

Scaling limit of a model of quantum electrodynamics with N -nonrelativistic particles

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1 INTRODUCTION

The main problem presented in this paper is to consider a scaling limit of a model in quantum electrodynamics which describes an interaction of N -nonrelativistic charged particles and a quantized radiation field in the Coulomb gauge with the dipole approximation. The model we consider is called “the Pauli-Fierz model”. Authors in [5,6] have studied a scaling limit of the Pauli-Fierz model with one-nonrelativistic charged particle. We may well extend the scaling limit of one-particle system to N -particles system.

The Pauli-Fierz Hamiltonians $H_{\vec{p}}$ with N -nonrelativistic charged particles in the Coulomb gauge with the dipole approximation are defined as operators acting in the Hilbert space $\underbrace{L^2(\mathbb{R}^d) \otimes \dots \otimes L^2(\mathbb{R}^d)}_N \otimes \mathcal{F}(\mathcal{W}) \cong L^2(\mathbb{R}^{dN}) \otimes \mathcal{F}(\mathcal{W})$ by

$$H_{\vec{p}} = \frac{1}{2m} \sum_{j=1}^N \sum_{\mu=1}^d \left(-i\hbar D_{\mu}^j \otimes I - eI \otimes A_{\mu}(\rho_j) \right)^2 + I \otimes H_b,$$

where D_{μ}^j is the differential operator with respect to the j -th variable in the μ -th direction, $A_{\mu}(\rho_j)$ the quantized radiation field in the μ -th direction with an ultraviolet cut-off function ρ_j in the Coulomb gauge, H_b the free Hamiltonian in $\mathcal{F}(\mathcal{W})$, and $m, e, \hbar$ the mass of the particles, the charge of the particles, the Planck constant divided 2π , respectively.

Note that A_{μ} is depend on the speed of light c . We introduce the following scaling.

$$c(\kappa) = c\kappa, e(\kappa) = e\kappa^{-\frac{1}{2}}, m(\kappa) = m\kappa^{-2}. \quad (1.1)$$

Then the scaled Hamiltonian $H_{\vec{p}}(\kappa)$ amounts to

$$-\frac{\hbar^2 \kappa^2}{2m} \Delta \otimes I + \kappa I \otimes H_b + \frac{1}{2m} \sum_{j=1}^N \sum_{\mu=1}^d \left(\kappa 2e\hbar i D_\mu^j \otimes A_\mu(\rho_j) + e^2 I \otimes A_\mu^2(\rho_j) \right).$$

Defining a pseudo differential operator $E^{REN}(D, \kappa)$ in $L^2(\mathbb{R}^{dN})$ with a symbol $E^{REN}(p, \kappa)$ such that $E^{REN}(p, \kappa) \rightarrow \infty$ as $\kappa \rightarrow \infty$, we define a Hamiltonian $H_{\vec{p}}^{REN}(\kappa)$ by

$$-E^{REN}(D, \kappa) \otimes I + \kappa I \otimes H_b + \frac{1}{2m} \sum_{j=1}^N \sum_{\mu=1}^d \left(\kappa 2e\hbar i D_\mu^j \otimes A_\mu(\rho_j) + e^2 I \otimes A_\mu^2(\rho_j) \right).$$

Consequently, we shall show the following for some $\vec{p} = (\rho_1, \dots, \rho_N)$ and scalar potentials V with some conditions (Theorem 3.7):

$$s - \lim_{\kappa \rightarrow \infty} (H_{\vec{p}}^{REN}(\kappa) + V \otimes I - z)^{-1} = \mathcal{U}(\infty) \left\{ (E^\infty(D) + V_{eff} - z)^{-1} \otimes P_0 \right\} \mathcal{U}^{-1}(\infty),$$

