

Remarks on Pointwise Nonlinear ergodic theorems in L_p

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1. Introduction

In [1], [2] Baillon considered a class of nonexpansive self-mappings T of a bounded closed convex subset C of a Hilbert space H or L_p with $1 < p < \infty$, formed the Cesàro $(C, 1)$ mean value process

$$C_n^{(1)}[T]f = \frac{1}{n+1} \sum_{k=0}^n T^k f, \quad n \geq 0$$

for $f \in C$ and established the weak nonlinear ergodic theorem for T . Then later, Krengel and Lin [5] considered another class of order preserving, L_∞ -norm decreasing and L_1 -nonexpansive operators in L_p and proved the following weak nonlinear ergodic theorem which can not be covered by Baillon's theorem: Let T be an operator in L_p ($1 < p < \infty$) which is order preserving, L_∞ -norm decreasing and L_1 -nonexpansive. Then for any $f \in L_p$, $C_n^{(1)}[T]f$ converges weakly in L_p . If the basic measure is finite, $C_n^{(1)}[T]f$ converges weakly in L_1 for $f \in L_1$. The same result holds for operators in L_1^+ .

In the settings of Baillon, Krengel and Lin, however, one can only expect weak convergence of the $(C, 1)$ process. Indeed, the example due to Krengel and Lin [5] shows that $C_n^{(1)}[T]f$ need not converge in the strong topology of L_p and the example given by Krengel [4] shows that the pointwise convergence of $C_n^{(1)}[T]f$ may fail to hold. Note here that the iteration process considered by Wittmann [7] has a different aspect. So, as suggested (implicitly) by Krengel and Lin, it seems to be a question of great significance to find those (extra) conditions under which $C_n^{(1)}[T]f$ converges almost everywhere or in the strong topology of L_p (cf. [7]). In [9] the author made an attempt to deal with this question in the strong topology of L_p . [By the way, in [4] Krengel dared to say: It therefore seems that the example essentially eliminates all hopes for general pointwise nonlinear ergodic theorems. Of course, the possibility of positive results for specific class of nonlinear operators remains.]

We are particularly interested in finding some conditions or in changing the settings under which the almost everywhere convergence of $C_n^{(1)}[T]f$ holds in both linear and nonlinear cases. In a forthcoming paper [10] the author proved the following theorem:

Theorem. Let T be an order preserving operator in L_p ($1 \leq p \leq \infty$) with $T(0) = 0$. Let $0 < \alpha < \infty$ and $f, f^* \in L_p^+$. Assume that E is a measurable set of measure zero such that for any $\omega \in E^c$, the generalized Dirichlet series $\sum_{n=0}^{\infty} \frac{A_n^{\alpha-1}(T^n f)(\omega)}{(A_n^\alpha)^\alpha}$ converges (absolutely) for each $z \in C$ with $\operatorname{Re}(z) > 1$. Here A_n^α , $n \geq 0$, denote the (C, α) coefficients of order α . Assume that for any $\omega \in E^c \cup \{f^* < \infty\}$, the analytic function

$$G_\omega(z) = \sum_{n=0}^{\infty} \frac{A_n^{\alpha-1}(T^n f)(\omega)}{(A_n^\alpha)^\alpha} - \frac{f^*(\omega)}{z-1}, \quad \operatorname{Re}(z) > 1$$

has an analytic or just continuous extension (also called $G_\omega(\cdot)$) to the closed half-plane $\{\operatorname{Re}(z) \geq 1\}$. Finally assume that for each $\omega \in E^c \cup \{f^* < \infty\}$, there exists a constant $M_\omega \geq 1$ such that

$$(*) \quad G_\omega(z) = O(|z|^{M_\omega}), \quad \operatorname{Re}(z) > 1.$$

Then

$$\lim_{n \rightarrow \infty} \frac{1}{A_n^\alpha} \sum_{k=0}^n A_k^{\alpha-1} (T^k f)(\omega) = f^*(\omega)$$

holds for almost all ω .

