

ON 3-MANIFOLDS UP TO BLOWING-UP

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INTRODUCTION

This paper has been expressly written for these Proceedings. It contains essentially the text of my talk , together with the report of some facts occurred after a first version entitled "Classification of 3-manifolds up to blowing-up and existence of rational real algebraic structures" containing complete proofs of the main results has circulated .

I was originally motivated , actually many years ago , to introduce and study smooth manifolds up to blowing-up along smooth centres , by an old question posed by J.Nash. In his pioneering paper [Na] he asked , among other , if every compact closed connected smooth manifold of dimension n admitted any structure of real algebraic variety birationally equivalent to S^n (see also [iv]) . As blowing-up along non singular centres is the basic construction in algebraic geometry to produce birationally equivalent varieties , and as this construction naturally extends to the smooth case as blowing-up along smooth centres , it suggested to me , by analogy , to consider the following equivalence relation on the set V_n of compact closed connected smooth manifolds of dimension n .

1.1 DEFINITION $X, Y \in V_n$ are said to be m -equivalent if there exists a finite sequence :

$$X = M_0 \leftrightarrow M_1 \leftrightarrow M_2 \leftrightarrow \dots \leftrightarrow M_k = Y$$

where each $\leftrightarrow$ is either a diffeomorphism or a blowing-up (or down) along a smooth centre (i.e. along a manifold $N \in V_m$, $m < n$, eventually embedded in M_i or in M_{i+1}) .

In other words we mean each blowing-up (down) as a simple modification and we consider the relation generated by diffeomorphisms and such simple modifications .

Thus a smooth analogous of Nash's question arises :

is every $X \in V_n$ m -equivalent to S^n ?

Probably more reasonably , one could ask :

how does the quotient set $V_n / \sim_m$ look ?

1.2 REMARK The relationship between Nash's question and its smooth analogous should be seen just at the analogy level . The implications between them is a subtler matter. Let $X \in V_n$ be a non singular projective real algebraic variety birationally equivalent to S^n . Are X and S^n m -equivalent as smooth manifolds ?

An answer should presumably make use of Hironaka's theorem on the resolution of singularities. However , even among people having familiarity with Hironaka's work , I have encountered two attitudes : someone seems to consider such a positive answer as a "moral" corollary of Hironaka proof ; someone else seems to be much more prudent ; shortly I did not get a definitive answer .

On the other hand , we shall see in the present paper that even in low dimension , the fact that $X \in V_n$ is m -equivalent to S^n allows only a weak answer to Nash's question (in fact the best possible) : actually Nash asked for non singular real algebraic models , and we get such models having in general non empty singular set .

I can state now the main results of the paper .

THEOREM A (main theorem-weak formulation) Every $M \in V_3$ is m -equivalent to S^3 .

For the strong formulation see the next paragraph 2 .

As an application I get

THEOREM B (An answer in low dimension to Nash's question)
Every $M \in V_3$ is homeomorphic to some affine real algebraic variety M' , having , in general , non empty singular set Σ , and containing a proper algebraic subset Y ($\Sigma \subset Y$) such that $M' - Y$ is algebraically isomorphic to a Zariski open set of S^3 .

The analogous results for V_2 hold in a easier way .

Actually we can obtain some more precision : we may get Σ of dimension 1 (0 for V_2) and the homeomorphism as a "stratified smooth isomorphism" with respect to the stratification $(M' - \Sigma, \Sigma)$ of M'

Moreover , the singularities seem to be not eliminable , in the general case ; for example , in dimension 2 :

a non singular rational real algebraic surface in V_2 is necessarily homeomorphic to S^2 or to the 2-torus , providing it is orientable (see [Kh] and also [S]) .

The case of V_3 was the first interesting open one ; for V_2 the result is an easy consequence of the classification of surfaces up to diffeomorphism . Moreover , it was not hopeless : a general well-known method to construct all the manifolds of V_3 (that is via surgery along framed links in S^3 or in $S^2 \times S^1$, the twisted S^2 -bundle over S^1 , according to the orientability or non-orientability) looked very suited to approach the problem : in fact one had , in particular , to blow-up along embedded circles. It was a nice fact (but actually not so hard , all one needed is contained in the simple lemma 3.1. below) to remark that the well-known Kirby's calculus on framed links , which preserves the diffeomorphism type , could be quite naturally extended to a very "flexible" m-calculus preserving the m-equivalence type . I remarked it with Alexis Marin ; using it we constructed several non-trivial examples of 3-manifolds m-equivalent to S^3 (enough to conjecture the main result) ; but after some fruitless attempts we were contented to collect our remarks (not so strong in our opinion) in the note published at the Dipartimento di Mat. of Pisa ([BM]) .

