

Linearity of general fibers of separable Gauss maps

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Abstract. We prove the linearity of general fibers of a separable Gauss map for a projective variety in arbitrary characteristic.

1. STARTING POINT

$X \subset \mathbb{P}^n$: a projective variety over $k = \bar{k}$ with $\text{ch}(k) \geq 0$.
The Gauss map γ is defined by:

$$\gamma = \gamma_X : X \dashrightarrow \mathbb{G}(\dim(X), \mathbb{P}^n) \\ x \mapsto [\mathbb{T}_x X].$$

(We assume that X is non-linear in $\mathbb{P}^n$.)

Denote by $F_\gamma \subset X$ the closure of a general fiber of γ .

Here, the following theorems are well known:

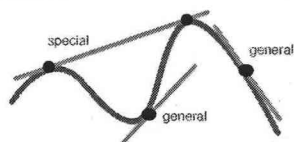
Griffiths and Harris, Zak:

F_γ is a linear subvariety of $\mathbb{P}^n$ in $\text{ch}(k) = 0$.

Zak:

γ is finite if X is smooth in $\text{ch}(k) \geq 0$.

(*) Thus γ is birational if X is smooth & $\text{ch}(k) = 0$.



(*) means that a general embedded tangent space is tangent to X at a unique point.

2. IN $\text{ch}(k) > 0 \dots$

• γ can be inseparable in $\text{ch}(k) > 0$ [Wallace '56].

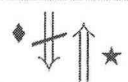
• $\exists$ examples s.t. F_γ is not a linear subvariety:

[Kaji '86, '89], [Rathmann '87], [Noma '01], [Fukasawa '05, '06].

Here, in all these examples, γ 's are inseparable.

Question [Kaji '03, '09]. Is a general fiber of a separable Gauss map a linear subvariety?

γ : separable



F_γ : a linear subvariety

X : reflexive

3. MAIN THEOREM

We give the answer to the question affirmatively:

(♣) **Theorem 1.** $X \subset \mathbb{P}^n$: a projective variety.
 γ : separable $\implies$ a general fiber F_γ of γ is a linear subvariety of $\mathbb{P}^n$.

By combining Zak's theorem and our result, we have:

Corollary 2.

γ : separable & X : smooth $\implies \gamma$: birational.

4. EARLIER WORKS RELATED TO THE LINEARITY

► **dim $X = 1$:** In the case of $\dim X = 1$,

γ : separable $\implies \gamma$: birational.

• This fact was classically known for projective plane curves.

• It was shown for any projective curve by [Kaji '89].

► **dim $X = 2$ (approach by reflexivity):**

(★) **Kleiman and Piene '89:**

X : reflexive $\implies F_\gamma$: a linear subvariety & γ : separable.

Conversely, " γ : separable $\implies X$: reflexive" holds if

• $\dim X = 1$ [Voloch '89], [Kaji '92],

• $\dim X = 2$ [Fukasawa and Kaji '07].

Thus, for $\dim X = 2$, " γ : separable $\implies F_\gamma$: linear" was shown in terms of reflexivity. On the other hand, ...

For $\dim X \geq 3$, $\exists$ examples of non-reflexive X

whose γ 's are birational: [Kaji '03(2)], [Fukasawa '06(2), '07].

Thus, in general,

(♦) γ : separable $\not\Rightarrow X$: reflexive.

5. OUTLINE OF THE PROOF OF THEOREM 1

▲ **Step 1:** $r := \dim(X)$, $c := n - r = \text{codim}(X, \mathbb{P}^n)$,

$L \subset \mathbb{P}^n$: a general $(c-2)$ -plane.

For the linear projection $\pi_L : X \rightarrow \mathbb{P}^{r+1}$, set

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$X_L := \pi_L(X) \subset \mathbb{P}^{r+1}$. Let us consider

$$\begin{array}{ccc} X & \xrightarrow{\gamma} & \mathbb{G}(r, \mathbb{P}^n) \\ \pi_L \downarrow & & \downarrow q_L \\ X_L & \xrightarrow{\gamma_{X_L}} & \check{\mathbb{P}}^{r+1}, \end{array}$$

where q_L is given by $[M] \mapsto [\pi_L(M)]$.

The inequivalence (♦) yields the following problem:

(♦') γ_{X_L} can be inseparable even if γ_X is separable.

To fix it, we give the following proposition, *without assuming that γ_Z is separable*:

Proposition 3. $Z \subset \mathbb{P}^m$: a hypersurface, $z \in Z$.

Consider $d_z \gamma_Z : T_z Z \dashrightarrow T_{\gamma_Z(z)} \check{\mathbb{P}}^m$, and set

$\mathbb{K} \subset T_z Z$: the linear subvariety $\leftrightarrow \ker(d_z \gamma_Z) \subset T_z Z$,

$\mathbb{I} \subset \check{\mathbb{P}}^m$: the linear subvariety $\leftrightarrow d_z \gamma_Z(T_z Z) \subset T_{\gamma_Z(z)} \check{\mathbb{P}}^m$.
 $\implies \mathbb{K} = \mathbb{I}^*$ in $\mathbb{P}^m$.

▲ **Step 2:** $L_1, L_2, \dots, L_c \subset \mathbb{P}^n$: general $(c-2)$ -planes,
 $Y := \gamma(X)^\perp$. For general $y \in Y$, we consider

$$d_y q_{L_i} : T_y Y \rightarrow T_{y_i} \check{\mathbb{P}}^{r+1} \quad (y_i := q_{L_i}(y)).$$

Set $\mathbb{J}_{y,i} \subset \check{\mathbb{P}}^{r+1}$: the linear subvariety

$$\leftrightarrow d_y q_{L_i}(T_y Y) \subset T_{y_i} \check{\mathbb{P}}^{r+1},$$

$\overline{\mathbb{J}}_{y,i} \subset \check{\mathbb{P}}^n$: the image of $\mathbb{J}_{y,i}$ under $\check{\mathbb{P}}^{r+1} \hookrightarrow \check{\mathbb{P}}^n : [N] \mapsto [\pi_L^{-1}(N)]$.

Now, we define a linear subvariety $\mathbb{D}_y \subset \mathbb{P}^n$ by:

$$\mathbb{D}_y = \mathbb{D}(T_y Y; L_1, L_2, \dots, L_c) := \bigcap_{i=1}^c (\overline{\mathbb{J}}_{y,i})^* \cap M,$$

where $M \subset \mathbb{P}^n$ with $y = [M]$. Then $\mathbb{D}_y$ is regarded as a "dual" of $T_y Y$ w.r.t. linear projections $\pi_{L_1}, \pi_{L_2}, \dots, \pi_{L_c}$.

In fact, by applying Proposition 3 to $Z = X_{L_i}$, we have:

γ : separable, $x \in \gamma^{-1}(y)$.

$\implies \mathbb{D}_y$: the linear subvariety $\leftrightarrow \ker(d_x \gamma) \subset T_x X$.

$\implies (\gamma^{-1}(y))^\perp = \mathbb{D}_y$.

Hence Theorem 1 follows. $\square$