where $E^\infty(D)$ is a pseudo differential operator in $L^2(\mathbb{R}^{dN})$, V_{eff} a multiplication operator, which is called “effective potential”, and P_0 a projection on $\mathcal{F}(\mathcal{W})$. Despite the fact that in the case of one-particle system the effective potential V_{eff} is the Gaussian transformation of a given scalar potential V , we shall show that in N -particles system, it is not necessary to be the Gaussian transformation. Actually it is determined by a matrix $\tilde{\Delta}^\infty = (\tilde{\Delta}_{ij}^\infty)_{1 \leq i, j \leq N}$ which is defined by the ultraviolet cut-off functions ρ_j ;

$$\tilde{\Delta}_{ij}^\infty = \frac{1}{2} \frac{d-1}{d} \left(\frac{\hbar}{mc} \right) \frac{e^2}{\hbar c} \int_{\mathbb{R}^d} dk \frac{\hat{\rho}_i(k) \hat{\rho}_j(k)}{\omega(k)^3}.$$

2 THE PAULI-FIERZ MODEL

To begin with, let us introduce some preliminary notations. Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$. We denote the inner product and the associated norm by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}}$ respectively. The inner product is linear in $\cdot$ and antilinear in $\ast$. The domain of an operator A in $\mathcal{H}$ is denoted by $D(A)$. A notation $\hat{f}$ (resp. $\check{f}$) denotes the Fourier transformation (resp. the inverse Fourier transformation) of f and $\bar{f}$ the complex conjugate of f . Let

$$\mathcal{W} \equiv \underbrace{L^2(\mathbb{R}^d) \oplus \dots \oplus L^2(\mathbb{R}^d)}_{d-1}.$$

We define the Boson Fock space over $\mathcal{W}$ by

$$\mathcal{F}(\mathcal{W}) \equiv \bigoplus_{n=0}^{\infty} \otimes_s^n \mathcal{W} \equiv \bigoplus \mathcal{F}_n(\mathcal{W}),$$

where $\otimes_s^0 \mathcal{W} \equiv \mathbb{C}$ and $\otimes_s^n \mathcal{W}$ ($n \geq 1$) denotes the n -fold symmetric tensor product. Put

$$\mathcal{F}^\infty(\mathcal{W}) \equiv \bigcup_{N=0}^{\infty} \bigoplus_{n=0}^N \mathcal{F}_n(\mathcal{W}) \bigoplus_{n \geq N+1} \{0\}.$$

The annihilation operator $a(f)$ and the creation operator $a^\dagger(f)$ ($f \in \mathcal{W}$) act on $\mathcal{F}^\infty(\mathcal{W})$ and leave it invariant with the canonical commutation relations (CCR): for $f, g \in \mathcal{W}$

$$\begin{aligned} [a(f), a^\dagger(g)] &= \langle \bar{f}, g \rangle_{\mathcal{W}}, \\ [a^\sharp(f), a^\sharp(g)] &= 0, \end{aligned}$$

where $[A, B] = AB - BA$, $a^\sharp$ denotes either a or $a^\dagger$. Furthermore,

$$\langle a^\dagger(f)\Phi, \Psi \rangle_{\mathcal{F}(\mathcal{W})} = \langle \Phi, a(\bar{f})\Psi \rangle_{\mathcal{F}(\mathcal{W})}, \quad \Phi, \Psi \in \mathcal{F}^\infty(\mathcal{W}).$$

We define polarization vectors e^r ($r = 1, \dots, d-1$) as measurable functions $e^r : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that

$$e^r(k)e^s(k) = \delta_{rs}, \quad e^r(k)k = 0, \quad a.e. k \in \mathbb{R}^d.$$

The μ -th direction time-zero smeared radiation field in the Coulomb gauge with the dipole approximation is defined as operators acting in $\mathcal{F}(\mathcal{W})$ by

$$A_\mu(f) = \frac{1}{\sqrt{2}} \left\{ a^\dagger \left(\bigoplus_{r=1}^{d-1} \frac{\sqrt{\hbar} e_\mu^r \hat{f}}{\sqrt{c\omega}} \right) + a \left(\bigoplus_{r=1}^{d-1} \frac{\sqrt{\hbar} e_\mu^r \tilde{\hat{f}}}{\sqrt{c\omega}} \right) \right\}, \quad (2.1)$$

where $\omega(k) = |k|$ and $\tilde{g}(k) = g(-k)$. Let $\Omega = (1, 0, 0, \dots) \in \mathcal{F}(\mathcal{W})$. For a nonnegative self-adjoint operator $h : \mathcal{W} \rightarrow \mathcal{W}$, we denote “the second quantization of h ” by $d\Gamma(h)$. Put

