

Parametrices for Degenerate Operators

of Grushin's Type

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Introduction. Grushin studied the hypoellipticity of an degenerate operator A of the form given in §1. In his paper [2] he used operator valued pseudo-differential operators essentially.

In this note we construct a left parametrix Q for the operator A as a pseudo-differential operator by symbol calculus instead of his method. For the construction of Q , we use the fundamental solution of parabolic equation studied in [4] and [5]. Also we discuss estimates for A .

We give main theorems and several examples in §1. In §2 and §3 the method of construction of Q will be shown. We devote §4 for proves of Theorem B and Theorem C.

§1. Main theorems and examples.

In this note we treat operators defined in R^3 for simplicity. Consider an operator

$$(1) \quad A = A(x_1, y, D_{x_1}, D_{x_2}, D_y) = \sum_{\substack{|\alpha| \leq m \\ (\tau, \gamma) \geq (\tau, \alpha) - m}} a_{\alpha, \gamma} x_1^{\gamma_1} y^{\gamma_2} D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} D_y^{\alpha_3}$$

where $a_{\alpha, \gamma}$ are constants, $D_j = \partial / \partial x_j$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\tau = (\tau_1, \tau_2, 1)$,

$\tau_j \geq 1$ ($j=1, 2$), $\gamma = (\gamma_1, 0, \gamma_2)$, $\sigma = (\sigma_1, 0, 1)$, $\min(\tau_1, \tau_2) > \sigma_1 > 0$

and $(\tau, \alpha) = \sum_{j=1}^3 \tau_j \alpha_j$

Let A_0 be the principal part of A in a sense, that is,

$$(2) \quad A = A(x_1, y, D_{x_1}, D_{x_2}, D_{x_3}) = \sum_{\substack{|\alpha| \leq m \\ (\tau, \gamma) = (\tau, \alpha) - m}} a_{\alpha, \gamma} x_1^{\gamma_1} y^{\gamma_2} D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} D_{x_3}^{\alpha_3}$$

We denote by $A(x_1, y, \xi, \eta)$ the differential polynomial correspond to A_0 .

Condition 1. $A_0(x_1, y, D_{x_1}, D_{x_2}, D_y)$ is elliptic for $|x_1| + |y| \neq 0$ i.e.

$$A_0(x_1, y, \xi, \eta) \neq 0$$

for $|x_1| + |y| \neq 0$, $\xi \in \mathbb{R}^2$, $\eta \in \mathbb{R}^1$, and $|\xi| + |\eta| \neq 0$.

Condition 2. For all $x_1 \in \mathbb{R}^1$ and for all nonzero vector $\xi \in \mathbb{R}^2$ the equation $A_0(x_1, y, \xi, D_y)v(y) = 0$ has no nonzero solution in $\mathcal{J}(\mathbb{R}^1)$.

We get main theorems as follows.

Theorem A. Under condition 1 and condition 2, there exist a neighbourhood Ω of $x_1=y=0$ and a left parametrix Q for A in a class of pseudo-differential operators;

$QA = I + W$ in Ω , where W is a smoothing operator.

By the construction of Q we get estimates for A and A_0 .

$$(3)_s \quad \|u\|_{m+s} \leq c_K (\|A_0 u\|_s + \|u\|_s) \quad \forall u \in C_0^\infty(K),$$

$$(4)_s \quad \|u\|_{m+s} \leq c (\|Au\|_{(s)} + \|u\|_{(s)}) \quad \forall u \in C_0^\infty(\Omega),$$

where $\|\cdot\|_m$ is the usual Sobolev norm, K is any compact set of $\mathbb{R}^n$ such that $K \cap \{0, x_2, 0\} = \emptyset$, $\|u\|_{(m)} = \|\Lambda_{\tau}^m(D)u\|_0$ and Λ_{τ} is a pseudo-differential operator with symbol $\sum_{j=1}^2 |\xi_j|^{\frac{2}{\tau}} + |\eta|^2 + 1$.

Moreover we get the following statement.

"If $Au \in H_{loc}^s(\Omega)$ and $u \in D'(\Omega)$, then $u \in H_{loc}^{s+m/\bar{\tau}}(\Omega)$

where $\bar{\tau} = \max(\tau_1, \tau_2)$.

Remark. If $\bar{\tau}=1$, then condition 1 means that A is elliptic in case $|x_1| + |y|$ is sufficient small.

We may assume $\bar{\tau} > 1$ by the above remark.

