

22 次 Mathieu 群の dual hyperoval を通じた簡明な構成について

$$m(x, y) = x \otimes y + \Delta(\iota(x \times y))$$

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1 Contents and some history

1.1 DHO

We first recall the notion of dimensional dual hyperovals. Let n be an integer with $n \geq 2$ and let U be a finite vector space over $\mathbb{F}_q$ of dimension at least $2n - 1$.

Definition 1 *A collection $\mathcal{S}$ of n -dimensional subspaces of U is called a **dual hyperoval** over $\mathbb{F}_q$ of **rank** n (abbreviated to n -DHO in the sequel), if it satisfies the following conditions (i), (ii) and (iii):*

- (i) $\dim(X \cap Y) = 1$ for distinct $X, Y \in \mathcal{S}$,
- (ii) $X \cap Y \cap Z = \{0\}$ for mutually distinct $X, Y, Z \in \mathcal{S}$,
- (iii) $|\mathcal{S}| = 1 + \{(q^n - 1)/(q - 1)\}$.

1.2 Contents of my talk

In this talk, I will first give

a simple construction of a 3-DHO $\mathcal{M}$ over $\mathbb{F}_4$ (so that $|\mathcal{M}| = 22$),

which is given inside the symmetric tensor product $S^2(\mathbb{F}_4^3)$ as a deformation of the Veronesean DHO. Based on this construction, then I will present

a self-contained introduction to M_{22} .

More precisely I will discuss

- how to see $\text{Aut}(\mathcal{M}) \cong 3.M.2$ with M a simple group of order $2^7.3^2.5.7.11$, and
- how to find explicit unitary matrices in $SU_6(\mathbb{F}_4)$ generating $L(\mathcal{M}) \cong 3.M$.

1.3 Known models of $\mathcal{M}$

As far as I know, the 3-DHO $\mathcal{M}$ over $\mathbb{F}_4$ associated with the simple Mathieu group M_{22} was first mentioned in paper [1] below. It is given in terms of the Leech lattice. Then it appears in [2] as a table in terms of MOG arrangement. The paper [3] characterizes $\mathcal{M}$ as a 3-DHO $\mathcal{M}$ over $\mathbb{F}_4$ of unitary polar type, in the sense that every member of the DHO is totally isotropic with respect to a nondegenerate hermitian form on the ambient space. It also recovers the seemingly miracle table given in [2] as explicit descriptions of all members.

- 1 W. Jónsson and J. McKay, “More about the Mathieu group M_{22} ”, Can.J.Math. 28 (1976), 929–937.
- 2 J.H.Conway, R.T.Curtis, S.P.Norton, R.A.Parker, W.A.Wilson, p.39 in “Atlas of Finite Groups”, 1985.
- 3 N.Nakagawa, “On 2-dimensional dual hyperovals of polar type”, Utilitas Mathematica 76 (2008), 101–114.

Below I repeat Nakagawa’s descriptions of members of $\mathcal{M}$, where $\mathbf{e}_i$ ($i = 0, \dots, 5$) are basis for a 6-dimensional vector space U over $\mathbb{F}_4$ equipped with a nondegenerate hermitian form $(\ , \)$ with $(\mathbf{e}_i, \mathbf{e}_j) = \delta_{i+j,5}$ ($0 \leq i, j \leq 5$). The letters α and θ denote nonzero elements in $\mathbb{F}_4$ with $\beta = \alpha + \theta$, and $\bar{x} = x^2$ for $x \in \mathbb{F}_4$.

