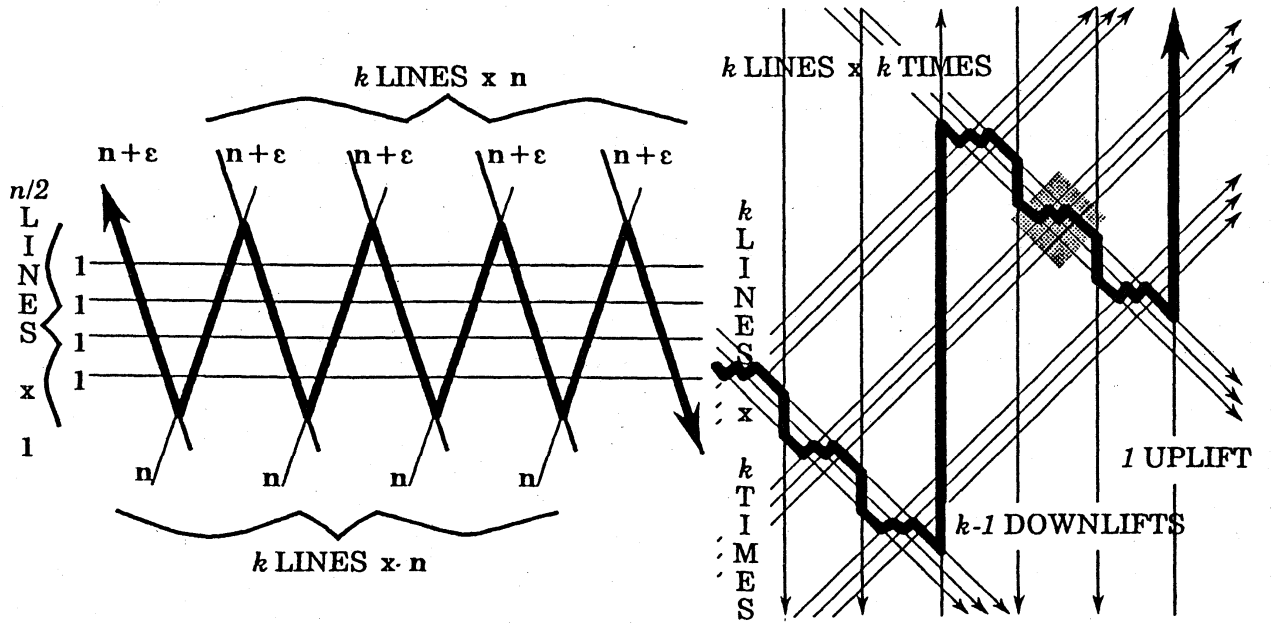


Figure 4.1.(a)  $\Omega(n^2)$  Degenerate vertices. Figure 4.1.(b)  $\Omega(n^{1.5})$  Non-degenerate vertices.

The complexity of  $n_w$  depends on how a vertex of the median-belt is defined. A vertex can be described as either any point on the boundary of the weighted median-belt which is incident to more than one line of the arrangement (called *degenerate vertices*), or any point on the boundary of the weighted median-belt whose incident boundary edges are distinct (*nondegenerate vertices*) (see Figure 4.1(a),(b)). Note that the nondegenerate vertices are a subset of the degenerate vertices.

Since the weighted median-belt is an  $x$ -monotone chain, the number of vertices can be bounded by results for monotone chains. The number of degenerate vertices is  $\Omega(n^2)$  [14] (see Figure 4.1(a)). The number of non-degenerate vertices is  $\Omega(n^{1.5})$  [12] (see Figure 4.1(b),  $n = \sqrt{k}$ , shaded area has  $k^2$  vertices); although an (unknown at the time of printing) improvement to this lower bound has been reportedly made by R. Cole and J. Matoušek [4].

Note, however, that not all  $x$ -monotone chains are weighted median-belts. In the case of the example for the degenerate bound, legal weight assignments (bold numbers) can be given to the lines, however, the particular example given for the nondegenerate bound cannot be assigned weights such that the chain becomes a weighted belt.