Theorem B. If $(4)_0$ hold, then we obtain the following estimate for all $x_1 \in \mathbb{R}^1$ and nonzero $\xi \in \mathbb{R}^2$.

$$\|v\| \leq C \|A_0(x_1, y, \xi, D_y)v\| \quad \forall v \in \mathcal{D}(R_y^1)$$

where $\|\cdot\|$ is the norm in $L_2(R_y^1)$.

Owing to the above theorem we get

Theorem C. Assume that A satisfies $(3)_0$ and $(4)_0$. Then, A holds condition 1 and condition 2.

Now let us give some examples

Example 1. $A=A_0=(-\Delta_y)^\ell + y^{2k}(-\Delta_{x_2})^\ell$ in $\mathbb{R}^2$,

where ℓ and k are positive integers. We can take $\mathcal{L}=(\ell+k, \ell)$, $\sigma=(0, 1)$ and $m=2$. In this case condition 1 and condition 2 hold.

Example 2. (Grushin [2]). $A=A_0=\frac{\partial^2}{\partial y^2} + (y^2+x_1^2)(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}) + i\lambda \frac{\partial}{\partial x_2}$. In this case we can choose $\mathcal{L}=(2, 2, 1)$, $\sigma=(1, 0, 1)$ and $m=2$. Condition 1 holds. Condition 2 holds if $|\lambda| < 1$ or $\text{Im} \lambda \neq 0$.

Example 3. $A=A_0=(\frac{\partial}{\partial y} - iay^k \frac{\partial}{\partial x_2})(\frac{\partial}{\partial y} - iby^k \frac{\partial}{\partial x_2}) + icy^{k-1} \frac{\partial}{\partial x_2}$ in $\mathbb{R}^2$, where k is a odd integer, a, b and c are real constants such that $ab < 0$. In this case $\mathcal{L}=(k+1, 1)$, $\sigma=(0, 1)$ and $m=2$. Condition 1

always holds. Condition 2 holds if and only if

$$\frac{c}{a-b} \not\equiv 0, 1 \pmod{(k+1)}.$$

This result is due to Gilioli and Treves [1].

§ 2. Notations and theorems for pseudo-differential operators

For a pair of real vectors $\rho = (\rho_1, \rho_2, \dots, \rho_n)$ and $\delta = (\delta_1, \delta_2, \dots, \delta_n)$ we say $\rho > \delta$ if $\rho_j > \delta_j$ for all j . In this section we fix ρ and δ such that $\rho > \delta \geq 0$.

Definition (cf. [5]). We say that a C^∞ -function $\lambda(x, \xi)$ define in $R_x^n \times R_\xi^n$ is a basic weight function when $\lambda(x, \xi)$ satisfies the conditions below:

$$(5) \quad |\lambda_{(\beta)}^{(\alpha)}(x, \xi)| < A_{\alpha, \beta} \lambda(x, \xi)^{1 - (\rho, \alpha) + (\delta, \beta)}$$

$$(6) \quad 1 \leq \lambda(x+y, \xi) \leq A \langle y \rangle^{\tau_0} \lambda(x, \xi) \quad (\tau_0 \geq 0),$$

where $\lambda_{(\beta)}^{(\alpha)}(x, \xi) = (\partial/\partial \xi_1)^{\alpha_1} \dots (\partial/\partial \xi_n)^{\alpha_n} (-i\partial/\partial x_1)^{\beta_1} \dots (-i\partial/\partial x_n)^{\beta_n}$

$\lambda(x, \xi)$ and $\langle y \rangle = (|y|^2 + 1)^{1/2}$.

We denote by $S_{\lambda, \rho, \delta}^m$ ($-\infty < m < \infty$) the set of all C^∞ -functions $p(x, \xi)$ defined in $R_x^n \times R_\xi^n$ which satisfies for any α and β

$$|p_{(\beta)}^{(\alpha)}(x, \xi)| \leq C_{\alpha, \beta} \lambda(x, \xi)^{m - (\rho, \alpha) + (\delta, \beta)}$$

for some constants $C_{\alpha, \beta}$. For a symbol $p(x, \xi) \in S_{\lambda, \rho, \delta}^m$ we define

a pseudo-differential operator by

$$Pu(x) = p(x, D_x)u(x) = \int e^{ix \cdot \xi} p(x, \xi) \hat{u}(\xi) d\xi,$$

where $d\xi = (2\pi)^{-n} d\xi$ and $\hat{u}(\xi)$ denote the Fourier transform of $u(x)$

in $\mathcal{S}$ defined by

$$\hat{u}(\xi) = \int e^{-ix \cdot \xi} u(x) dx.$$

We call a pseudo-differential operator a smoothing operator when its symbol belongs to $S_{\langle \xi \rangle}^m, \rho, \delta$ for any m .