$$\begin{aligned}
A &:= \langle \mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2 \rangle, \\
A[\mathbf{e}_0] &:= \langle \mathbf{e}_0, \mathbf{e}_3, \mathbf{e}_4 \rangle, \\
A[\mathbf{e}_1] &:= \langle \mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_5 \rangle, \\
A[\mathbf{e}_2] &:= \langle \mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_5 \rangle, \\
A[\alpha\mathbf{e}_0 + \mathbf{e}_1] &:= \langle \mathbf{e}_0 + \alpha\mathbf{e}_3, \mathbf{e}_1 + \bar{\alpha}\mathbf{e}_3, \mathbf{e}_2 + \alpha\mathbf{e}_4 + \bar{\alpha}\mathbf{e}_5 \rangle, \\
A[\alpha\mathbf{e}_0 + \mathbf{e}_2] &:= \langle \mathbf{e}_0 + \alpha\mathbf{e}_4, \mathbf{e}_2 + \bar{\alpha}\mathbf{e}_4, \mathbf{e}_1 + \alpha\mathbf{e}_3 + \bar{\alpha}\mathbf{e}_5 \rangle, \\
A[\alpha\mathbf{e}_1 + \mathbf{e}_2] &:= \langle \mathbf{e}_1 + \alpha\mathbf{e}_5, \mathbf{e}_2 + \bar{\alpha}\mathbf{e}_5, \mathbf{e}_0 + \alpha\mathbf{e}_3 + \bar{\alpha}\mathbf{e}_4 \rangle, \\
A[\alpha\theta\mathbf{e}_0 + \alpha\mathbf{e}_1 + \mathbf{e}_2] &:= \langle \theta\mathbf{e}_0 + \mathbf{e}_1 + \bar{\alpha}\mathbf{e}_2, \bar{\beta}\mathbf{e}_0 + \mathbf{e}_3 + \alpha\mathbf{e}_4, \beta\mathbf{e}_2 + \bar{\theta}\mathbf{e}_4 + \mathbf{e}_5 \rangle.
\end{aligned}$$

1.4 Motivation to find another description

All descriptions obtained in papers [1], [2] and [3] are just tables. Thus in analogy with coding theory, they are just “generator matrices”. This gives difficulties in finding the intersection of two members, specifically between $A[\alpha\theta\mathbf{e}_0 + \alpha\mathbf{e}_1 + \mathbf{e}_2]$ ’s in the description by Nakagawa. Consequently, it is not straightforward to find automorphisms based only on these tables.

Thus we need more concise description of members of $\mathcal{M}$, which corresponds “parity-check conditions” in analogy with coding theory.

2 A construction of $\mathcal{M}$

2.1 Reviews on $\mathbb{F}_4^3$ and $S^2(\mathbb{F}_4^3)$

2.1.1 Notation

We use the letter V to denote a 3-dimensional vector space over $\mathbb{F}_4$ with a specified basis e_i ($i = 0, 1, 2$). Consider the symmetric square tensor product $S^2(V)$ of V . (which is obtained as $(V \otimes V)/W$ with W the subspace of $V \otimes V$ spanned by $x \otimes y + y \otimes x$ for all $x, y \in V$. We denote the image of $x \otimes y$ in this factor space by the same symbol $x \otimes y$, so that we have $x \otimes y = y \otimes x$ for all $x, y \in V$.) The vector space $S^2(V)$ is a 6-dimensional vector space over $\mathbb{F}_4$ with a basis Δ_i, ∇_i ($i = 0, 1, 2$), where

$$\begin{aligned}\Delta_i &:= e_i \otimes e_i \quad (i = 0, 1, 2), \\ \nabla_i &:= e_j \otimes e_k \quad (\{i, j, k\} = \{0, 1, 2\}).\end{aligned}$$

Explicitly, $\nabla_0 = e_1 \otimes e_2$, $\nabla_1 = e_2 \otimes e_0$, $\nabla_2 = e_0 \otimes e_1$.

Delta-map We denote by Δ a map from V to $S^2(V)$ given by $\Delta(x) := x \otimes x$. Δ is a $\mathbb{F}_4$ -semilinear injection, because for all $x, y \in V$ we have

$$\Delta(x + y) = \Delta(x) + \Delta(y), \quad \Delta(\alpha x) = \alpha^2 \Delta(x).$$

2.1.2 A quadratic map on $V = \mathbb{F}_4^3$

The map $\iota : V \rightarrow V$ sending $x = x_0 e_0 + x_1 e_1 + x_2 e_2 \in V$ to

$$\sum_i x_j x_k e_i = x_1 x_2 e_0 + x_2 x_0 e_1 + x_0 x_1 e_2 \in V$$

is a quadratic map, in the sense that the associated map $(x, y) \mapsto \iota(x + y) + \iota(x) + \iota(y) + \iota(0)$ is a bilinear (in fact, alternating bilinear) map from $V \times V$ to V . The associated alternating map

$$\begin{aligned}& \iota(x + y) + \iota(x) + \iota(y) + \iota(0) \\ &= (x_1 y_2 + x_2 y_1) e_0 + (x_2 y_0 + x_0 y_2) e_1 + (x_0 y_1 + x_1 y_0) e_2\end{aligned}$$

is the **exterior product** on $V = \mathbb{F}_4^3$, which I shall denote $x \times y$, following the common notation in college mathematics.