$\tilde{\omega} = \underbrace{\omega \oplus \dots \oplus \omega}_{d-1}$. The free Hamiltonian H_b in $\mathcal{F}(\mathcal{W})$ is defined by

$$H_b \equiv \hbar c d\Gamma(\tilde{\omega}).$$

The Pauli-Fierz Hamiltonians with N -nonrelativistic charged particles interacting with the quantized radiation field with the dipole approximation in the Coulomb gauge read as follows:

$$H_{\vec{\rho}} \equiv H_{\rho_1, \dots, \rho_N} \equiv \frac{1}{2m} \sum_{j=1}^N \sum_{\mu=1}^d \left(-i\hbar D_{\mu}^j \otimes I - eI \otimes A_{\mu}(\rho_j) \right)^2 + I \otimes H_b,$$

acting in

$$\underbrace{L^2(\mathbb{R}^d) \otimes \dots \otimes L^2(\mathbb{R}^d)}_N \otimes \mathcal{F}(\mathcal{W}) \cong L^2(\mathbb{R}^{dN}) \otimes \mathcal{F}(\mathcal{W}) \cong \int_{\mathbb{R}^{dN}}^{\oplus} \mathcal{F}(\mathcal{W}) dx.$$

We introduce the scaling (1.1). For objects A containing of the parameters c, e, m , we denote the scaled object by $A(\kappa)$ throughout this paper. We define classes P and $\tilde{P}$ of sets of functions as follows:

Definition 2.1 $\vec{\rho} = (\rho_1, \dots, \rho_N)$ is in P if and only if

- (1) $\hat{\rho}_j, j = 1, \dots, N$ are rotation invariant, $\hat{\rho}_j(k) = \hat{\rho}_j(|k|)$, and real-valued,
- (2) $\hat{\rho}_j/\omega, \hat{\rho}_j/\sqrt{\omega}, \hat{\rho}_j, \sqrt{\omega}\hat{\rho}_j \in L^2(\mathbb{R}^d)$.

Moreover $\vec{\rho}$ is in $\tilde{P}$ if and only if in addition to (1) and (2) above

- (3) $\hat{\rho}_j/\omega\sqrt{\omega} \in L^2(\mathbb{R}^d)$ and there exist $0 < \alpha < 1$ and $1 \leq \epsilon$ such that $\hat{\rho}_i(\sqrt{\cdot})\hat{\rho}_j(\sqrt{\cdot})(\sqrt{\cdot})^{d-2} \in \text{Lip}(\alpha) \cap L^{\epsilon}([0, \infty))$, where $\text{Lip}(\alpha)$ is the set of the Lipschitz continuous functions on $[0, \infty)$ with the degree α ,
- (4) $\sup_k |\hat{\rho}_j(k)\omega^{\frac{d}{2}-\frac{3}{2}}(k)| < \infty, \sup_k |\hat{\rho}_j(k)\omega^{\frac{d}{2}-\frac{1}{2}}(k)| < \infty, j = 1, \dots, N$.

Put

$$H_0 = -\frac{1}{2m}\hbar^2\Delta \otimes I + I \otimes H_b,$$

where Δ is the Laplacian in $\mathbb{R}^{dN}$. It is well known that H_0 is a nonnegative self-adjoint operator on $D(H_0) = D\left(-\frac{1}{2m}\hbar^2\Delta \otimes I\right) \cap D(I \otimes H_b)$.