In this paper we will show that the above theorem is also valid even without assuming the growth condition (*). Our general approach to understanding analytic conditions will be via Dirichlet series concerning operators in L_p (cf. [10]). The proof will make heavy use of Landau's Tauberian technique in Landau-Wiener-Ikehara's Tauberian theorem for Dirichlet series (cf. [3]).

2. Pointwise nonlinear ergodic theorems in L_p

Let $L_p = L_p(\Omega, \mathfrak{E}, \mu)$, $1 \leq p \leq \infty$, be the usual Lebesgue spaces, where $(\Omega, \mathfrak{E}, \mu)$ is a σ -finite measure space. A operator T in L_p is said to be L_p -norm decreasing if $\|Tf\|_p \leq \|f\|_p$ holds for all $f \in L_p$. T is called order preserving in L_p if $f, g \in L_p$ and $f \leq g$ imply $Tf \leq Tg$. T is called nonexpansive in L_p if $\|Tf - Tg\|_p \leq \|f - g\|_p$ holds for all $f, g \in L_p$. We say that T is positively homogeneous if $T(cf) = cTf$ for all $f \in L_p$ and any constant $c \geq 0$. For a real number $\alpha > -1$ and each integer $n \geq 0$, let A_n^α denote the (C, α) coefficient of order α , which is defined by the generating function

$$\frac{1}{(1-\lambda)^{\alpha+1}} = \sum_{n=0}^{\infty} A_n^\alpha \lambda^n, \quad 0 < \lambda < 1$$

with $A_0^\alpha = 1$. We also let $A_0^{-1} = 1$ and $A_n^{-1} = 0$ for all $n \geq 1$. Then for $\alpha > -1$, we have $A_n^\alpha > 0$, $A_n^0 = 1$, $A_n^\alpha \sim \frac{n^\alpha}{\Gamma(\alpha+1)}$, and

$$A_n^\alpha = \sum_{k=0}^n A_{n-k}^{\alpha-1} = \sum_{k=0}^n A_k^{\alpha-1} = \frac{(\alpha+1)(\alpha+2) \cdots (\alpha+n)}{n!}.$$

Moreover, it follows that A_n^α is increasing in n for $\alpha > 0$ and decreasing in n for $-1 < \alpha < 0$. We will prove

Theorem 1. Let T be an order preserving operator in L_p ($1 \leq p \leq \infty$) with $T(0) = 0$ and let $0 < \alpha < \infty$, $f, f^* \in L_p$. Assume that E is a set in $\mathfrak{E}$ with $\mu(E) = 0$ such that for any $\omega \in \Omega - E$, the generalized Dirichlet series $\sum_{n=0}^{\infty} \frac{A_n^{\alpha-1} (T^n f)(\omega)}{(A_n^\alpha)^\alpha}$ converges (absolutely) for each $z \in \mathbb{C}$ with $\operatorname{Re}(z) > 1$. Assume that for any $\omega \in \Omega - (E \cup E_0)$, where $E_0 = \{f^* = \infty\}$, the

analytic function

$$G_{\omega}(z) = \sum_{n=0}^{\infty} \frac{A_n^{\alpha-1}(T^n f)(\omega)}{(A_n^{\alpha})^z} - \frac{f^{\alpha}(\omega)}{z-1}, \quad \operatorname{Re}(z) > 1$$

has an analytic or just continuous extension (also called $G_{\omega}(\cdot)$) to the closed half-plane $\{\operatorname{Re}(z) \geq 1\}$. Then $\frac{1}{A_n^{\alpha}} \sum_{k=0}^n A_k^{\alpha-1}(T^k f)(\omega)$ converges as $n \rightarrow \infty$ to $f^{\alpha}(\omega)$ for almost all $\omega \in \Omega$.

We need some lemmas.