2.1.3 Basic equations

For all $x, y, z \in V = \mathbb{F}_4^3$, we have

$$\begin{aligned}x \times y &= \iota(x + y) + \iota(x) + \iota(y), \\ (x \times y) \cdot z &= \det(r(x, y, z)), \\ x \times (y \times z) &= (x \cdot z)y + (x \cdot y)z,\end{aligned}$$

where $x \cdot y := \sum_{i=0}^2 x_i y_i$ (**dot product**) and

$$r(x, y, z) = \begin{pmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{pmatrix}.$$

2.2 Construction

2.2.1 Vector $m(x, y)$ in $S^2(\mathbb{F}_4^3)$

Definition 2 For $x, y \in V = \mathbb{F}_4^3$, define a vector $m(x, y) \in S^2(V)$ by

$$m(x, y) := x \otimes y + \Delta(\iota(x \times y)). \quad (1)$$

The expression of $m(x, y)$ as a linear combination of basis ∇_i, Δ_i ($i = 0, 1, 2$) of $S^2(V)$ is given as follows for $x = \sum_{i=0}^2 x_i e_i$ and $y = \sum_{i=0}^2 y_i e_i$:

$$m(x, y) = \sum_{i=0}^2 (x \times y)_i \nabla_i + \sum_{i=0}^2 (x_i y_i + \overline{(x \times y)_j (x \times y)_k}) \Delta_i,$$

where $\{i, j, k\} = \{0, 1, 2\}$ and $\bar{\alpha} = \alpha^2$ ($\alpha \in \mathbb{F}_4$).

Observe $m(x, y) = m(y, x)$, $m(x, x) = \Delta(x)$.

2.2.2 Subsets $A[v]$, A of $S^2(\mathbb{F}_4^3)$

Observe $m(\alpha x, y) = \alpha m(x, y)$ for $\alpha \in \mathbb{F}_4^\times$; because $\alpha^4 = \alpha$ (we are working with $\mathbb{F}_4$!). Thus a subset

$$A[v] := \{m(x, v) \mid x \in V\}$$

of $S^2(V)$ depends only on the projective point $[v]$ (1-space containing v).

We set

$$A := \{\Delta(x) \mid x \in V\},$$

which forms a subspace of $S^2(V)$ by the semilinearity of Δ .

In fact, $A[v]$ is a subspace of $S^2(V)$, as we shall see below.

2.2.3 Additive formula

Lemma 1 For $a, b, v \in V$, the following equations hold with $\delta := \det(r(a, b, v))$:

$$m(a, v) + m(b, v) = m(a + b, v) + \Delta(\delta v), \quad (2)$$

$$m(a, v) + m(b, v) = m(a + b + \bar{\delta} v, v). \quad (3)$$

Equation (3) follows from equation (2), because $\Delta(\delta v) = m(\delta v, \delta v) = \bar{\delta} m(v, v)$. Moreover, Equation (3) implies that $A[v]$ is a subspace of $S^2(V)$.

2.2.4 Proof of additive formula

As $m(x, y) = x \otimes y + \Delta(\iota(x \times y))$ and Δ is $\mathbb{F}_2$ -linear, in order to prove equation (2) it suffices to show

$$\iota((a + b) \times v) + \iota(a \times v) + \iota(b \times v) = \delta v. \quad (4)$$

As ι is quadratic with the associated form $\times$ (that is, $\iota(v + w) + \iota(v) + \iota(w) = v \times w$), the left hand side of equation (4) is $(a \times v) \times (b \times v)$, which equals $\{(a \times v) \cdot b\}v + \{(a \times v) \cdot v\}b = \det(r(a, b, v))v$.