It has turned out that essentially the same path had been followed , independently and with the same motivations , by S.Akbulut who also remarked the m-calculus and used it to produce examples ; this is what one can deduce from the Abstract of A.M.S. (1/30/1987 832-57-118) .

Since 1987 no progress had been made .

On last September I was aware of Nakanishi's paper ([N]) concerning (unframed) links in S^3 up to Fox congruences : I realized

soon that his result for the congruence mod(2,1) , together with the m-calculus actually gave a proof of the main theorem in the orientable case. By a non immediate extra-work not depending on Nakanishi's theorem in [N] , I reduced the non-orientable case to the orientable case , finally completing the proof .

This is the proof I had presented in my talk .

While it has to my eyes the nice feature of the first complete solution of a longstanding problem I had been interested in from the very beginning of its own formulation , however it could be considered not completely satisfactory : one could try to find a more self-contained proof , relaxing the dependence on [N] ; moreover Nakanishi's theorem , while being very nice , is largely proved by "pictures" in 2-dimensional space of links projections . One could look for a more conceptual and purely 3-dimensional proof .

After our symposium I had realized that the argument of the reduction from the non-orientable to the orientable case , could be used to get a proof of the main theorem , at least in its weak formulation , without using the m-calculus and Nakanishi's result ; but it was not yet completely satisfactory .

Finally I received a letter (dated 3 Nov.1989) from Alexis Marin , containing several friendly comments on my paper and a sketch of a proof , extracting the basic 3-dimensional meaning of the business , not using [N] , which I consider the definitive proof as it is short and clean . I will reproduce this sketch .

On the other hand , I believe that the formulation of the results in terms of m-calculus (see the parag. 4 and 5), allowed by the use of [N] , maintains its own interest and deserves to be pointed out .

I take the occasion to thank the JSPS for giving me the occasion to spend some time in Japan , the Nagoya University and mostly Masahiro Shiota for the friendly hospitality .

2. Strong formulation of the main theorem .

Decompose $V_3 = V_3^+ \cup V_3^-$, "+" means orientable and "-" non-orientable . For every $M \in V_3$ let $b = b_M = \dim H_2(M, \mathbb{Z}_2)$.

In fact we shall obtain :

THEOREM A⁺ (main theorem-strong formulation for V_3^+)

For every $M \in V_3^+$, there exist a link L in M and a link L' in $\#_{1 \leq i \leq b} \mathbb{P}^3$ (i.e. the connected sum of b copies of the projective 3-space) such that by blowing-up along the components of L and L' respectively one obtains diffeomorphic 3-manifolds .

Note that $\#_i \mathbb{P}^3$ is nothing else than the result of a blowing-up of S^3 at b points .

Before stating the analogous result for V_3^- we need a definition.

2.1. DEFINITION A simple tower of blowing-up of lenght s over S^3 is obtained as follows : consider an ordered family $C_1, \dots, C_s$ of unknots in S^3 , looking like in this picture :



We mean that at the common point C_i and C_{i+1} have independent tangents . Then make a sequence of blowing-up :

$$S^3 = B_0 \leftarrow B_1 \leftarrow B_2 \leftarrow \dots \leftarrow B_s$$

such that C_1 is the centre of the first one π_1 , the "strict transform" of C_{i+1} in B_i (i.e. the closure in B_i of $(\pi_i \dots \pi_1)^{-1}(C_{i+1} - C_i)$) is the centre of π_{i+1} .

For example $S^2 \times S^1$ is diffeomorphic to B_1 , $\mathbb{P}^2 \times S^1$ to B_2 .

THEOREM A⁻ (main theorem-strong formulation for V_3^-)

For every $M \in V_3^-$, there exist a link L in M and a link L' in $B_s \# (\#_{1 \leq i \leq h} \mathbb{P}^3)$, $s+h=b$, every component of L and L' being a non-reversing orientation circle, such that by blowing-up along the components of L and L' respectively, one obtains diffeomorphic 3-manifolds .

The proof shall give further nice corollaries . We limit to remark the following one, not at all evident a priori .