Proposition 2.2 ([3,4]) *For $\vec{\rho} \in P$ and $\kappa > 0$, the operator $H_{\vec{\rho}}(\kappa)$ is self-adjoint on $D(H_0)$ and essentially self-adjoint on any core of H_0 and nonnegative.*

Let $\mathbf{F} = F \otimes I$, where F denotes the Fourier transform in $L^2(\mathbb{R}^{dN})$. It is clear that operators $\mathbf{F}H_{\vec{\rho}}\mathbf{F}^{-1}$ can be decomposable as follows:

$$\mathbf{F}H_{\vec{\rho}}(\kappa)\mathbf{F}^{-1} = \int_{\mathbb{R}^{dN}}^{\oplus} H_{\vec{\rho}}(p, \kappa) dp,$$

where

$$H_{\vec{\rho}}(p, \kappa) = \frac{1}{2m} \sum_{j=1}^N \sum_{\mu=1}^d \left(\kappa \hbar p_{\mu}^j - e A_{\mu}(\rho_j) \right)^2 + \kappa H_b.$$

Proposition 2.3 ([3,4]) *For $\vec{\rho} \in P$ and $\kappa > 0$, the operator $H_{\vec{\rho}}(p, \kappa)$ is self-adjoint on $D(H_b)$ and essentially self-adjoint on any core of H_b and nonnegative.*

Set Hilbert spaces $M_d = \left\{ f \mid \int |f(k)|^2 \omega(k)^d dk < \infty \right\}$ and put $\mathcal{W}_{\alpha} = \underbrace{M_{\alpha} \oplus \dots \oplus M_{\alpha}}_{d-1}$, $\alpha \in \mathbb{R}$.

The following lemma is the key lemma to investigating the scaling limits.

Lemma 2.4 ([9]) *Let $\vec{\rho} \in \tilde{P}$ and $\kappa > 0$ be sufficiently large. Then there exist a Hilbert Schmidt operator $\mathbf{W}_{-}$, a bounded operator $\mathbf{W}_{+}$, and $\mathbf{L}_j = (\mathbf{L}_j^1, \dots, \mathbf{L}_j^d)$, $\mathbf{L}_j^{\mu} \in \mathcal{W}$, $j = 1, \dots, N$, $\mu = 1, \dots, d$ such that, if we put for $p^j \in \mathbb{R}^d$, $j = 1, \dots, N$*

$$\begin{aligned} B(\mathbf{f}, p) &= a^{\dagger}(\mathbf{W}_{-}\mathbf{f}) + a(\mathbf{W}_{+}\mathbf{f}) + \sum_{j=1}^N \langle \mathbf{L}_j p^j, \mathbf{f} \rangle_{\mathcal{W}}, \\ B^{\dagger}(\mathbf{f}, p) &= a^{\dagger}(\overline{\mathbf{W}}_{+}\mathbf{f}) + a(\overline{\mathbf{W}}_{-}\mathbf{f}) + \sum_{j=1}^N \langle \overline{\mathbf{L}}_j p^j, \mathbf{f} \rangle_{\mathcal{W}}, \end{aligned}$$

then

$$\begin{aligned} [B(\mathbf{f}, p), B^{\dagger}(\mathbf{g}, p)] &= \langle \mathbf{f}, \mathbf{g} \rangle_{\mathcal{W}}, \\ [B^{\sharp}(\mathbf{f}, p), B^{\sharp}(\mathbf{g}, p)] &= 0, \text{ on } \mathcal{F}^{\infty}(\mathcal{W}), \end{aligned}$$

and for $\Phi, \Psi \in \mathcal{F}^{\infty}(\mathcal{W})$,

$$\langle B^{\dagger}(\mathbf{f}, p)\Phi, \Psi \rangle_{\mathcal{F}(\mathcal{W})} = \langle \Phi, B(\bar{\mathbf{f}}, p)\Psi \rangle_{\mathcal{F}(\mathcal{W})},$$

moreover

$$[H_{\vec{p}}(p), B^\sharp(\mathbf{f}, p)] = \pm B^\sharp(\hbar \tilde{\alpha} \tilde{\omega} \mathbf{f}, p), \text{ on } \mathcal{F}^\infty(\mathcal{W}) \cap D(H_b^{\frac{3}{2}}),$$

where $\mathbf{f} \in \mathcal{W}_0 \cap \mathcal{W}_2$ and $+$ (resp. $-$) corresponds to $B^\dagger$ (resp. B).