The geometry of the (unweighted) median-belt and of the weighted median-belt can be quite different. In the unweighted case, the median-belt only has nondegenerate vertices since the belt switches lines at every intersection. In the weighted case, however, the weighted median-belt may not switch at an intersection if the current edge has a relatively large weight.

An algorithm based only on the properties given by Lemma 2.1 and Lemma 2.2 must check the objective function at all degenerate vertices. Hence, the complexity of applying the  $k$ -belt construction algorithm, mentioned above, to the weighted problem is in  $O(n^2 \log^2 n)$ . However, the weighted orthogonal  $L_1$  problem can be solved in  $O(n^2)$  time by the following "efficient" brute force algorithm.

An optimal solution is found by performing an  $O(n^2)$  time,  $O(n)$  space plane sweep as described in [5]. Note that this plane sweep differs from the plane sweep algorithm used for the unweighted case; the plane sweep used above only computes points of interest in order from left to right, whereas the plane sweep used here reports all  $O(n^2)$  vertices of the arrangement in an order defined by a topological sweep from left to right.

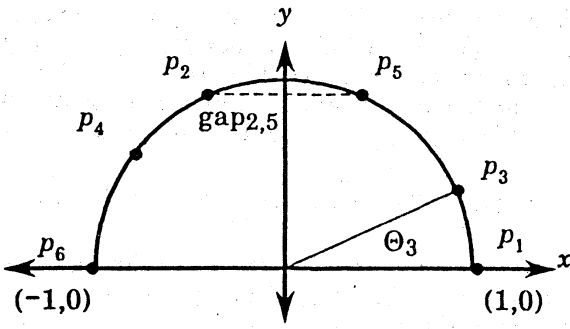


Figure 5.1. Uniform gap problem on the half unit circle.

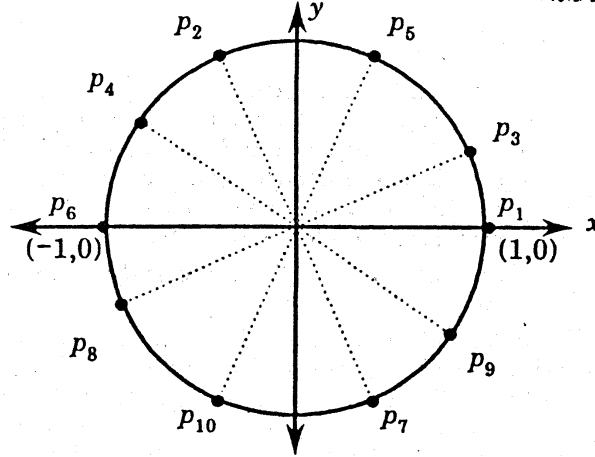


Figure 5.2. Transformed input.

First, note that the leftmost edge of the weighted median-belt can be determined in linear time. Second, the plane sweep algorithm insures that, theoretically, each vertex is incident to only two lines; hence, determining whether a switch to a new line should take place at a vertex of the belt can be decided in constant time by maintaining  $D_o(a, b)$  as described in Section 2. Third, since the next event vertex in the sweep is identified by its incident edges (lines), the vertex can be tested to see if it is on the belt by simply comparing the current edge of the belt to the incident lines in constant time.

Since the plane sweep is performed in  $O(n^2)$  time and the extra cost to maintain the weighted median-belt and the function  $D_o(a, b)$  is  $O(1)$  time for each vertex, an optimal solution may be found in  $O(n^2)$  time. The algorithm is called an efficient brute force algorithm since the computations are performed efficiently, reducing the complexity of a pure brute force search by a factor of  $O(n)$ . Also note that the above algorithm implies that the algorithm for constructing  $k$ -belts used in the unweighted problem is not, at least in one case, the most efficient algorithm for computing weighted  $k$ -belts.

## 5. $\Omega(n \log n)$ Lower Bound

This section provides an  $\Omega(n \log n)$  lower bound for the orthogonal  $L_1$  problem under the algebraic computation tree model [2]. (Due to space limitations not all the proofs are included in this paper). First, the lower bound of the complexity of a certain uniform gap problem of  $n$  points on a circle is shown to be in  $\Omega(n \log n)$  under the algebraic computation tree model. Second, that uniform gap problem is reduced to the unweighted orthogonal  $L_1$  problem in linear time, thus proving an  $\Omega(n \log n)$  lower bound for the orthogonal  $L_1$  problem.