Let us denote by m^+ -equivalence the relation generated on V_3 by diffeomorphisms and blowing-up along non-reversing orientation embedded circles . In particular no blowing-up at points is allowed . Then we have :

2.2 COROLLARY m -equivalence and m^+ -equivalence coincide on V_3 . That is every $M \in V_3$ is m^+ -equivalent to S^3 .

3. The m -calculus.

Dehn surgery data on a manifold $M \in V_3$ consist of $\Delta = (C, U, m, \ell, q/p)$ where : C is an embedded circle in M with U as orientable tubular neighbourhood ; m and ℓ are fixed meridian and longitude in ∂U ; q/p is a rational number with $(q, p) = 1$ and also $\infty = q/0$ is allowed . The manifold M_Δ obtained from M by surgery with data Δ , is well defined (up to diffeo.) as :

$$M_\Delta = \overline{M-U} \cup_h U$$

$h: \partial U \rightarrow \partial U$ being any diffeomorphism such that $h_*([m]) = q[m] + p[\ell]$ in $\pi_1(\partial U)$. The key remark is contained in the following simple lemma .

3.1 LEMMA Let A, B be obtained from $M \in V_3$ by surgeries with data differing only by q/p and b/a . Let A', B' be the blowing-up of A and B respectively along C . Assume that either :

- 1) $p \equiv a \equiv 0 \pmod{2}$; or
- 2) $p \equiv a \equiv 1 \pmod{2}$ and $q \equiv b \pmod{2}$.

Then there exists a diffeomorphism between A' and B' which is the identity on $M-U$.

The proof is an easy consequence of the fact that the inverse image of U , via the blowing-up , U' say , is diffeomorphic to $\mathcal{M} \times S^1$, $\mathcal{M}$ being the Moebius strip , and that a diffeomorphism $g: \partial U' \rightarrow \partial U'$ extends to a diffeomorphism $G: U' \rightarrow U'$ if $g_*([m]) \equiv [m] \pmod{2}$.

By definition , a framed link in $M \in V_3$, is a link L embedded in M with a system of surgery data for each its component , with the restriction that q/p is actually equal to $q/1$. If a component C of a link L is contained in an embedded ball , then its framing is completely determined by an integer number , that is the linking number $lk(C, C')$ between C and its suitable "parallel" C' .

It is well-known $([L_1], [L_2])$, that every orientable $M \in V_3^+$ is diffeomorphic to some manifold $\chi(L)$ obtained from S^3 by surgery along the components of some framed link L , and that every $M \in V_3^-$ can be obtained as such a $\chi(L)$ for some framed link in $S^2 \times S^1$. Such representation being not unique, Kirby's calculus on framed links, preserving the diffeomorphism type of $\chi(L)$ is well-known (we refer to $[K]$, $[R]$, and $[FR]$). We consider the following extension of this calculus, called in the sequel χ -calculus, to an m-calculus preserving the m-equivalence type of $\chi(L)$. The "moves" generating the m-calculus we need are:

- m_1 : the "moves" of χ -calculus ;
- m_2 : introducing (or deleting) $\bigcirc_2$ contained in a separating 3-ball .
- m_3 : changing the framing according to lemma 3.1. . In particular if the framing is given by an integer number (like in the case of links in S^3) k , then we may replace k by any $k' \equiv k \pmod{2}$.

Note that m_2 corresponds to the blowing-up at a point, because this is equivalent to make the connected sum with $\mathbb{P}^3$ and $\bigcirc_2$ is the simplest representation of $\mathbb{P}^3$ via framed links in S^3 .

We shall denote by m_R -calculus the reduced m-calculus generated only by the moves m_1 and m_3 .

3.2. LEMMA $\dim H_2(\chi(L), \mathbb{Z}_2)$ is invariant by m_R -calculus.

The proof is immediate.

3.3. LEMMA If two framed links L, L' (for fixing the ideas in S^3 or in $S^2 \times S^1$) are related by the m-calculus, then $\chi(L)$ and $\chi(L')$ are actually m^+ -equivalent.

Proof. If L and L' are related by an m_3 move, then they are evidently m^+ -equivalent. Then $\bigcirc_2$ is m^+ -equivalent to $\bigcirc_0$ which is diffeomorphic to $S^2 \times S^1$. Then it is easy to see that $S^2 \times S^1$ is m^+ -equivalent to $(S^2 \times S^1) \# (S^2 \times S^1)$ which is diffeomorphic to $(S^2 \times S^1) \# (S^2 \times S^1)$, that is to $B_1 \# B_1$ (with the notation of def. 2.1.), thus it is m^+ -equivalent to S^3 .