By virtue of Lemma 2.4, we see the following.

Corollary 2.5 *Let $\vec{p} \in \tilde{P}$ and κ be sufficiently large. Then for $\Phi \in D(H_b)$,*

$$\exp\left(i\frac{t}{\hbar}H_{\vec{p}}(p)\right) B^\sharp(\mathbf{f}, p) \exp\left(-i\frac{t}{\hbar}H_{\vec{p}}(p)\right) \Phi = B^\sharp(e^{i\tilde{\alpha}\tilde{\omega}t}\mathbf{f}, p) \Phi$$

3 SCALING LIMITS

In this section, we construct a unitary operator which implements unitary equivalence of the Pauli-Fierz Hamiltonian and a decoupled Hamiltonian. Moreover we investigate a scaling limit of the Pauli-Fierz Hamiltonian. Unless otherwise stated in this section, we suppose that $\kappa > 0$ is sufficiently large. From Lemma 2.4 (1) it follows that there exist two unitary operators $U(\kappa)$ (p independent) and $S(p, \kappa)$ such that ([6, Section III])

$$U^{-1}(\kappa)S(p, \kappa)^{-1}B^\sharp(\mathbf{f}, p, \kappa)S(p, \kappa)U(\kappa) = a^\sharp(\mathbf{f}), \quad \mathbf{f} \in \mathcal{W}. \quad (3.1)$$

Concretely $S(p, \kappa)$ is given by

$$S(p, \kappa) = \exp\left(\sum_{i,j=1}^N \frac{e\hbar}{\kappa^2} p_\mu^i \left\{ a\left(\oplus_{r=1}^{d-1} \frac{e_\mu^r M_{ij}(\kappa) \hat{\rho}_j}{\sqrt{2\hbar c^3 \omega^3}}\right) - a^\dagger\left(\oplus_{r=1}^{d-1} \frac{e_\mu^r M_{ij}(\kappa) \hat{\rho}_j}{\sqrt{2\hbar c^3 \omega^3}}\right) \right\}\right),$$

where $(M_{ij}(\kappa))_{1 \leq i,j \leq N}$ is a matrix such that

$$\lim_{\kappa \rightarrow \infty} \frac{M_{ij}(\kappa)}{\kappa^2} = \delta_{ij} \frac{1}{m}.$$

Theorem 3.1 *Suppose $\vec{p} \in \tilde{P}$. Then putting $S(p, \kappa)U(\kappa) = \mathcal{U}(p, \kappa)$, we see that $\mathcal{U}(p, \kappa)$ maps $D(H_b)$ onto itself with*

$$\mathcal{U}(p, \kappa)H_{\vec{p}}(p, \kappa)\mathcal{U}^{-1}(p, \kappa) = \kappa H_b + E(p, \kappa), \quad (3.2)$$