2.2.5 DHO $\mathcal{M}$

We shall now define a DHO $\mathcal{M}$. We denote by $\mathbf{PG}(V)$ the set of projective points in V .

Proposition 1 (i) *The collection $\mathcal{M} := \{A[v] \mid [v] \in \mathbf{PG}(V)\} \cup \{A\}$ is a DHO of rank 3 over $\mathbb{F}_4$.*

(ii) *For $[v] \in \mathbf{PG}(V)$, $A \cap A[v]$ is a 1-subspace spanned by $\Delta(v)$.*

(iii) *For distinct $[v], [w]$ in $\mathbf{PG}(V)$, $A[v] \cap A[w]$ is a 1-subspace spanned by $m(w, v) = m(v, w)$.*

2.2.6 Proof of (iii)

Assume $0 \neq c := m(x, v) = m(y, w) \in A[v] \cap A[w]$. Comparing the coefficients of ∇_i in the expressions of $m(x, v)$ and $m(y, w)$, this implies that $(x \times v)_i = (y \times w)_i$ ($i \in \{0, 1, 2\}$) and thus $x \times v = y \times w$ ($=: a$).

It's easy to show $a \neq 0$.

Then $a^\perp := \{z \in V \mid z \cdot a = 0\}$ is a hyperplane of V , which is spanned by v and w , as $[v] \neq [w]$. Thus $a^\perp \ni x = \alpha v + \beta w$, $y = \gamma v + \delta w$ for some $\alpha, \beta, \gamma, \delta$ in $\mathbb{F}_4$.

Then the additive formula implies

$$\begin{aligned} m(x, v) &= m(\alpha v + \beta w, v) \\ &= m(\alpha v, v) + m(\beta w, v) + \Delta(\det(r(\alpha v, \beta w, v))v) \\ &= \alpha \Delta(v) + \beta m(w, v). \end{aligned}$$

Similarly, we have $m(y, w) = \gamma m(v, w) + \delta \Delta(w)$. As $m(x, v) = m(y, w)$, these expressions imply

$$(\beta + \gamma)m(v, w) = \Delta(\bar{\alpha}v + \bar{\delta}w).$$

This holds iff $\beta + \gamma = 0 = \bar{\alpha}v + \bar{\delta}w$, or equivalently $\beta = \gamma$ and $\alpha = \delta = 0$. Thus $c = m(x, v) = m(y, w) = \beta m(v, w)$.

3 Automorphisms

3.1 Basic idea

3.1.1 Automorphisms

Definition 3 $\text{Aut}(\mathcal{M})$ and $L(\mathcal{M})$ respectively denote the groups of $\mathbb{F}_4$ -semilinear and linear bijections on $S^2(V)$ permuting the members of $\mathcal{M}$.

It is not difficult to establish the following facts:

- $\text{Aut}(\mathcal{M})$ contains $L(\mathcal{M})$ with index two: a field automorphism lies in $\text{Aut}(\mathcal{M}) \setminus L(\mathcal{M})$.
- The kernel of the action of $L(\mathcal{M})$ on $S^2(V)$ is $Z := \langle \omega I_6 \rangle$, a central subgroup of order 3 of $SL_6(4)$, where ω denotes a primitive cubic root of unity in $\mathbb{F}_4$.
- The stabilizer of A in $L(\mathcal{M})/Z$ is a subgroup of $GL(V) \cong GL_3(4)$.

3.1.2 A method to find an automorphism of a DHO

Assume $\lambda \in L(\mathcal{M})$ stabilizes $A = \{\Delta(x) \mid x \in V\}$. We shall explain a basic idea to find the action of λ on $m(x, y)$. It is easy to see that there is a linear bijection g on V such that $\Delta(x)^\lambda = \Delta(x^g)$ for all $x \in V$.