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4. Formulation of THEOREM A^+ in terms of m -calculus .

For every framed link L in S^3 , set $b=b_L = \dim H_2(\chi(L), \mathbb{Z}_2)$.

THEOREM C^+ Framed links in S^3 up to m_R -calculus are classified by b_L . Every link L is m_R -related to the framed link :

$$\bigcirc_2 \quad \dots \quad \bigcirc_2$$

b-times

Consequently every framed link L in S^3 is related to the empty link by the m -calculus .

It is not hard that Theorem C^+ implies Theorem A^+ .

Moreover , by lemma 3.3. we get immediately the corollary 2.2. in the orientable case .

This is a good point to recall Nakanishi's result ($[N]$) . We shall formulate it in a convenient way to our purpose .

4.1. DEFINITION Let L be a link in S^3 . Let $H:D^2 \times [0,1] \rightarrow S^3$ be a smooth embedding such that $H|(S^1 \times [0,1])$ (here $S^1 = \partial D^2$) is an embedding in $S^3 - L$. Then for every finite set $\{x_1, \dots, x_k\} \subset [0,1]$, $\bigcup_i H(S^1 \times \{x_i\})$ is called a set of parallel components of the link $L \cup (\bigcup_i H(S^1 \times \{x_i\})) = L \cup P$.

Let Λ_n be the set of (unframed) links in S^3 with n components.

4.2. DEFINITION Fox congruence mod $(2,1)$ on Λ_n is the relation generated by the following elementary "move" :

- (a) First introduce an even number k of parallel components getting $L \cup P \in \Lambda_{n+k}$. Give all components of P the framing 1 (or -1) .
- (b) Make k Kirby's moves along the so framed components of P (forgetting the framing for L) , getting $L' \in \Lambda_n$.

If L and L' are related by a finite sequence of such moves , we say shortly that they are congruent .

4.3. DEFINITION $L = C_1 \cup \dots \cup C_n$, $L' = C'_1 \cup \dots \cup C'_n$ in Λ_n are $\mathbb{Z}_2$ -link-homologous if the linking numbers $lk(C_i, C_j) = lk(C'_i, C'_j)$ for $i \neq j$.

4.4. THEOREM ([N]) $L, L' \in \Lambda_n$ are congruent if and only if they are $\mathbb{Z}_2$ -link-homologous . In particular every knot is congruent to the unknot .

The key remark to us is :

4.5. LEMMA The elementary move generating the congruence is composition of moves of the m_R -calculus .

Proof. By χ -calculus we can introduce any number of pairs of parallel components one with framing 1 , the other with -1 .
By m_3 we can change 1 in -1 .

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Sketch of proof of theorem C^+ .

It is a rather easy induction on the number of components of framed links in S^3 , denote it by n . If $n=1$ then (by 4.4.) the framed knot is m_R -equivalent either to $\bigcirc_1$ or $\bigcirc_0$.
For general n , (again by 4.4.) , we are free to work in the $\mathbb{Z}_2$ -link-homology class of any given link L with n components (forgetting the framing) ; considering a simple representative and reintroducing the possible framings reduced mod 2 , it is not hard , using the χ -calculus , to pass to a framed link with a strictly smaller number of components , unless we are in the final "normal form" .

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This concludes the discussion in the orientable case .

5. The non-orientable case .

We want to prove :

5.1. PROPOSITION Every $M \in V_3^-$ is m -equivalent to some $M' \in V_3^+$.

I will sketch the proof.

First it is convenient to consider two models for $S^2 \times S^1$.