Lemma 1. Let T be an order preserving operator in L_p ($1 \leq p \leq \infty$) with $T(0) = 0$ and let $0 < \alpha < \infty$, $f \in L_p^+$. Assume that E is a set in Ξ with $\mu(E) = 0$ such that for any $\omega \in \Omega - E$, (the abscissa of convergence)

$$a_{\omega}(\alpha; f) = \limsup_{n \rightarrow \infty} \frac{\log \left[\sum_{k=0}^n A_k^{\alpha-1}(T^k f)(\omega) \right]}{\log A_n^{\alpha}} \leq 1.$$

Then for each $\omega \in \Omega - E$, the generalized Dirichlet series $\sum_{n=0}^{\infty} \frac{A_n^{\alpha-1}(T^n f)(\omega)}{(A_n^{\alpha})^z}$ converges (absolutely) for $z \in \mathbb{C}$ with $\operatorname{Re}(z) > 1$.

Proof. Let $\omega \in \Omega - E$ be fixed and let $z \in \mathbb{C}$, $\operatorname{Re}(z) > 1$. We choose some $\delta > 0$ (which may depends on (α, ω, z)) such that

$$a_{\omega}(\alpha; f) + \frac{\delta}{2} < a_{\omega}(\alpha; f) + \delta < \operatorname{Re}(z).$$

Then there exists a sufficiently large number $N_0 = N_0(\delta, a_{\omega})$ (where $a_{\omega} = a_{\omega}(\alpha; f)$) such that

$$\sum_{n=0}^m A_n^{\alpha-1}(T^n f)(\omega) < (A_m^{\alpha})^{a_{\omega} + \frac{\delta}{2}}, \quad m \geq N_0.$$

Thus, letting

$$D_{\omega, n}(s) = \sum_{k=0}^n \frac{A_k^{\alpha-1}(T^k f)(\omega)}{(A_k^{\alpha})^s}, \quad s \geq 0$$

and using the partial summation formula of Abel, we have, for $m \geq n+1 > N_0$

$$\begin{aligned} \sum_{k=n+1}^m \frac{A_k^{\alpha-1}(T^k f)(\omega)}{(A_k^{\alpha})^{a_{\omega} + \delta}} &= \left(\sum_{k=0}^m - \sum_{k=0}^n \right) \frac{A_k^{\alpha-1}(T^k f)(\omega)}{(A_k^{\alpha})^{a_{\omega} + \delta}} \\ &= \sum_{k=n}^{m-1} \left\{ \frac{1}{(A_k^{\alpha})^{a_{\omega} + \delta}} - \frac{1}{(A_{k+1}^{\alpha})^{a_{\omega} + \delta}} \right\} D_{\omega, k}(0) + \frac{D_{\omega, m}(0)}{(A_m^{\alpha})^{a_{\omega} + \delta}} - \frac{D_{\omega, n}(0)}{(A_n^{\alpha})^{a_{\omega} + \delta}} \\ &\leq \sum_{k=n}^{m-1} (A_k^{\alpha})^{a_{\omega} + \frac{\delta}{2}} \left\{ \frac{1}{(A_k^{\alpha})^{a_{\omega} + \delta}} - \frac{1}{(A_{k+1}^{\alpha})^{a_{\omega} + \delta}} \right\} + \frac{1}{(A_m^{\alpha})^{a_{\omega} + \delta - (a_{\omega} + \frac{\delta}{2})}} + \frac{1}{(A_n^{\alpha})^{a_{\omega} + \delta - (a_{\omega} + \frac{\delta}{2})}} \end{aligned}$$

$$\begin{aligned}
&\leq (a_\omega + \delta) \sum_{k=n}^{m-1} (A_k^\alpha)^{a_\omega + \frac{\delta}{2}} \int_{\log A_k^\alpha}^{\log A_{k+1}^\alpha} e^{-(a_\omega + \delta)u} du + \frac{2}{(A_n^\alpha)^{\frac{\delta}{2}}} \\
&\leq (a_\omega + \delta) \sum_{k=n}^{m-1} \int_{\log A_k^\alpha}^{\log A_{k+1}^\alpha} e^{-\frac{\delta}{2}u} du + \frac{2}{(A_n^\alpha)^{\frac{\delta}{2}}} \\
&\leq \frac{2(a_\omega + \delta)}{\delta} \left\{ \frac{1}{(A_n^\alpha)^{\frac{\delta}{2}}} - \frac{1}{(A_m^\alpha)^{\frac{\delta}{2}}} \right\} + \frac{2}{(A_n^\alpha)^{\frac{\delta}{2}}}.
\end{aligned}$$