Now consider a pseudo-differential operator of parabolic type.

$$L = \frac{\partial}{\partial t} + p(x, D_x)$$

We call an operator $E(t)$ a fundamental solution for L when $E(t)$ satisfies

$$\begin{cases} L E(t) = 0 & \text{in } 0 < t < \infty, \\ E(0) = I. \end{cases}$$

By [5], we get the next theorem

Theorem 1. Assume that $p(x, \xi) \in S_{\lambda}^{\ell}$, satisfies

$$\operatorname{Re} p(x, \xi) + c \geq c_0 \lambda(x, \xi)^{\ell}$$

for positive constants c and c_0 . Then, there exists a fundamental solution $E(t)$ which belongs to $S_{\lambda, \rho, \delta}^0$ with parameter t and whose

symbol $e(t; x, \xi)$ has the following expansion

$$e(t; x, \xi) = \sum_{j=0}^{N-1} e_j(t; x, \xi) + r_N(t; x, \xi)$$

where $e_j(t; x, \xi) \in S_{\lambda, \rho, \delta}^{-\omega_j}$, $r_N(t; x, \xi) \in S_{\lambda, \rho, \delta}^{\ell - \omega_N}$, $\omega = \min(\rho_j - \delta_j)$ and

N is any number such that $\omega_N \geq \ell$. Moreover we get $e_0(t; x, \xi)$

$$= \exp \left\{ -tp(x, \xi) \right\} \text{ and}$$

$$e_j \begin{pmatrix} \alpha \\ \beta \end{pmatrix} (t; x, \xi) = a_{j, \alpha, \beta} (t; x, \xi) e_0(t; x, \xi) \quad (j \geq 1),$$

where $a_{j, \alpha, \beta} (t; x, \xi)$ satisfy

$$\begin{aligned} \left| a_{j, \alpha, \beta} (t; x, \xi) \right| &\leq c_{j, \alpha, \beta} \lambda(x, \xi)^{-\omega_j - (\rho, \alpha) + (\delta, \beta)} \\ &\times \sum_{k=2}^{|d|+|\beta|+2j} \left\{ t \operatorname{Re} p(x, \xi) \right\}^k \end{aligned}$$

About the behavior of $e(t; x, \xi)$ for large t , we get

Theorem 2 (Tsutsumi [5]). Let $p(x, \xi)$ satisfy the assumption of theorem 1 and

$$\operatorname{Re}(p(x, D_x)u, u) \geq c_1 \|u\|^2 \quad u \in \mathcal{S}(\mathbb{R}^n)$$

with a positive constant c_1 . Moreover let $t > 0$ and $\lambda(x, \xi)$ satisfy

$$\lambda(x, \xi) \geq a(|\xi| + |x| + 1)^a$$

for some positive constant a . Then the symbol $e(t; x, \xi)$ constructed in theorem 1 holds the following estimate for any integers j and k and positive constant ε

$$\lambda(x, \xi)^j \left| \frac{\partial^k}{\partial t^k} e_{(\beta)}^{(\alpha)}(t+\varepsilon; x, \xi) \right| \leq C(j, k, \alpha, \beta, \varepsilon) \exp(-c_2 t) \quad (t \geq 0)$$

for any α and β ,

where c_2 is any constant less than c_1 and $C(j, k, \alpha, \beta, \varepsilon)$ is independent of t .

Now assume that the basic weight function $\lambda(x, \xi)$ is independent of x and $b^{-1} |\xi|^{k_1} \leq \lambda(\xi) \leq b |\xi|^{k_2}$ ($k_1, k_2 > 0$).

We denote the Sobolev norm $\|\cdot\|_{m, \lambda}$ by

$$\|u\|_{m, \lambda} = \|\lambda(D_x)^m u\|$$

where $\|\cdot\|$ is the norm in $L^2(\mathbb{R}^n)$.

Let Ω be a open set in $\mathbb{R}^n$ and let $p(x, \xi)$ hold the estimate below

$$(7) \quad \left| p_{(\beta)}^{(\alpha)}(x, \xi) \right| \leq c_{K, \alpha, \beta} \lambda(\xi)^{M - (\rho, \alpha) + (\delta, \beta)} \quad \forall x \in K$$

for any compact set K in Ω .