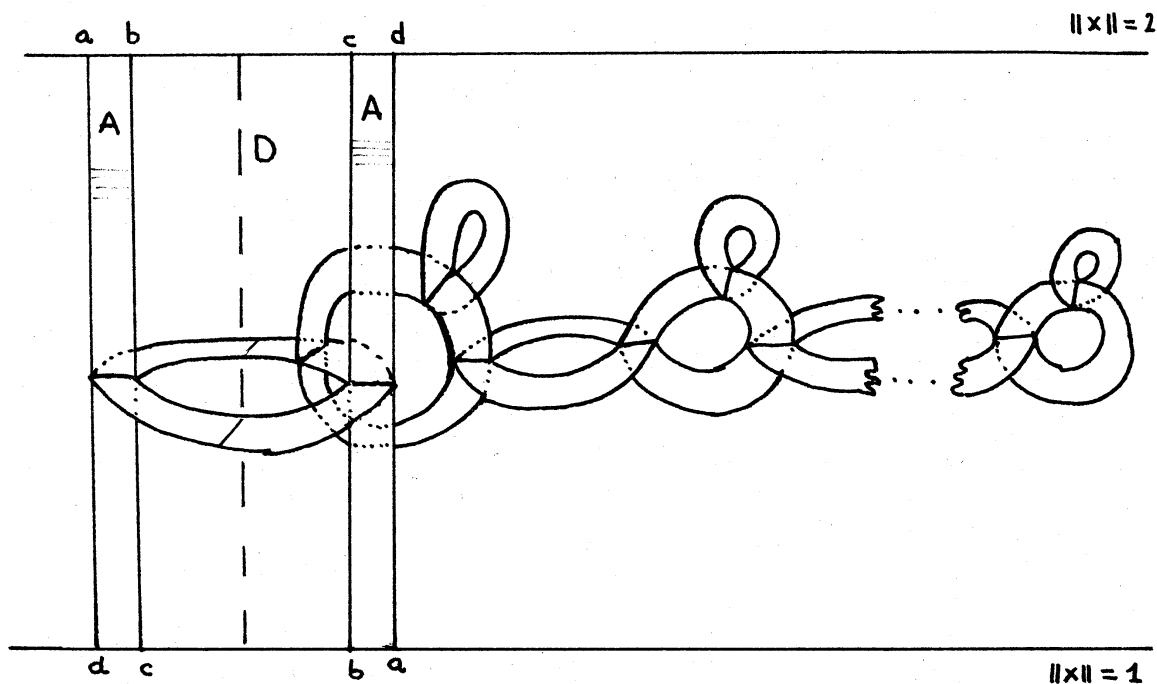
One consists in considering it as B_1 (with the notation of def. 2.1.) , that is as the result of the blowing-up of S^3 along an unknot K .



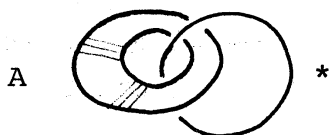
"*" means to be centre of the blowing-up .

The other one is obtained by identifying the boundary components of $S = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid 1 \leq \|x\| \leq 2\}$, via a dilatation followed by the reflection along the plane $\{x_2=0\}$. Thus we mean $S^2 \times S^1$ as $S/\sim$. Call D the inverse image of the unknot K , by the blowing-up in the former model. In $S/\sim$, D can be identified with $(\{x_2=0\} \cap S)/\sim$.

Then, using the finite set of generators for the mapping class group of any given non-orientable surface of fixed genus, constructed in [C], and arguing similarly to the orientable case as made, for example, in [R], we deduce that every $M \in V_3^-$ can be obtained by surgery along some framed link in $S^2 \times S^1$, looking in a certain special configuration. More precisely consider the family of annuli embedded in $S/\sim$ as shown in this picture :



For simplicity S is shown "straight" ; $k-1$ annuli looks "round and planar" in the picture and are fibred by concentric circles which are homotopically trivial in $S^2 \times S^1$. Exactly one annulus, called A , is fibred by non-trivial circles and looks "round and planar" in the other model :



Note that A cuts only one of the other annuli (the one intersecting D) along two segments .

Every component of our special link is contained in and concentric to one of these annuli . The components on A are called of A-type , the other of N-type .

Evidently , if no A-type components do occur , then we could isotopy the link far from D and , after a blowing-down , to see it as a link in S^3 . In some sense the problem is to eliminate the A-type components . We may assume that at least one of these A-type components has the property that the Moebius strip "naturally bounded" by it does not intersect the other components of the link . For the framing of such a component we have (mod 2) two possibilities : if it is 1 we can eliminate the component by χ -calculus . If the framing is 0 , we use the following argument .



and B_2 (see def. 2.1.)

(in fact B_2 is the result of a blowing-up of $S^2 \times S^1$ along a section over S^1) are diffeomorphic and in fact diffeomorphic to $\mathbb{P}^2 \times S^1$. Using a suitable diffeomorphism , we see that the image of the remaining components of the link are contained in a copy of $\mathcal{M} \times S^1$, embedded in B_2 and look inside it like a link in special configuration , preserving the type . So we have one A-type component less and we can conclude by induction .

This concludes the sketch of the proof of 5.1. .