This gives

$$0 \leq \lim_{n, m \rightarrow \infty} \sum_{k=n+1}^m \frac{A_k^{\alpha-1}(T^k f)(\omega)}{(A_k^\alpha)^{\operatorname{Re}(z)}} \leq \lim_{n, m \rightarrow \infty} \sum_{k=n+1}^m \frac{A_k^{\alpha-1}(T^k f)(\omega)}{(A_k^\alpha)^{a_\omega + \delta}} = 0,$$

and the lemma follows.

Lemma 2. Let $\omega \in \Omega - E$ be fixed. Then

$$\int_1^\infty \frac{1}{v^{s+1}} \left[\sum_{0 \leq k \leq n, A_n^\alpha \leq v} A_k^{\alpha-1}(T^k f)(\omega) \right] dv = \frac{1}{z} \sum_{n=0}^\infty \frac{A_n^{\alpha-1}(T^n f)(\omega)}{(A_n^\alpha)^z}, \quad \operatorname{Re}(z) > 1.$$

Proof. Let $\varepsilon > 0$ be fixed sufficiently small and let $\omega \in \Omega - E$, $z \in C$, $\operatorname{Re}(z) > 1$. By assumption there exists a number N_0 (which may depend on $(f, \varepsilon, \alpha, \omega, z)$) large enough so that

$$\sum_{k=N_0+1}^\infty \frac{A_k^{\alpha-1}(T^k f)(\omega)}{(A_k^\alpha)^{\operatorname{Re}(z)}} < \varepsilon.$$

We then have for sufficiently large v

$$\frac{1}{v^{\operatorname{Re}(z)}} \sum_{N_0+1 \leq k \leq n, A_n^\alpha \leq v} A_k^{\alpha-1}(T^k f)(\omega) \leq \sum_{k=N_0+1}^n \frac{A_k^{\alpha-1}(T^k f)(\omega)}{(A_k^\alpha)^{\operatorname{Re}(z)}} \leq \sum_{k=N_0+1}^\infty \frac{A_k^{\alpha-1}(T^k f)(\omega)}{(A_k^\alpha)^{\operatorname{Re}(z)}} < \varepsilon.$$

This implies that $\lim_{N \rightarrow \infty} \frac{1}{(A_{N+1}^\alpha)^z} \sum_{k=0}^N A_k^{\alpha-1}(T^k f)(\omega) = 0$. Now let us define

$$\begin{aligned}
S_\omega(v) &= \sum_{0 \leq k \leq n, A_n^\alpha \leq v} A_k^{\alpha-1}(T^k f)(\omega), \quad v \geq 1, \quad (n \geq 0) \\
&= 0, \quad v < 1.
\end{aligned}$$

Then it follows that

$$\begin{aligned}
z \int_1^{A_{N+1}^\alpha} \frac{S_\omega(v)}{v^{s+1}} dv &= z \sum_{j=0}^N \int_{A_j^\alpha}^{A_{j+1}^\alpha} \frac{S_\omega(v)}{v^{s+1}} dv \\
&= z \sum_{j=0}^N S_\omega(A_j^\alpha) \int_{A_j^\alpha}^{A_{j+1}^\alpha} \frac{1}{v^{s+1}} dv \\
&= \sum_{j=0}^N S_\omega(A_j^\alpha) \left[\frac{1}{(A_j^\alpha)^z} - \frac{1}{(A_{j+1}^\alpha)^z} \right]
\end{aligned}$$

$$= \frac{S_{\omega}(A_0^q)}{(A_0^q)^z} + \frac{S_{\omega}(A_1^q) - S_{\omega}(A_0^q)}{(A_1^q)^z} + \dots + \frac{S_{\omega}(A_N^q) - S_{\omega}(A_{N-1}^q)}{(A_N^q)^z} - \frac{S_{\omega}(A_N^q)}{(A_{N+1}^q)^z}.$$