Theorem 3 (Tsutsumi [6]). Let $P = p(x, D_x)$ satisfy (7) and

$$\|u\|_{m, \lambda}^2 \leq c_K (\|Pu\| + \|u\|_{m', \lambda}) \quad u \in C_0^\infty(K).$$

where $M \geq m \geq 0$, $m > m'$ and K is any compact set in Ω . Then, for every K and every integer N one can find C so that when $x \in K$ and $\xi \in \mathbb{R}^n$

$$\lambda(\xi)^{2m} \|\psi\|^2 \leq c \int \left| \sum_{|\alpha+\beta| \leq N} p_{(\beta)}^{(\alpha)}(x, \xi) \lambda(\xi)^{(\alpha, \alpha-\beta)} y^\beta D_y^\alpha \psi / \alpha! \beta! \right|^2 dy \\ + \lambda(\xi)^{2M-2\varepsilon_0 N+d_1} \left(\sum_{\substack{|\alpha+\beta| \leq N+d_2 \\ |\alpha| \leq |\beta|}} \int |y^\beta D_y^\alpha \psi|^2 dy \right), \quad \psi \in C_0^\infty(\mathbb{R}^n)$$

Here $\mathcal{O}=(\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_n)$ is chosen such that $\varepsilon_0 = \min(\rho_j - \mathcal{O}_j, \mathcal{O}_j - \delta_j, \mathcal{O}_j) > 0$ and both d_1 and d_2 are constants independent of N .

§ 3. Outline of construction of a left parametrix Q .

We introduce some notations.

$$h=h(x_1, y) = (|x|^{1/\tau_1} + |y|) \\ \nu=\nu(x_1, y, \xi, \eta) = \sum_{j=1}^2 h(x_1, y)^{\tau_j-1} |\xi_j| + |\eta| \\ \mu=\mu(\xi) = \sum_{j=1}^2 |\xi_j|^{1/\tau_j} + 1 \\ \chi=\chi(\xi, \eta) = \sum_{j=1}^2 |\xi_j| + |\eta| + 1$$

We denote by $S^m(\nu, \mu, \chi)$ ($-\infty < m < \infty$) the set of all C^∞ -functions

$p(x_1, y, \xi, \eta)$ defined in $\mathbb{R}_{x_1}^1 \times \mathbb{R}_y^1 \times \mathbb{R}_\xi^2 \times \mathbb{R}_\eta^1$ which satisfy

$$(8) \quad \left| p_{(\beta)}^{(\alpha)}(x_1, y, \xi, \eta) \right| \leq c_{\alpha, \beta} (\nu + \mu)^{m-|\beta|} \mu^{|\beta|+|\alpha|} \chi^{-|\alpha|+|\alpha_2|}$$

for any $\alpha=(\alpha_1, \alpha_2, \alpha_3)$, $\beta=(\beta_1, 0, \beta_3)$. We define a pseudo-differential operator $Pu(x_1, x_2, y)$ with a symbol $\sigma(P)=p(x_1, y, \xi, \eta) \in S^m(\nu, \mu, \chi)$ by

$$Pu(x_1, x_2, y) = \int e^{i(x_1 \xi_1 + x_2 \xi_2 + y \eta)} p(x_1, y, \xi, \eta) \hat{u}(\xi, \eta) d\xi d\eta$$

for $u \in \mathcal{S}(\mathbb{R}^3)$.

We get the following lemma for operators of this class.

Lemma 1. (Tsutsumi [6]) (i) If P_j belongs to $S^{m_j}(\nu, \mu, \chi)$ ($j=1, 2$) then $P_1 P_2$ belongs to $S^{m_1+m_2}(\nu, \mu, \chi)$ and

$$\sigma(P_1 P_2)(x_1, y, \xi, \eta) = \sigma(P_1(x_1, y, \xi, D_y)) P_2(x_1, y, \xi, D_y)$$

belongs to $S^{-\delta_0}(\nu, \mu, \chi)$, where $\delta_0 = \min(l_1, l_2) - \sigma_1$.

(ii) If P belongs to $S^0(\nu, \mu, \chi)$, then P is a bounded operator in $L^2(\mathbb{R}^3)$

(iii) $P \in S^m(\nu, \mu, \chi)$ has pseudo-local property, i.e. For any $\phi, \psi \in C^\infty(\mathbb{R}^3)$

P is a smoothing operator when $\text{supp } \phi \cap \text{supp } \psi = \emptyset$.

The main result of this section is the proposition below.