As $\langle \Delta(x) \rangle = A \cap A[x]$ for $x \neq 0$, $\langle \Delta(x)^\lambda \rangle = A^\lambda \cap A[x]^\lambda = A \cap A[x]^\lambda$. On the other hand, $\langle \Delta(x)^\lambda \rangle = \langle \Delta(x^g) \rangle = A \cap A[x^g]$. As $\mathcal{M}$ is a DHO, we have

$$A[x]^\lambda = A[x^g] \quad (x \in V, x \neq 0).$$

Then for $x, y \in V$ with $[x] \neq [y]$ we have

$$\begin{aligned} \langle m(x, y)^\lambda \rangle &= (A[x] \cap A[y])^\lambda = A[x]^\lambda \cap A[y]^\lambda \\ &= A[x^g] \cap A[y^g] = \langle m(x^g, y^g) \rangle. \end{aligned}$$

Thus we have the following “Key Equation”:

$$m(x, y)^\lambda = \gamma_{x,y} m(x^g, y^g) \quad \text{for some } \gamma_{x,y} \in \mathbb{F}_4^\times. \quad (5)$$

This restricts the shape of g , as $m(x, y)$ ($x, y \in V$) span $S^2(V)$.

3.2 The stabilizer of A

3.2.1 Unitary form

Define a unitary form $(,)$ on $S^2(V)$ by

$$\begin{aligned} (\Delta_i, \Delta_j) &:= 0 =: (\nabla_i, \nabla_j) \quad (i, j \in \{0, 1, 2\}), \\ (\Delta_i, \nabla_j) &:= 1 \text{ or } 0 \text{ according as } i = j \text{ or not.} \end{aligned}$$

Lemma 2 *We can verify the following facts:*

- (1) *Every member of $\mathcal{M}$ is totally isotropic.*
- (2) *$L(\mathcal{M})$ preserves $(,)$.*

3.2.2 An important property of $m(x, y)$

For $x = \sum_{i=0}^2 x_i e_i$ and $y = \sum_{i=0}^2 y_i e_i$ in V , we already saw

$$m(x, y) = \sum_{i=0}^2 (x \times y)_i \nabla_i + \sum_{i=0}^2 (x_i y_i + \overline{(x \times y)_j (x \times y)_k}) \Delta_i.$$

Observe that $\nabla_i = e_j \otimes e_k = m(e_j, e_k)$, $\Delta_i = \Delta(e_i)$, $\det(r(e_0, e_1, e_2)) = 1$.

Lemma 3 *The same formula holds for another basis u_i ($i = 0, 1, 2$) of V ; namely, for $x = \sum_{i=0}^2 x_i u_i$, $y = \sum_{i=0}^2 y_i u_i$ in V with $\delta := \det(r(u_0, u_1, u_2))$, we have*

$$\begin{aligned} m(x, y) &= \sum_{i=0}^2 (x \times y)_i m(u_j, u_k) \\ &\quad + \sum_{i=0}^2 (x_i y_i + \overline{\delta (x \times y)_j (x \times y)_k}) \Delta(u_i). \end{aligned}$$

3.2.3 The stabilizer of A

We shall now state the main result and give its proof.

Proposition 2 *The stabilizer of A in $L(\mathcal{M})$ coincides with $\{\tilde{g} \mid g \in SL(V)\}$, where, if $g_i := e_i g = g_{i0}e_0 + g_{i1}e_1 + g_{i2}e_2$ ($i = 0, 1, 2$), the action of $\tilde{g}$ is given as follows:*

$$\begin{aligned}\Delta_i \tilde{g} &= \overline{g_{i0}}\Delta_0 + \overline{g_{i1}}\Delta_1 + \overline{g_{i2}}\Delta_2, \\ \nabla_i \tilde{g} &= g_j \otimes g_k + \Delta(\iota(g_j \times g_k)),\end{aligned}$$

for $\{i, j, k\} = \{0, 1, 2\}$. Moreover $m(x, y)^{\tilde{g}} = m(x^g, y^g)$ and $A[x]^{\tilde{g}} = A[x^g]$ ($x, y \in V$).

3.2.4 Proof of Proposition

Take $\lambda \in L(\mathcal{M})$ stabilizing A . Then there is $g \in GL(V)$ such that $\Delta(x)^\lambda = \Delta(x^g)$ ($x \in V$). The vectors $g_i := e_i^g$ form a basis of V and $\Delta_i^\lambda = \Delta(g_i)$ ($i = 0, 1, 2$).