It is not hard that this proof with the one of Theorem C^+ , actually implies the theorem A^- . Using Theorem A^- , and "reversing" the proof of 5.1. , one can obtain the proof of corollary 2.2. also in the non-orientable case .

Finally we can deduce the following version in terms of m-calculus on framed links in $S^2 \tilde{\times} S^1$.

Consider the m_E -calculus , that is the extended calculus generated by the usual m-calculus of paragraph 3 together with the further move :

- " introduce or delete any A-type component with framing 0 , such that the Moebius strip naturally bounded by it does not intersect the other components " .

THEOREM C⁻ Every framed link L in $S^2 \times S^1$ is related to the empty link by the m_E -calculus .

6. A sketch of proof of THEOREM B .

It uses arguments currently employed in Nash-Tognoli type theorem. I refer to [AK],[BCR],[Iv],[To] for an ample account about this circle of ideas .

Let $M \in V_3$. Use , to simplify , the strong version of the main theorem. First we may assume that the blowing-up over S^3 is made along non singular algebraic centres . Thus we get a non singular projective real algebraic variety M'' birationally equivalent to S^3 . Moreover it has the property that $H_2(M'', \mathbb{Z}_2)$ is generated by the algebraic hypersurfaces in M'' . Thus we can approximate in M'' any compact smooth closed hypersurface by some non singular algebraic ones ; moreover also a relative version of this fact holds. Assume , for simplicity , that we have only one blowing-down to do in order to pass from M'' to M . Let C be its smooth centre in M and D be its inverse image in M'' . D is a smooth 2-torus and , by the above discussion , we may assume that D is a non singular algebraic subvariety of M'' having the property that $H_1(D, \mathbb{Z}_2)$ is generated by the algebraic curves contained in D . Give C the algebraic structure of the standard unit circle S^1 . By the so recalled assumptions on D , we know that the map $\pi|_D$ (π is the blowing-down) can be approximated by rational regular maps $g: D \rightarrow S^1$. Note that , topologically, $M = (M'' \cup C)/\pi|_D$ and that if g is close enough to $\pi|_D$, then M and $(M'' \cup C)/g$ are homeomorphic . Finally one recalls the fact that , since g is a regular rational map , the last quotient space can be realized by a (singular) real algebraic, affine variety M' and that there exists a regular rational map from M'' onto M' which is an algebraic isomorphism between $M'' - D$ and $M' - C$.

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7. A sketch of A.Marin's proof of the main theorem .

Here is the key remark :

7.1. REMARK Let C_1 and C_2 be knots in $M \in V_3$, making the boundary of an embedded surface F having an orientable neighbourhood in M . Then C_1 and C_2 become isotopic in a manifold M' obtained from M by some Dehn surgeries having in the data q/p of the form $2k+1/2$, along circles embedded in a neighbourhood of $F - (C_1 \cup C_2)$ in $M - (C_1 \cup C_2)$.

In fact, we may assume that F is non-orientable (like the unknot in this picture bounds a Moebius strip ).

Then F is diffeomorphic to the connected sum of some copies of $\mathbb{P}^2$, with two disks removed.

Thus we can pass from F to the 2-disk with two holes by replacing some Moebius strip embedded in F by 2-disks. Actually we can realize this operation in a manifold M' obtained by Dehn surgeries of the type said above along the base line of each Moebius strip.

Actually it is enough to prove the main theorem in the orientable case. We may assume that $M \in V_3^+$ is obtained by surgery along a framed link in S^3 having at least one component with odd framing. Using the classification of odd bilinear forms over $\mathbb{Z}_2$ and by "handle slides" (if one prefers, by χ -calculus), we may assume that the linking numbers of any pair of different components of the link, is even. Thus we can see M as the last of a sequence $S^3 = M_1, M_2, \dots, M_k = M$, of 3-manifolds such that M_{i+1} is obtained from M_i , by a surgery along a knot C_i which bounds a surface F_i (not necessarily orientable) embedded in M_i . Now the proof of the main theorem is an easy consequence of the above remark and of lemma 3.1.: we see, by induction, that M is m -equivalent (in fact m^+ -equivalent) to $\#_{1 \leq i \leq d} \mathbb{P}^3$, where $d = \dim H_1(M, \mathbb{Z}_2)$, that is the nullity of the bilinear form over $\mathbb{Z}_2$ of the given link. Note that $d = \dim H_2(M, \mathbb{Z}_2)$, so we have obtained nothing else than Theorem A^+ .

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