Proposition 1. Under condition 1 and condition 2 there exist a neighbourhood $\Omega' \subset \mathbb{R}^2$ of $x_1=y=0$ and $\tilde{A}(x_1, y, \xi, \eta) \in S^m(\nu, \mu, \chi)$ which satisfy the following properties.

(i) We can construct a left parametrix $Q \in S^{-m}(\nu, \mu, \chi)$ for $\tilde{A}$, i.e.

$$Q\tilde{A} = I + W, \text{ where } W \text{ is a smoothing operator.}$$

(ii) $(A - \tilde{A})(x_1, y, \xi, \eta) = 0$ if $(x_1, y) \in \Omega'$.

By this proposition and lemma 1 one can prove theorem A, noting that Λ_τ belongs to $S^1(\nu, \mu, \chi)$ and that Q is elliptic for $|x_1| + |y| \neq 0$, i.e. The symbol $q(x_1, y, \xi, \eta)$ of Q satisfy $|q| \geq c(|\xi| + |\eta| + 1)^{-m}$ for $|x_1| + |y| \neq 0$.

We need several steps to show proposition 1.

Lemma 2. For all α and β we have

$$|\partial_\eta^\alpha \partial_y^\beta A_0(x_1, y, \xi, \eta)| \leq C_{\alpha, \beta} (\nu + \mu)^{m-|\alpha|} \mu^{|\beta|} \quad \text{for } |\xi| \geq c > 0$$

Moreover if we fix a $C^\infty(\mathbb{R}^1)$ -function $\mathcal{G}$, then $\mathcal{G}(|x_1|^2 + |y|^2) A_0(x_1, y, \xi, \eta)$ belongs to $S^m(\nu, \mu, \chi)$ and $\mathcal{G}(|x_1|^2 + |y|^2) (A - A_0)(x_1, y, \xi, \eta)$ satisfies (3) for $C_{\alpha, \beta} h(x_1, y)^{\varepsilon_1}$ ($\varepsilon_1 > 0$) instead of the constants $C_{\alpha, \beta}$ in (3).

Lemma 3. If condition 1 hold, then we get for some constant $C > 0$

$$C^{-1} |A_0(x_1, y, \xi, \eta)| \leq \mathcal{A}(x_1, y, \xi, \eta) \leq C |A_0(x_1, y, \xi, \eta)|.$$

Lemma 4. (Grushin [2]). We find a constant $C_1 > 0$ such that

$$\|u\| \leq C_1 \|A_0(x_1, y, \xi, D_y)u\| \quad \forall u \in \mathcal{S}(\mathbb{R}^1)$$

for any ξ when $|\xi| = 1$.

Set $\lambda_z(y, \eta)$ with parameters $z = (x_1, \xi)$ by

$$\lambda_z(y, \eta) = (|A_0(x_1, y, \xi, \eta)|^2 + 1)^{1/2m}$$

then by lemma 2 and lemma 3, we get

Lemma 5. If $|\xi| = 1$, then we get

(i) $\lambda_z(y, \eta)$ is a basic weight function which holds (5) and (6) for

$A_{\alpha, \beta}$ and A independent of parameters z .

(ii) $|A_0(x_1, y, \xi, \eta)|^2 + 1 \geq \lambda_z(y, \eta)^{2m}$

(iii) $|\partial_\eta^\alpha \partial_y^\beta A_0(x_1, y, \xi, \eta)| \leq C_{\alpha, \beta} \lambda_z(y, \xi)^{m-|\alpha|}$

where $C_{\alpha, \beta}$ are independent of z .

Now, by lemma 5 we can apply theorem 1 for $L_z = L_{x_1, \xi}$ ($|\xi|=1$)

defined by

$$L_z = L_{x_1, \xi} = \frac{\partial}{\partial t} + p(x_1, y, \xi, D_y),$$

where $p(x_1, y, \xi, D_y) = A_0(x_1, y, \xi, D_y) A_0^*(x_1, y, \xi, D_y)$ and $A_0^*(x_1, y, \xi, D_y)$ is the adjoint operator of $A(x_1, y, \xi, D_y)$.

Let $e(t; y, D_y; z) \in S_{\lambda_z, 1, 0}^0$ be the fundamental solution for L_z with parameters z . ($|\xi|=1$).