By the previous argument given in Subsubsection 3.1.2, $A[e_i]^\lambda = A[e_i^g] = A[g_i]$ and

$$(e_i \otimes e_j)^\lambda = \gamma_{e_i, e_j} m(g_i, g_j) = \gamma_{e_i, e_j} \{g_i \otimes g_j + \Delta(\iota(g_i \times g_j))\}.$$

As λ preserves the unitary form $(\ , \)$, we can show that $\gamma_{e_i, e_j} = \overline{\det(g)}$.

Thus the action of λ on the basis Δ_i and ∇_i for $S^2(V)$ is determined as follows: for any $i \in \{0, 1, 2\} = \{i, j, k\}$,

$$\Delta_i^\lambda = \Delta(g_i), \quad \nabla_i^\lambda = (e_j \otimes e_k)^\lambda = \overline{\det(g)} m(g_j, g_k).$$

Take any distinct $[x], [y] \in \mathbf{PG}(V)$ with $x = \sum_{i=0}^2 x_i e_i$, $y = \sum_{i=0}^2 y_i e_i$. As we noticed above in equation (5),

$$m(x, y)^\lambda = \gamma_{x, y} m(x^g, y^g) \text{ for some } \gamma_{x, y} \in \mathbb{F}_4^\times.$$

The left hand side of equation (5) is calculated as

$$\begin{aligned}m(x, y)^\lambda &= \sum_{i=0}^2 \overline{\det(g)} (x \times y)_i m(g_j, g_k) \\ &\quad + \sum_{i=0}^2 \{x_i y_i + \overline{(x \times y)_j (x \times y)_k}\} \Delta(g_i),\end{aligned}\tag{6}$$

in view of the above action of λ on Δ_i, ∇_i applied to $m(x, y) = \sum_{i=0}^2 (x \times y)_i \nabla_i + \sum_{i=0}^2 \{x_i y_i + \overline{(x \times y)_j (x \times y)_k}\} \Delta_i$.

On the other hand, the right hand side of equation (5) is given by Lemma 3 applied to basis g_i for V ($\delta := \det(r(g_0, g_1, g_2)) = \det(g)$):

$$\begin{aligned}m(x^g, y^g) &= m\left(\sum_{i=0}^2 x_i g_i, \sum_{i=0}^2 y_i g_i\right) \\ &= \sum_{i=0}^2 (x \times y)_i m(g_j, g_k) \\ &\quad + \sum_{i=0}^2 \{x_i y_i + \overline{\delta (x \times y)_j (x \times y)_k}\} \Delta(g_i).\end{aligned}\tag{7}$$

As $\Delta(g_i)$ and $m(g_j, g_k)$ ($i \in \{0, 1, 2\} = \{i, j, k\}$) form a basis for $S^2(V)$, “KeyEquation” (equation (5)) together with equations (6) and (7) implies

$$\begin{aligned} \overline{\det(g)}(x \times y)_i &= \gamma_{xy}(x \times y)_i, \text{ and} \\ x_i y_i + \overline{(x \times y)_j (x \times y)_k} &= \gamma_{xy} \{x_i y_i + \overline{\det(g)(x \times y)_j (x \times y)_k}\} \end{aligned}$$

for all $i \in \{0, 1, 2\}$. As $[x] \neq [y]$, there exists $i \in \{0, 1, 2\}$ with $(x \times y)_i \neq 0$. Thus we have $\gamma_{xy} = \det(g)$ from the first equation above. Then the second equation above reads

$$(x_i y_i)(1 + \overline{\det(g)}) = (1 + \det(g))\overline{(x \times y)_j (x \times y)_k}$$

for all $i \in \{0, 1, 2\}$.

This conclusion holds for every distinct $[x], [y] \in \mathbf{PG}(V)$. Take $x = e_0 + e_1 + e_2$ and $y = e_0 + \omega e_1 + \overline{\omega} e_2$. Then the above conclusion for these x, y reads $1 + \overline{\det(g)} = 1 + \det(g)$, whence $\det(g) = \overline{\det(g)} = 1$.