Thus, since $\lim_{N \rightarrow \infty} \frac{S_{\omega}(A_N^q)}{(A_{N+1}^q)^z} = 0$ for $\operatorname{Re}(z) > 1$, we obtain the desired equality

$$z \int_1^{\infty} \frac{S_{\omega}(v)}{v^{z+1}} dv = \sum_{n=0}^{\infty} \frac{A_n^{q-1}(T^n f)(\omega)}{(A_n^q)^z}, \quad \operatorname{Re}(z) > 1.$$

Lemma 3. Let $\omega \in \Omega - (E \cup E_0)$ be fixed and put

$$H_{\omega}(y) = e^{-y} S_{\omega}(e^y), \quad y \geq 0.$$

Then for some real $a > 0$,

$$\lim_{y \rightarrow \infty} \int_{-\infty}^{ay} H_{\omega}(y - \frac{v}{a}) K_1(v) dv = f^*(\omega) \int_{-\infty}^{\infty} K_1(v) dv$$

holds with the Fejér kernel $K_{\rho}(t) = \frac{\sin^2(\rho t)}{\rho t^2}$, $\rho > 0$.

Proof. By Lemma 2 we see that

$$\int_0^{\infty} H_{\omega}(y) e^{-(z-1)y} dy = \frac{1}{z} \sum_{n=0}^{\infty} \frac{A_n^{q-1}(T^n f)(\omega)}{(A_n^q)^z}, \quad \operatorname{Re}(z) > 1,$$

and so

$$\int_0^{\infty} (H_{\omega}(y) - f^*(\omega)) e^{-(z-1)y} dy = \frac{G_{\omega}(z) - f^*(\omega)}{z}, \quad \operatorname{Re}(z) > 1.$$

Note that the Fourier transform of $K_a(t)$ becomes

$$\frac{1}{\pi} \int_{-\infty}^{\infty} K_a(t) e^{-iyt} dt = \begin{cases} 1 - \frac{|y|}{2a}, & \text{if } |y| \leq 2a, \\ 0, & \text{if } |y| > 2a. \end{cases}$$

Using the Fejér kernel

$$\frac{1}{2} \int_{-2a}^{2a} \left(1 - \frac{|t|}{2a}\right) e^{i(y-u)t} dt = \frac{1}{4a} \int_0^{2a} \left\{ \int_{-\tau}^{\tau} e^{it(y-u)} dt \right\} d\tau = \frac{1 - \cos(2a(y-u))}{2a(y-u)^2} = K_a(y-u),$$

we have for $\sigma > 1$ and every $a > 0$

$$\begin{aligned} & \frac{1}{2a} \int_{-2a}^{2a} e^{iyt} \left(1 - \frac{|t|}{2a}\right) \frac{G_{\omega}(\sigma+it) - f^*(\omega)}{\sigma+it} dt \\ &= \frac{1}{2} \int_{-2a}^{2a} \left[\int_0^{\infty} (H_{\omega}(u) - f^*(\omega)) e^{-(\sigma-1+it)u} du \right] \left(1 - \frac{|t|}{2a}\right) dt \\ &= \frac{1}{2} \int_0^{\infty} (H_{\omega}(u) - f^*(\omega)) e^{-(\sigma-1)u} du \int_{-2a}^{2a} \left(1 - \frac{|t|}{2a}\right) e^{it(y-u)t} dt \end{aligned}$$

$$= \int_0^\infty H_\omega(u) e^{-(\sigma-1)u} K_a(y-u) du - f^*(\omega) \int_0^\infty e^{-(\sigma-1)u} K_a(y-u) du.$$

So, letting $\sigma \rightarrow 1$ gives

$$\begin{aligned} & \frac{1}{2} \int_{-2a}^{2a} e^{iyt} \left(1 - \frac{|t|}{2a}\right) \frac{G_\omega(1+it) - f^*(\omega)}{1+it} dt \\ &= \int_0^\infty H_\omega(u) K_a(y-u) du - f^*(\omega) \int_0^\infty K_a(y-u) du \\ &= \int_{-\infty}^{ay} H_\omega(y - \frac{v}{a}) K_1(v) dv - f^*(\omega) \int_{-\infty}^{ay} K_1(v) dv. \end{aligned}$$

Consequently, the desired conclusion follows immediately from this and the Riemann-Lebesgue theorem.