The uniform gap problem on the unit circle is defined as:

**Problem 5.1. Uniform Gap Problem on the (Half) Unit Circle.** Given  $m+1$  points  $p_1, p_2, \dots, p_m, p_{m+1}$  on the unit circle  $x^2 + y^2 = 1$  such that  $p_1 = (1, 0)$ ,  $p_{m+1} = (-1, 0)$  and the  $y$ -coordinates of points  $p_i$  ( $i = 2, \dots, m$ ) are positive (see Figure 5.1), the uniform gap problem on the unit circle answers whether the distances of every consecutive pair of points are equal when these  $m+1$  points are arranged in increasing order of their polar angles.

Denote the  $(x, y)$ -coordinates of point  $p_i$  by  $(x_i, y_i)$ , where  $x_1 = 1$ ,  $x_{m+1} = -1$  and  $y_i \geq 0$ . For a permutation  $\sigma$  on  $\{2, \dots, m\}$ , define  $W_\sigma \subset \mathbb{R}^{m-1}$  by

$$W_\sigma = \left\{ (x_2, \dots, x_m) \in \mathbb{R}^{m-1} \mid 1 = x_1 > x_{\sigma(2)} > x_{\sigma(3)} > \dots > x_{\sigma(m)} > x_{m+1} = -1, \right. \\ \left. (x_{\sigma(i+1)} - x_{\sigma(i)})^2 + (\sqrt{1 - x_{\sigma(i+1)}^2} - \sqrt{1 - x_{\sigma(i)}^2})^2 = \epsilon^2 \right\},$$

where  $\sigma(1) = 1$  and  $\sigma(m+1) = m+1$ . Define  $W \subset \mathbb{R}^{m-1}$  by  $W = \bigcup_\sigma W_\sigma$  where the union is

taken over all permutations  $\sigma$  on  $\{2, \dots, m\}$ . Then the answer for the uniform gap problem for points  $p_1, p_2, \dots, p_{m+1}$  is yes if and only if  $(x_2, \dots, x_m) \in W$ , which implies that, by Ben-Or's theorem [2], a lower bound for the complexity of the uniform gap problem under the algebraic computation tree is  $\Omega(\log \#W)$  where  $\#W$  is the number of connected components of  $W$ .  $\#W$  can be shown to be equal to  $(m-1)!$ , hence, the following result holds under the algebraic computational model of Ben-Or.

**Lemma 5.1.** *The complexity of the uniform gap problem on the unit circle is  $\Omega(m \log m)$ .*

Next, the input for the uniform gap problem on a circle is transformed, in linear time, to an input for the orthogonal linear  $L_1$  approximation of points. Given the points  $p_1, p_2, \dots, p_{m+1}$  of the uniform gap problem, assume that, without loss of generality,  $0 = \theta_1 < \theta_2 < \dots < \theta_m < \theta_{m+1} = \pi$ , where  $\theta_i$  denotes the polar angle of  $p_i$ . For each  $p_i$  ( $i = 2, \dots, m$ ), construct the point  $p_{i+m}$  on the unit circle whose polar angle  $\theta_{i+m}$  is equal to  $\theta_i + \pi$  (see Figure 5.2). The set,  $S$ , of  $n = 2m$  points  $p_1, p_2, \dots, p_n$  is then used as the input for the orthogonal  $L_1$  problem. The transformation is then completed by showing the following result.