Thus we showed that if λ is a linear automorphism of $\mathcal{M}$ stabilizing A , then λ is of the form $\tilde{g}$ for some $g \in SL(V)$.

Conversely, we can show that $\tilde{g}$ for $g \in SL(V)$ in fact lies in $L(\mathcal{M})$.

3.2.5 Matrix form

For $g \in SL(V)$, we also use g to denote the matrix representing g with respect to e_i . Then the matrix representing $\tilde{g}$ in Proposition 2 with respect to Δ_i, ∇_i is given by

$$\begin{pmatrix} \overline{g} & 0 \\ L(g) & {}^t g^{-1} \end{pmatrix}, \quad L(g) = {}^t \iota({}^t g) + \iota({}^t \overline{g})^{-1},$$

where $\iota(h) = \begin{pmatrix} h_{01}h_{02} & h_{02}h_{00} & h_{00}h_{01} \\ h_{11}h_{12} & h_{12}h_{10} & h_{10}h_{11} \\ h_{21}h_{22} & h_{22}h_{20} & h_{20}h_{21} \end{pmatrix}$ for $h = (h_{ij})$. (The following property of ι may be of some interest: for matrices a, b of degree 3, we have $\iota(ab) = \overline{a}\iota(b) + \iota(a)({}^t b^{-1})$.)

For example, take the following matrices in $SL(V)$ generating $3_+^{1+2} : Q_8$, where we adopt the usual convention to denote monomial matrices t_1 and t_2 .

$$\begin{aligned} t_1 &:= (e_0, e_1, e_2), & t_2 &:= \text{diag}(1, \omega, \overline{\omega}); \\ q_1 &:= \begin{pmatrix} 1 & 1 & 1 \\ 1 & \overline{\omega} & \omega \\ 1 & \omega & \overline{\omega} \end{pmatrix}, & q_2 &:= \begin{pmatrix} 1 & \overline{\omega} & \overline{\omega} \\ \omega & \omega & \overline{\omega} \\ \omega & \overline{\omega} & \omega \end{pmatrix} \end{aligned}$$

The the corresponding matrices in the stabilizer of A in $L(\mathcal{M})$ can be obtained as follows:

$$\begin{aligned} \tilde{t}_1 &= (\Delta_0, \Delta_1, \Delta_2)(\nabla_0, \nabla_1, \nabla_2), & \tilde{t}_2 &= \text{diag}(1, \overline{\omega}, \omega, 1, \overline{\omega}, \omega); \\ \tilde{q}_1 &= \begin{pmatrix} \overline{q_1} & 0 \\ 0 & \overline{q_1} \end{pmatrix}, & \tilde{q}_2 &= \begin{pmatrix} \overline{q_2} & 0 \\ \overline{q_2} & \overline{q_2} \end{pmatrix}. \end{aligned}$$

3.3 Structure of $L(\mathcal{M})$

3.3.1 An involution moving A

The above arguments can also be applied to find a linear automorphism moving A . For example, consider the following involutive linear automorphism σ on $S^2(V)$ (represented with respect to basis Δ_i and ∇_i ($i = 0, 1, 2$).) Then σ sends A to $A[e_2]$.

$$\sigma = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Lemma 4 *Let σ be a linear bijection on $S^2(V)$ which fixes ∇_2 and Δ_2 and interchanges the pairs (∇_i, Δ_i) for $i = 0$ and 1 . Then σ is an automorphism of $\mathcal{M}$. Moreover $\Delta(x)\sigma = m(e_2, \delta(x))$ and $m(x, y)\sigma = m(\delta(x), \delta(y))$, and hence $A\sigma = A[e_2]$, $A[e_2]\sigma = A$, $A[x]\sigma = A[\delta(x)]$ for every $x, y \in V \setminus [e_2]$, where $\delta(x)$ is given by:*

$$\delta(x) := \overline{x_1}e_0 + \overline{x_0}e_1 + (x_0x_1 + \overline{x_2})e_2.$$