Proof of Theorem 1. After observing that $y_2 \geq y_1 > 0$ implies $H_\omega(y_2)e^{y_2} \geq H_\omega(y_1)e^{y_1}$, it follows from Lemma 3 that

$$\begin{aligned} f^*(\omega) \int_{-\infty}^\infty K_1(v) dv &\geq \limsup_{y \rightarrow \infty} \int_{-\sqrt{a}}^{\sqrt{a}} H_\omega(y - \frac{v}{a}) K_1(v) dv \\ &\geq \limsup_{y \rightarrow \infty} \int_{-\sqrt{a}}^{\sqrt{a}} H_\omega(y - \frac{1}{\sqrt{a}}) e^{-\frac{2}{\sqrt{a}}} K_1(v) dv \\ &= \limsup_{y \rightarrow \infty} H_\omega(y - \frac{1}{\sqrt{a}}) e^{-\frac{2}{\sqrt{a}}} \int_{-\sqrt{a}}^{\sqrt{a}} K_1(v) dv. \end{aligned}$$

Therefore

$$\limsup_{y \rightarrow \infty} H_\omega(y) \leq \frac{f^*(\omega) e^{\frac{2}{\sqrt{a}}}}{\int_{-\sqrt{a}}^{\sqrt{a}} K_1(v) dv} \int_{-\infty}^\infty K_1(v) dv.$$

Moreover, letting $a \rightarrow \infty$ yields

$$\limsup_{y \rightarrow \infty} H_\omega(y) \leq f^*(\omega).$$

This also implies that $H_\omega(y)$ is bounded, so we may write $H_\omega(y) \leq C_\omega$ with a suitably chosen constant C_ω . On the other hand, for $y \geq \frac{1}{\sqrt{a}} > 0$,

$$\begin{aligned} \int_{-\infty}^{ay} H_\omega(y - \frac{v}{a}) K_1(v) dv &\leq \left(\int_{-\infty}^{-\sqrt{a}} + \int_{-\sqrt{a}}^{\sqrt{a}} + \int_{\sqrt{a}}^{ay} \right) H_\omega(y - \frac{v}{a}) K_1(v) dv \\ &\leq 2C_\omega \int_{\sqrt{a}}^\infty K_1(v) dv + \int_{-\sqrt{a}}^{\sqrt{a}} H_\omega(y + \frac{1}{\sqrt{a}}) e^{\frac{2}{\sqrt{a}}} K_1(v) dv. \end{aligned}$$

Thus by Lemma 3 again we have

$$f^*(\omega) \int_{-\infty}^\infty K_1(v) dv \leq 2C_\omega \int_{\sqrt{a}}^\infty K_1(v) dv + \liminf_{y \rightarrow \infty} \int_{-\sqrt{a}}^{\sqrt{a}} H_\omega(y + \frac{1}{\sqrt{a}}) e^{\frac{2}{\sqrt{a}}} K_1(v) dv$$

$$= 2C_\omega \int_{\sqrt{a}}^{\infty} K_1(v)dv + \liminf_{y \rightarrow \infty} H_\omega(y) e^{\frac{2}{\sqrt{a}}} \int_{-\sqrt{a}}^{\sqrt{a}} K_1(v)dv,$$

and hence

$$\liminf_{y \rightarrow \infty} H_\omega(y) \geq \frac{f^*(\omega) \int_{-\infty}^{\infty} K_1(v)dv - 2C_\omega \int_{\sqrt{a}}^{\infty} K_1(v)dv}{\int_{-\sqrt{a}}^{\sqrt{a}} K_1(v)dv} e^{-\frac{2}{\sqrt{a}}}.$$

Finally, let $a \rightarrow \infty$ to get

$$\liminf_{y \rightarrow \infty} H_\omega(y) \geq f^*(\omega).$$

The above two parts together shows that $\lim_{y \rightarrow \infty} H_\omega(y) = f^*(\omega)$. Hence we may take $y = \log A_n^a$ to conclude that the theorem follows. The proof of the theorem has hereby completed.