**Lemma 5.2.** *The minimum objective function value of the orthogonal  $L_1$  linear approximation for the set  $S$  of points is at most  $2 \cot \frac{\pi}{n}$ , and is  $2 \cot \frac{\pi}{n}$  if and only if the answer for the uniform gap problem of points  $p_1, p_2, \dots, p_m, p_{m+1}$  is yes.*

The proof is developed as follows. For the orthogonal linear  $L_1$  approximation problem, it is known that there is an optimal approximation line such that the line passes two points among the given points and  $|N_A - N_B| < N_O$  where  $N_A$ ,  $N_B$  and  $N_O$  are the numbers of points above, below, and on the line [11]. By the definition of the transformed problem, there is an optimal approximation line among the  $m$  lines  $l_i$  connecting points  $p_i$  and  $p_{i+m}$  ( $i = 1, \dots, m$ ). The function value of  $l_i$ , the summation of the orthogonal distances from points  $p_j$  ( $j = 1, \dots, n$ ) to  $l_i$ , is given by

$$\sum_{j=1}^n |\sin(\theta_j - \theta_i)|. \quad (1)$$

Hence, the minimum function value of the orthogonal linear  $L_1$  approximation for  $S$  is given by

$$\min_{i=1, \dots, m} \sum_{j=1}^n |\sin(\theta_j - \theta_i)|. \quad (2)$$

We are considering to maximize (2) for  $0 = \theta_1 < \theta_2 < \dots < \theta_m < \pi$  and  $\theta_{i+m} = \theta_i$  ( $i = 1, \dots, m$ ). However, this is an optimization problem of maximin type, and rather difficult to handle directly. Instead, we will consider to maximize

$$\sum_{i=1}^m \sum_{j=1}^n |\sin(\theta_j - \theta_i)|. \quad (3)$$

Here, observe that the function values for lines  $l_i$  are the same when the set  $S$  of points are uniformly placed on the circle. Hence, if (3) is maximized when and only when the set  $S$  of points are uniformly placed, then (2) is maximized only in the same uniform case.

Let us prove that (3) is maximized when and only when the set  $S$  of points are uniformly placed. (3) is further expressed as

$$\sum_{i=1}^m \sum_{j=1}^n |\sin(\theta_j - \theta_i)| = 2 \sum_{i=1}^m \sum_{j=i+1}^{i+m} \sin(\theta_j - \theta_i) = 2 \sum_{k=1}^m \left( \sum_{i=1}^m \sin(\theta_{i+k} - \theta_i) \right)$$

For any  $k = 1, \dots, m$ , we have

$$0 < \theta_{i+k} - \theta_i < \pi, \sum_{i=1}^m (\theta_{i+k} - \theta_i) = k\pi.$$

Since  $\sin x$  is strictly concave on the interval  $[0, \pi]$ , we have

$$\sum_{i=1}^m \sin(\theta_{i+k} - \theta_i) \leq m \sin \frac{k\pi}{m}$$

and the equality holds if and only if all  $\theta_{i+k} - \theta_i$  ( $i = 1, \dots, m$ ) are equal. All  $\theta_{i+k} - \theta_i$  ( $i = 1, \dots, m$ ) are equal for any  $k = 1, \dots, m$  if and only if all  $\theta_{i+1} - \theta_i$  ( $i = 1, \dots, m$ ) are equal. Hence, (3) is maximized when and only when the set  $S$  of points are located uniformly on the circle.

When the set  $S$  of points are placed uniformly, the function value of the orthogonal linear  $L_1$  approximation is expressed by

$$\sum_{j=1}^n \left| \sin \frac{j\pi}{m} \right| = 2 \sum_{j=1}^m \sin \frac{j\pi}{m} = 2 \cot \frac{\pi}{2m} = 2 \cot \frac{\pi}{n}.$$

Thus, we have shown Lemma 5.2.

Using the above lemmas, the following result provides a lower bound for the orthogonal  $L_1$  problem.

**Theorem 5.1.** *Under the algebraic computation tree model, the complexity of the orthogonal  $L_1$  linear approximation of  $n$  points is  $\Omega(n \log n)$ .*

This result is mainly of interest to the unweighted orthogonal problem since the actual bound for the algorithm presented above is in  $O(b_k(n) \log^2 n)$ , where  $b_k(n)$  is the number of  $k$ -sets of  $n$  points ( $k = \lceil n/2 \rceil$ ). As mentioned above, the bounds for  $b_k(n)$  are in  $\Omega(n \log k)$  and  $O(nk^{1.5})$  ( $\Omega(n \log n)$  and  $O(n^2)$ , respectively, for  $k = \lceil n/2 \rceil$ ), but Erdős, Lovász, Simmons, and Strauss [7] conjecture that the upper bound is actually closer to the lower bound of  $\Omega(n \log k)$  ( $\Omega(n \log n)$  for  $k = \lceil n/2 \rceil$ ).