For any ξ ($|\xi| \geq 1$) we get

Lemma 6. If condition 1 and condition 2 hold, then there exists the fundamental solution $u(t; y, D_y; z)$ for L_z which admits the expansion below for $N \geq 2m$,

$$u(t; y, \eta; z) = \sum_{j=0}^{N-1} u_j(t; y, \eta; z) + v_N(t; y, \eta; z)$$

$$|\partial_\eta^\alpha \partial_y^\beta u_j(t; y, \eta; z)| \leq C_{j, \alpha, \beta} (\nu + \mu)^{-j-|\alpha|} \mu^{j+|\beta|} \exp[-c_0(\nu + \mu)^{2m} t]$$

$$|\partial_\eta^\alpha \partial_y^\beta v_N(t; y, \eta; z)| \leq C_{N, \alpha, \beta} (\nu + \mu)^{-N-|\alpha|} \mu^{N+|\beta|},$$

$$(9) \quad |\partial_\eta^\alpha \partial_y^\beta u(t; y, \eta; z)| \leq C'_{\alpha, \beta} (\nu + \mu)^{-k-|\alpha|} \mu^{k+|\beta|} \exp[-c_0 \mu^{2m} t]$$

for $\mu^m t \geq \varepsilon > 0$ and all k .

Proof. For $|\xi|=1$ by lemma 4 and theorem 2, one can find a constant C independent of z such that

$$\max_{|\alpha|+|\alpha'|+|\beta|+|\beta'| \leq k, y \in \mathbb{R}^1, \eta \in \mathbb{R}^1} \sup_{\eta \in \mathbb{R}^1} |y^\alpha \eta^{\alpha'} D_y^\beta \partial_\eta^{\beta'} e(t+\varepsilon; y, \eta; z)| \leq C \exp[-C_2 t] \quad (t \geq 0)$$

for any $\varepsilon > 0$, k and $C_2 < C_1$.

$A_0(x_1, y, \xi, \eta)$ is quasihomogeneous in the sense

$$A_0(\lambda^{-\sigma_1 x_1}, \lambda^{-1} y, \lambda^{\tau_1} \xi_1, \lambda^{\tau_2} \xi_2, \lambda \eta) = \lambda^m A_0(x_1, y, \xi_1, \xi_2, \eta)$$

then $u(t; y, \eta; z)$ is given by

$$(10) \quad u(t; y, \eta; x_1, \xi_1, \xi_2) = e(\tilde{\mu}^{2m} t; \tilde{\mu} y, \tilde{\mu}^{-1} \eta; \tilde{\mu}^{\sigma_1} x_1, \tilde{\mu}^{-\tau_1} \xi_1, \tilde{\mu}^{-\tau_2} \xi_2)$$

where $\tilde{\mu} = \tilde{\mu}(\xi)$ is the positive root of the equation $\sum_{j=1}^2 \tilde{\mu}^{-2\tau_j} \xi_j^2 = 1$.

By (10) and theorem 1 we get the assertion.

$$\text{Put } r(x_1, y, \xi, \eta) = r(y, \eta; x_1, \xi) = \int_0^\infty u(t; y, \eta; x_1, \xi) dt \quad \text{for } |\xi| \geq 1.$$

Then $R(x_1, \xi) = r(y, D_y; x_1, \xi)$ is a left and right inverse of

$p(x_1, y, \xi, D_y)$. Let us define a pseudo-differential operator $K(x_1, \xi)$ with parameters by

$$K(x_1, \xi) = R(x_1, \xi) A_0(x_1, y, \xi, D_y).$$

Then for $|\xi| \geq 1$,

Lemma 7. $K(x_1, \xi)$ is a left inverse of $A_0(x_1, y, \xi, D_y)$ with a symbol $k(y, \eta; x_1, \xi) = k(x_1, y, \xi, \eta)$ which admits for any α and β

$$|\partial_\eta^\alpha \partial_y^\beta k(y, \eta; x_1, \xi)| \leq C_{\alpha, \beta} (\nu + \mu)^{-m - |\alpha|} \mu^{|\beta|}.$$

Moreover $\mathcal{G}(|x_1|^2 + |y|^2) k(x_1, y, \xi, \eta)$ belongs to $S^{-m}(\nu, \mu, \chi)$ for any $\mathcal{G} \in C_0^\infty(\mathbb{R}^1)$.

Proof. It is sufficient to show that $r(y, \eta; x_1, \xi)$ satisfies

$$|\partial_\eta^\alpha \partial_y^\beta r(y, \eta; x_1, \xi)| \leq C'_{\alpha, \beta} (\nu + \mu)^{-2m - |\alpha|} \mu^{|\beta|}$$

and that $\mathcal{G}(|x_1|^2 + |y|^2) r(x_1, y, \xi, \eta)$ belongs to $S^{-2m}(\nu, \mu, \chi)$, by lemma 2.