3.3.2 Structure of $\text{Aut}(\mathcal{M})$

By Proposition 2, the stabilizer of A in $L(\mathcal{M})$ induces a permutation group isomorphic to $SL_3(4)/Z(SL_3(4)) \cong PSL_3(4)$ on the 21 members in $\mathcal{M} \setminus \{A\}$. This group is a non-abelian simple group and is doubly transitive on $\mathcal{M} \setminus \{A\}$, as this action is equivalent to the 2-transitive action of $PSL_3(4)$ on 21 points of $\mathbf{PG}(V)$. Then the existence of σ in $L(\mathcal{M})$ moving A to $A[e_2]$ implies that $L(\mathcal{M})/Z$ is a triply transitive permutation group on $\mathcal{M}$ with stabilizer $PSL_3(4)$. Hence $L(\mathcal{M})/Z$ is a simple group of order $22|PSL_3(4)| = 2^7 3^2 \cdot 5 \cdot 7 \cdot 11$ acting 3-transitively on $\mathcal{M}$.

Summarizing, the structure of the automorphism group $\text{Aut}(\mathcal{M})$ is determined as follows:

- $[\text{Aut}(\mathcal{M}) : L(\mathcal{M})] = 2$ and $\text{Aut}(\mathcal{M}) \setminus L(\mathcal{M})$ contains an involution (the field automorphism).
- $L(\mathcal{M})$ is a subgroup of the special unitary group $SU_6(\mathbb{F}_4)$ containing the group Z of scalars (of order 3 inverted by the field automorphism).
- $L(\mathcal{M})/Z$ is a non-abelian group of order $2^7 3^2 \cdot 5 \cdot 7 \cdot 11$ acting 3-transitively on $\mathcal{M}$. (As a Sylow 3-subgroup of $L(\mathcal{M})$ is 3_+^{1+2} , which is not split over its center Z , the extension $L(\mathcal{M})/Z$ does not split by a theorem of Gashütz.)

The explicit identification of the simple group $L(\mathcal{M})/Z$ with the Mathieu simple group M_{22} can also be given as follows. We first recall the fact that there is a unique block design with parameters $t = 3, v = 22, k = 6$ and $\lambda = 1$ and that the Mathieu group M_{22} is defined to be the automorphism group of such a block design. Thus it suffices to construct a block design with parameters $(t, v, k, \lambda) = (3, 22, 6, 1)$ on which $L(\mathcal{M})/Z$ acts faithfully. As the set of points, we take $\mathcal{M}$. We shall construct a block design on $\mathcal{M}$ by defining a block as follows: take three distinct members X_i ($i \in \{0, 1, 2\}$) of $\mathcal{M}$.

Then it can be uniquely extended to a 6-subset $B(X_0, X_1, X_2) := \{X_k \mid k \in \{0, \dots, 5\}\}$ of $\mathcal{M}$ with the following property: for any 3-subset $\{p, q, r\}$ of $\{0, \dots, 5\}$, the 2-subspace spanned by $X_p \cap X_q$ and $X_p \cap X_r$ contains $X_p \cap X_j$ for all $j \in \{0, \dots, 5\} \setminus \{p, q, r\}$. To verify this claim, we may assume that $X_0 = A$, $X_i = A[e_i]$ ($i = 1, 2$) by the triply transitivity of $\text{Aut}(\mathcal{M})$ on the members of $\mathcal{M}$. It can be verified that $B(X_0, X_1, X_2) = \{X_k \mid k \in \{0, \dots, 5\}\}$ with $X_{3+j} := A[e_1 + \omega^j e_2]$ ($j = 0, 1, 2$). The above property implies that $B(X_0, X_1, X_2) = B(X_p, X_q, X_r)$ for every 3-subset $\{p, q, r\}$ of $\{0, \dots, 5\}$. We adopt as blocks all 6-subsets $B(X_0, X_1, X_2)$ determined by 3-subsets $\{X_0, X_1, X_2\}$ of $\mathcal{M}$. Then the 22-set $\mathcal{M}$ together with the set $\mathcal{B}$ of all blocks forms a design with parameters $t = 3$, $v = 22$, $k = 6$ and $\lambda = 1$. As $L(\mathcal{M})/Z$ acts faithfully on this block design $(\mathcal{M}, \mathcal{B})$, this establishes the claim $L(\mathcal{M}) \cong M_{22}$.