

The Jørgensen number of the Whitehead link

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ABSTRACT. In this paper we will sketch out the result obtained recently: the Jørgensen number of the Whitehead link is two. Furthermore we will represent points corresponding to the Whitehead link by using the coordinates introduced in Sato [7]. The details will appear in Sato [9].

1. In 1976 Jørgensen obtained the following important theorem called Jørgensen's inequality, which gives a necessary condition for a non-elementary Möbius transformation group $G = \langle A, B \rangle$ to be discrete.

THEOREM A (Jørgensen [1]). *Suppose that the Möbius transformations A and B generate a non-elementary discrete group. Then*

$$J(A, B) := |\operatorname{tr}^2(A) - 4| + |\operatorname{tr}(ABA^{-1}B^{-1}) - 2| \geq 1.$$

The lower bound 1 is best possible.

DEFINITION 1. Let A and B be Möbius transformations. The *Jørgensen number*

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$J(A, B)$ of the ordered pair (A, B) is defined as

$$J(A, B) := |\operatorname{tr}^2(A) - 4| + |\operatorname{tr}(ABA^{-1}B^{-1}) - 2|.$$

We denote by $\operatorname{Möb}$ the set of all Möbius transformations. Throughout this paper we will always write elements of $\operatorname{Möb}$ as matrices with determinant 1. We recall that $\operatorname{Möb} (= PSL(2, \mathbb{C}))$ acts on the upper half space H^3 of $\mathbb{R}^3$ as the group of conformal isometries of hyperbolic 3-space. A subgroup G of $\operatorname{Möb}$ is said to be *elementary* if there exists a finite G -orbit in $\mathbb{R}^3$.

DEFINITION 2. Let G be a non-elementary two-generator subgroup of $\operatorname{Möb}$. The *Jørgensen number* $J(G)$ for G is defined as

$$J(G) := \inf\{J(A, B) \mid A \text{ and } B \text{ generate } G\}.$$

DEFINITION 3. A non-elementary two-generator subgroup G of $\operatorname{Möb}$ is a *Jørgensen group* if G is a discrete group with $J(G) = 1$.

THEOREM B (Jørgensen-Kiikka [2]). *Let $\langle A, B \rangle$ be a non-elementary discrete group with $J(A, B) = 1$. Then A is elliptic of order at least seven or A is parabolic.*

If $\langle A, B \rangle$ is a Jørgensen group such that A is parabolic, then we call it a *Jørgensen group of parabolic type*. Here we only consider Jørgensen groups of parabolic type.

2. Let $\langle A, B \rangle$ be a marked two-generator group such that A is parabolic. Then we can normalize A and B as follows:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B := B_{\sigma, \mu} = \begin{pmatrix} \mu\sigma & \mu^2\sigma - 1/\sigma \\ \sigma & \mu\sigma \end{pmatrix},$$

where $\sigma \in \mathbb{C} \setminus \{0\}$ and $\mu \in \mathbb{C}$. We denote by $G_{\sigma, \mu}$ the marked group generated by A and $B_{\sigma, \mu}$: $G_{\sigma, \mu} = \langle A, B_{\sigma, \mu} \rangle$. We say that (σ, μ) is the point representing a marked group $G_{\sigma, \mu}$ and that $G_{\sigma, \mu}$ is the marked group corresponding to a point (σ, μ) .

In particular, we consider the case of $\mu = ik$ ($k \in \mathbf{R}$). Namely, we consider marked two-generator group $G_{\sigma,ik} = \langle A, B_{\sigma,ik} \rangle$ generated by

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B := B_{\sigma,ik} = \begin{pmatrix} ik\sigma & -k^2\sigma - 1/\sigma \\ \sigma & ik\sigma \end{pmatrix},$$

where $\sigma \in \mathbf{C} \setminus \{0\}$ and $k \in \mathbf{R}$.

3. Let C be the following cylinder: $C = \{(\sigma, ik) \mid |\sigma| = 1, k \in \mathbf{R}\}$.

THEOREM C (Sato [7]). *If a marked two-generator group $G_{\sigma,ik}$ is a Jørgensen group, then the point (σ, ik) representing $G_{\sigma,ik}$ lies on the cylinder C .*

By Theorem C we consider marked two-generator groups $G_{\sigma,\mu} = \langle A, B_{\mu,\sigma} \rangle$ with $\sigma = -ie^{i\theta}$ ($0 \leq \theta < 2\pi$) and $\mu = ik$ ($k \in \mathbf{R}$). For simplicity we set $B_{\theta,k} := B_{\sigma,ik}$ and $G_{\theta,k} = \langle A, B_{\sigma,ik} \rangle$ for $\sigma = -ie^{i\theta}$.

4. There are infinite number of Jørgensen groups (see Jørgensen-Lascurain-Pignataro [3], Sato [7]). The following families of groups are all Jørgensen groups: The modular group, the Picard group (Jørgensen-Lascurain-Pignataro [3], Sato [8, 9], Sato-Yamada [10]), the figure-eight knot group (Sato [7]), "the Gehring-Maskit group" (Sato [7]), where "the Gehring-Maskit group" is the group studied in Maskit [5]. Namely, we have the following theorem:

THEOREM D (Jørgensen-Lascurain-Pignataro [3], Sato [7, 8], Sato-Yamada [10]).

Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_{\theta,k} = \begin{pmatrix} ke^{i\theta} & ie^{-i\theta}(k^2e^{2i\theta} - 1) \\ -ie^{i\theta} & ke^{i\theta} \end{pmatrix}$$

and let $G_{\theta,k} = \langle A, B_{\theta,k} \rangle$ be the group generated by A and $B_{\theta,k}$, where $0 \leq \theta < 2\pi$ and $k \in \mathbf{R}$. Then

- (i) $G_{\pi/2,0}$ is a Jørgensen group.
- (ii) $G_{\pi/2,1/2}$ is a Jørgensen group.
- (iii) $G_{\pi/6,\sqrt{3}/2}$ is a Jørgensen group.
- (iv) $G_{0,\sqrt{3}/2}$ is a Jørgensen group.

REMARK (1) The groups $G_{\pi/2,0}$, $G_{\pi/2,1/2}$, $G_{\pi/6,\sqrt{3}/2}$ and $G_{0,\sqrt{3}/2}$ are conjugate to the modular group, the Picard group, the figure-eight knot group and "the Gehring-Maskit group", respectively.

- (2) See Sato [7] for other Jørgensen groups of parabolic type.

5. Now it gives rise to the following problem.

PROBLEM. Is the Whitehead link a Jørgensen group ?

Here we can give the answer to the problem, that is, we have the following theorems.

THEOREM 1 (Sato [9]). *The Jørgensen number of the Whitehead link is two.*

COROLLARY (Sato [9]). The Whitehead link is not a Jørgensen group.

THEOREM 2 (Sato [9]). *The Whitehead link is conjugate to the marked two-generator group $G_{\sigma,\mu}$ where $\sigma = \sqrt{2}e^{3\pi i/4}$ and $\mu = -1/2$.*

6. The proofs of the theorems will appear elsewhere. Here we only give sketches of the proofs.

THEOREM E (cf. Wielenberg [11], Krushkal', Apanasov and Gusevskiĭ,[4]). *The Whitehead link G_W has the following presentation:*

$$G_W = \langle A, B \mid (A^{-1}BAB^{-1})(ABA^{-1}B^{-1})(AB^{-1}A^{-1}B)(A^{-1}B^{-1}AB) = 1 \rangle,$$

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 1-i & 1 \end{pmatrix}.$$

PROPOSITION 1 *Let G_W be the Whitehead link defined in Theorem E. Then an element X of G_W has the following form:*

$$X = \begin{pmatrix} 1 + (1-i)a & b_1 + (1-i)b_2 \\ (1-i)c & 1 + (1-i)d \end{pmatrix}.$$

where $a, b_1, b_2, c, d \in \mathbf{Z} + i\mathbf{Z}$, $a + d - b_1c + (1-i)(ad - b_2c) = 0$.

PROPOSITION 2. *Let G_W be the Whitehead link defined in Theorem E and let $\langle X, Y \rangle$ be a non-elementary subgroup generated by X and Y , where $X, Y \in G_W$. Then the Jørgensen number of (X, Y) is greater than or equal to two: $J(X, Y) \geq 2$.*

PROPOSITION 3. *Let A, B be the matrices in Theorem E. Set $C = AB$. Then A and C generate the Whitehead link G_W and $J(A, C) = 2$.*

Theorem 1 follows from Propositions 2 and 3.

6. Next we will give a sketch of the proof of Theorem 2.

Let P be the regular ideal octahedron in Ratcliffe [6, p.454]. Let the sides $S_A, S_B, S_C, S_D, S_{A'}, S_{B'}, S_{C'}$ and $S_{D'}$ be the sides of P . Let f_A, f_B, f_C and f_D be the side pairing transformations of S_A to $S_{A'}$, of S_B to $S_{B'}$, of S_C to $S_{C'}$, and of S_D to $S_{D'}$, respectively.

PROPOSITION 4. *Let f_A, f_B, f_C and f_D be the side pairing transformations defined in the above. Then f_A, f_B, f_C and f_D generate the Whitehead link $G_{W,R}$ in the sense of Ratcliffe.*

PROPOSITION 5. *Let*

$$G_{W,R}^* = \langle A^*, B^* \mid A^*(B^*)^{-2}A^*B^*(A^*)^{-1}(B^*)^{-1} \\ (A^*)^{-1}(B^*)^2(A^*)^{-1}(B^*)^{-1}A^*B^* = 1 \rangle,$$

where

$$A^* = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B^* = \begin{pmatrix} 1/2 + i/2 & 3/4 + i/4 \\ -1 + i & 1/2 + i/2 \end{pmatrix}.$$

Then $G_{W,R}^*$ is conjugate to the Whitehead link $G_{W,R}$ in the sense of Ratcliffe.

(ii) $J(A^*, B^*) = 2$.

PROPOSITION 6. *The marked group $G_{W,R}^* = \langle A^*, B^* \rangle$ in Proposition 5 corresponds to the point $(-1 + i, -1/2)$.*

Theorem 2 follows from Propositions 5 and 6,

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