Remarks. It should be noticed that an essential role in the proof of Theorem 1 is played by Wiener's general Tauberian theorem ([6], Theorem VIII) which guarantees that the following equation to hold for some $a > 0$

$$\lim_{y \rightarrow \infty} \frac{1}{\pi} \int_{-\infty}^{\infty} K_a(y-u) H_\omega(u) du = \frac{f^*(\omega)}{\pi} \int_{-\infty}^{\infty} K_a(u) du$$

is in fact valid for all real $a > 0$. We next demonstrate two cases realizing all the conditions of Theorem 1.

(1) We consider the function space $C[0, 1]$ consisting of functions $f(t)$ continuous for $0 \leq t \leq 1$ such that $\|f\| = \max |f(t)|$. Let T be an order preserving, positively homogeneous and norm decreasing operator in $C[0, 1]$ with $T(0) = 0$. Let $T_r = rT$ for some r , $0 < r < 1$. Let $0 < \alpha < \infty$ and $f \in C[0, 1]^+$. Then one gets

$$\limsup_{n \rightarrow \infty} \frac{\log \left[\sum_{k=0}^n A_k^{\alpha-1} (T_k^\beta f)(t) \right]}{\log A_n^\alpha} = 0.$$

We can thus define the function $G_t(z)$ by the convergent Dirichlet series $\sum_{n=0}^{\infty} \frac{A_n^{\alpha-1} (T_n^\beta f)(t)}{(A_n^\alpha)^z}$ for $z \in \{\operatorname{Re}(z) \geq 1\}$. Clearly $G_t(z)$ is analytic in the closed half-plane $\{\operatorname{Re}(z) \geq 1\}$.

(2) Let $\beta > 0$ be fixed positive and define an operator T_β in $C[0, 1]$ by the fractional integral

$$(T_\beta f)(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-u)^{\beta-1} f(u) du, \quad 0 \leq t \leq 1$$

for $f \in C[0, 1]$. Let $0 < \alpha < \infty$ and $f \in C[0, 1]^+$. Then we see that

$$\limsup_{n \rightarrow \infty} \frac{\log \left[\sum_{k=0}^n A_k^{\alpha-1} (T_k^\beta f)(t) \right]}{\log A_n^\alpha} = 0.$$

Thus we may obtain a function $G_t(z)$ analytic in $\{\operatorname{Re}(z) \geq 1\}$ which is defined by the convergent Dirichlet series $\sum_{n=0}^{\infty} \frac{A_n^{\alpha-1} (T_n^\beta f)(t)}{(A_n^\alpha)^z}$ for $z \in \{\operatorname{Re}(z) \geq 1\}$.

Applying a modified Karamata's argument for series to nonlinear operators (see [9]), we have

Theorem 2. Let T be an order preserving and L_∞ -norm decreasing operator in L_p ($1 \leq p \leq \infty$) with $T(0) = 0$. Let $0 < \alpha < \infty$ and $f \in L_p^+ \cap L_\infty$. Define

$$\Psi_\alpha(t; f) = A_n^{\alpha-1} T^n f, \quad n \leq t < n+1, \quad n \geq 0.$$

If $\lambda^\alpha \int_0^\infty e^{-\lambda t} \Psi_\alpha(t; f) dt$ ($\lambda > 0$) converges a.e. as $\lambda \rightarrow 0+$ to some $f_0 \in L_p^+$, then $\frac{1}{A_n} \sum_{k=0}^n A_k^{\alpha-1} T^k f$ converges a.e. as $n \rightarrow \infty$ to the function f_0 .

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