## 6. Conclusion

Several results were given concerning the computation and analysis of the unweighted and weighted orthogonal  $L_1$  linear approximation problems. The results are significant from theoretical, practical, and historical viewpoints.

The results are of interest from a theoretical viewpoint since they relate the complexities of the problems to  $k$ -sets and  $k$ -belts. The complexities of those problems is an open problem in combinatorial geometry. The results presented here motivate further improvements on the bounds for the number of vertices in a  $k$ -belt. Another open problem is an improved algorithm for constructing the median belt.

The results are of practical interest since the algorithms provide efficient algorithms for solving the most popular forms of the  $L_1$  approximation problem. The results are of particular interest for the linear facility problem and the linear regression problem since the algorithms provide practical and efficient alternatives to the currently used methods (for example, the  $L_2$  and  $L_\infty$  approximations).

The results presented here, and also in [8], are of historical interest since this is the last of the three most popular  $L_p$  approximation problems ( $p = 1, 2, \infty$ ) to succumb to efficient algorithms. [3] notes that alternative criteria to the  $L_2$  norm have been investigated since the mid-1750s when R.J. Boscovitch proposed a geometric method for solving a special case of the  $L_1$  approximation problem. Interestingly enough, the efficient algorithms for both the  $L_1$  and the  $L_\infty$  problems have been derived from applying basic paradigms used in computational geometry.

Continuing research includes the  $L_1$  problem in higher dimensions, which is of particular interest to econometricians since they often consider the linear and non-linear  $L_p$  problems in higher dimensions [1]. Solutions to those problems under parallel processing models is also being considered. Furthermore, the paper has raised some questions about the complexity of computing median-belts and weighted median-belts.

### Acknowledgements

The authors would especially like to acknowledge the contributions and references provided by Emo Welzl while he was visiting Kyushu University. The authors are indebted to Professor Yahiko Kambayashi of Kyushu University for supplying a productive research environment. The first author acknowledges the support of a Japanese Government (Monbusho) Scholarship under which this research is being carried out.

### References

- [1] G. Appa and C. Smith: On  $L_1$  and Chebyshev estimation, *Mathematical Programming*, 5 (1973), 73–87.
- [2] M. Ben-Or: Lower bounds for algebraic computation trees. *Proc. 15th ACM Symp. on Theory of Computing*, 1983, 80–86.
- [3] V. Chvátal: *Linear programming*. Freeman Press, 1983.
- [4] R. Cole and J. Matoušek, result communicated by E. Welzl.
- [5] H. Edelsbrunner and L.J. Guibas: Topologically sweeping an arrangement, Digital Systems Research Center, April, 1986.
- [6] H. Edelsbrunner and E. Welzl: Constructing belts in two-dimensional arrangements with applications. *SIAM J. Comput.*, 15 (1986), 271–284.
- [7] P. Erdős, L. Lovász, A. Simmons and E. G. Straus: Dissection graphs of planar point sets. In “*A Survey of Combinatorial Theory*” (J. N. Srivastava et al., eds.), North-Holland, 1973, 139–149.
- [8] H. Imai, K. Kato, and P. Yamamoto: A linear time algorithm for linear  $L_1$  approximation of points, Kyushu University Technical Report CSCE-87-C30, August, 1987.
- [9] D.T. Lee and Y.F. Wu: Geometric complexity of some location problems. *Algorithmica*, 1 (1986), 193–211.
- [10] N. Megiddo: Linear programming in linear time when the dimension is fixed. *J. ACM*, 31 (1984), 114–127.
- [11] J. G. Morris and J. P. Norback: A simple approach to linear facility location. *Transportation Science*, 14 (1980), 1–8.
- [12] M. Sharir, result communicated by E. Welzl.
- [13] H. Späth and G.A. Watson: On orthogonal linear  $L_1$  approximation, *Numer. Math.*, 51 (1987), 531–543.
- [14] E. Welzl, personal communication.