We write

$$\begin{aligned} r(y, \eta; x_1, \xi) &= \int_0^{\mu^m(\xi)} u(t; y, \eta; x_1, \xi) dt + \int_{-\mu^m(\xi)}^{\infty} u(t; y, \eta; x_1, \xi) dt \\ &= I_1 + I_2. \end{aligned}$$

For I_1 we fix N such that $N \geq 2m$ and use the expansion in lemma 6.

We apply (9) taking $k=2m$ for I_2 . For derivatives with respect to parameters x_1 and ξ , we can write for example

$$r_{\xi_j}(y, D_y; x_1, \xi) = -r(y, D_y; x_1, \xi) p_{\xi_j}(x_1, y, \xi, D_y) r(y, D_y; x_1, \xi)$$

Then, by lemma 2 and symbol calculus of pseudo-differential operator with parameters $z=(x_1, \xi)$ and also noting that $R(x_1, \xi)$ has pseudo-local property with respect to y , one get the assertion.

By lemma 2 we can construct $\widetilde{A}_0(x_1, y, \xi, \eta) \in S^m(\gamma, \mu, \chi)$ such that $|\widetilde{A}_0(x_1, y, \xi, \eta)| \geq c\chi^m$ for $|x_1| + |y| \geq 1$ and $\widetilde{A}_0(x_1, y, \xi, \eta) = A_0(x_1, y, \xi, \eta)$ if $|x_1| + |y| \leq 1$. Now let us construct a pseudo-differential operator $Q_0 = q_0(x_1, y, D_{x_1}, D_{x_2}, D_y)$ which is a left parametrix for $\widetilde{A}_0(x_1, y, D_{x_1}, D_{x_2}, D_y)$ by using $k(x_1, y, \xi, \eta)$.

Put $q'_0(x_1, y, \xi, \eta) = \varphi_1(|x_1|^2 + |y|^2) \varphi_2(|\xi|^2) k(x_1, y, \xi, \eta) + \varphi_1(|x_1|^2 + |y|^2) \varphi_1(|\xi|^2) \varphi_2(|\xi|^2 + |\eta|^2) k_1(x_1, y, \xi, \eta) + \varphi_2(|x_1|^2 + |y|^2) k_1(x_1, y, \xi, \eta)$.

Here $k_1(x_1, y, \xi, \eta) = A_0(x_1, y, \xi, \eta)^{-1}$, $\varphi_1 \in C_0^\infty(\mathbb{R}^1)$ such that $\varphi_1(t) = 1$ $|t| \leq 1$

$\varphi_1(t)=0$ $|t| \geq 2$ and $\varphi_2=1-\varphi_1$. It is clear that $q'_0(x_1, y, \xi, \eta)$ belongs $S^{-m}(\nu, \mu, \chi)$.

$$(11) \quad q'_0(x_1, y, \xi, D_y) \widetilde{A}_0(x_1, y, \xi, D_y) - I \in S^{-\delta'_0}(\nu, \mu, \chi)$$

where $\delta'_0 = \min(\delta_0, 1) > 0$ using (i) of lemma 1.

From (11) one can get a left parametrix $Q_0 \in S^{-m}(\nu, \mu, \chi)$.

For any $\varepsilon > 0$ there exist Ω' and $\widetilde{A}(x_1, y, \xi, \eta) \in S^m(\nu, \mu, \chi)$ such that

$$(12) \quad \begin{aligned} \widetilde{A}(x_1, y, \xi, \eta) &= A(x_1, y, \xi, \eta) \quad \text{if } (x_1, y) \in \Omega' \\ \left| (\widetilde{A} - \widetilde{A}_0) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} (x_1, y, \xi, \eta) \right| &\leq \varepsilon (\nu + \mu)^{m-|\alpha|} \mu^{|\beta|+|\beta_3|} \chi^{-|\alpha|-|\alpha_2|} \\ &\quad \text{for } \alpha = (\alpha_1, \alpha_2, \alpha_3) \text{ and } \beta = (\beta_1, 0, \beta_3). \end{aligned}$$

If ε is sufficient small, then by (12) there exists $R_0 \in S^0(\nu, \mu, \chi)$ the inverse of $I + Q_0(\widetilde{A} - \widetilde{A}_0)$. Set $Q = R_0 Q_0$, then Q is a required left parametrix.

§ 4. Prooves of theorem B and theorem C.

At first we show theorem B.

Define a weight function $\lambda(\xi, \eta)$ by

$$\lambda(\xi, \eta) = \varphi(\xi, \eta) \left(\sum_{j=1}^2 |\xi_j|^{1/\tau_j} + |\eta|^{1/\overline{\tau}} \right) + 1$$

where $\varphi \in C^\infty(\mathbb{R}^3)$ such that $\varphi(\zeta) = 1$ ($|\zeta| \geq 1$), $\varphi(\zeta) = 0$ ($|\zeta| \leq 1/2$) and

$\overline{\tau} = \max(\tau_1, \tau_2)$. Then by (4)₀ it is clear that

$$(13) \quad \|u\|_{m, \lambda} \leq C(\|Au\| + \|u\|) \quad \forall u \in C_0^\infty(\Omega).$$

2

It is easy to see that $\tilde{A}(x_1, y, \xi, \eta)$ belongs to $S_{\lambda, \rho, \delta}^m$ if $\rho = (\tau_1, \tau_2, \tau)$

and $\delta = (0, 0, 0)$. Now from (13) we can apply theorem 3 in §1 for

$\ell = 0$, choosing θ_j such that $\theta_j = (\tau_j + \sigma_j)/2$ ($j=1, 2$) and $\theta_3 = 1$.

Then for any N

$$(14) \quad \lambda(\xi, 0)^{2m} \|\psi\|^2 \leq C \left\{ \int_{|\alpha+\beta| \leq N} \left| \sum A_{\beta}^{(\alpha)}(x_1, y, \xi, 0) \lambda(\xi, 0)^{(|\alpha|+|\beta|)\theta_1 + (|\alpha|-|\beta|)\theta_2 + |\alpha|-|\beta|} \right. \right. \\ \left. \left. \zeta^{\beta} D_{\xi}^{\alpha} \psi(\zeta) / \alpha! \beta! \right|^2 d\zeta + \lambda(\xi, 0)^{2m\tau - 2\varepsilon_0 N + d_1} \sum_{|\alpha+\beta| \leq N+d_2} \|\zeta^{\beta} D_{\xi}^{\alpha} \psi\|^2 \right\}$$

Now take N such that $2m\tau - 2\varepsilon_0 N + d_1 = 2m + \varepsilon_0$. Note that (14) is true for $\mu(\xi)$ instead of $\lambda(\xi, 0)$ and put $x_1 = t^{-\sigma_1} x_1^0, y=0$ and

$\xi_j = t^{\tau_j} \xi_j^0$ ($j=1, 2$) such that $\mu(\xi^0) = 1$. Then quasihomogeneity of $\mu(\xi)$ and $A_{\beta}^{(\alpha)}(x_1, y, \xi, 0)$ we get the following estimate

$$\|\psi\|^2 \leq C \left\{ \int_{|\alpha+\beta| \leq N} \left| \sum A_{\beta}^{(\alpha)} \left(\begin{smallmatrix} 0, 0, \alpha \\ 0, 0, \beta \end{smallmatrix} \right) (x_1^0, 0, \xi^0, 0) \zeta^{\beta} D_{\xi}^{\alpha} \psi / \alpha! \beta! \right|^2 d\zeta \right\} \\ + O(t^{-r})$$

where $r = \min(\xi_1, (\tau_j - \sigma_j)/2)$.

Note that $\sum_{|\alpha+\beta| \leq N} A_{\beta}^{(\alpha)} \left(\begin{smallmatrix} 0, 0, \alpha \\ 0, 0, \beta \end{smallmatrix} \right) (x_1^0, 0, \xi^0, 0) \zeta^{\beta} D_{\xi}^{\alpha} \psi / \alpha! \beta! = A_0(x_1^0, \zeta, \xi^0, D_{\xi}) \psi$ and

quasihomogeneity of $A_0(x_1, y, \xi, \eta)$ we get the assertion of theorem B.

To show theorem C we take $\lambda'(\xi, \eta) = (\sum_{j=1}^2 |\xi_j|^2 + |\eta|^2 + 1)^{1/2}$.

Then by (3)₀ and theorem 3 we get for $(x_1, y) \in K$

$$|\chi(\xi, \eta)|^m \leq c_K |A_0(x_1, y, \xi, \eta)| \quad \text{for } |\xi| + |\eta| \geq c > 0$$

By quasihomogeneity of $A_0(x_1, y, \xi, \eta)$, it is elliptic for $|x_1| + |y| \neq 0$.

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