where

$$\begin{aligned}
E(p, \kappa) &= \frac{\hbar^2}{2m} \sum_{i=1}^N \sum_{\mu=1}^d \left(\kappa p_\mu^i + \kappa \sum_{j=1}^N p_\nu^j \Delta_{\nu\mu}^{ji}(\kappa) \right)^2 + \square(\kappa), \\
\Delta_{\nu\mu}^{ji}(\kappa) &= \frac{1}{\kappa^3} \frac{e^2}{2c^2} \sum_{k=1}^N \sum_{r,s=1}^{d-1} \left\langle \frac{e_\nu^r M_{ij}(\kappa) \hat{\rho}_k}{\sqrt{\omega^3}}, \left(I + \mathbf{W}_-(\kappa) \mathbf{W}_+^{-1}(\kappa) \right)^{(r,s)} \frac{e_\mu^s \hat{\rho}_i}{\sqrt{\omega}} \right\rangle_{L^2(\mathbb{R}^d)}, \\
\square(\kappa) &= \frac{e^2 \hbar}{4mc} \sum_{i=1}^N \sum_{r,s=1}^{d-1} \left\langle \frac{e_\mu^r \hat{\rho}_i}{\sqrt{\omega}}, \left(I - \mathbf{W}_-(\kappa) \mathbf{W}_+^{-1}(\kappa) \right)^{(r,s)} \frac{e_\mu^s \hat{\rho}_i}{\sqrt{\omega}} \right\rangle_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Proof: For simplicity, we omit the symbol κ . Put $\mathcal{U}(p)\Omega \equiv \Omega(p)$. From [6, Proposition 2.4, Lemma 5.9] it follows that $\Omega(p) \in D(H_b)$. Then $\Omega(p) \in D(B(\mathbf{f}, p))$. By virtue of Corollary 2.5 and (3.1), we can see that for all $\mathbf{f} \in \mathcal{W}$

$$B(\mathbf{f}, p) \exp \left(i \frac{t}{\hbar} H_{\vec{p}}(p) \right) \Omega(p) = 0. \quad (3.3)$$

The equation (3.3) implies that there exists a positive constant $E(p)$ such that

$$\exp \left(i \frac{t}{\hbar} H_{\vec{p}}(p) \right) \Omega(p) = \exp \left(i \frac{t}{\hbar} E(p) \right) \Omega(p). \quad (3.4)$$

Hence from Corollary 2.5, (3.1), (3.4) and the denseness of

$$\mathcal{L} \left\{ B^\dagger(\mathbf{f}_1) \dots B^\dagger(\mathbf{f}_n) \Omega(p), \Omega(p) \mid \mathbf{f}_j \in \mathcal{W}, j = 1, \dots, n, n \geq 1 \right\},$$

one can get (3.2). The constant $E(p)$ is explicitly given by

$$E(p) = \frac{\langle H_{\vec{p}}(p) \Omega(p), \Omega \rangle_{\mathcal{F}(\mathcal{W})}}{\langle \Omega(p), \Omega \rangle_{\mathcal{F}(\mathcal{W})}}.$$

It completes the proof. □

The positive constant $E(p, \kappa)$ can be rewritten by:

$$E(p, \kappa) = \frac{\kappa^2 \hbar^2}{2m} p^2 + E^{REN}(p, \kappa) + \tilde{E}(p, \kappa),$$

where

$$\tilde{E}(p, \kappa) = \frac{\kappa^2 \hbar^2}{2m} \sum_{i,j=1}^N \sum_{\mu,\nu=1}^d p_\mu^i b_{\mu\nu}^{ij}(\kappa) p_\nu^j, \quad (3.5)$$

$$b_{\mu\nu}^{ij}(\kappa) = \sum_{k=1}^N \sum_{\alpha=1}^d \left(\frac{\Delta_{\nu\alpha}^{jk}(\kappa) + \overline{\Delta_{\nu\alpha}^{jk}(\kappa)}}{2} \right) \left(\frac{\Delta_{\mu\alpha}^{ik}(\kappa) + \overline{\Delta_{\mu\alpha}^{ik}(\kappa)}}{2} \right),$$

$$E^{REN}(p, \kappa) = E(p, \kappa) - \frac{\kappa^2 \hbar^2}{2m} p^2 - \tilde{E}(p, \kappa).$$

Note that since $(b_{\mu\nu}^{ij}(\kappa))_{1 \leq i, j \leq N, 1 \leq \mu, \nu \leq d}$ is nonnegative and symmetric $dN \times dN$ matrix, we have $\tilde{E}(p, \kappa) \geq 0$ for any $p \in \mathbb{R}^{dN}$. We define

$$H_{\vec{p}}^{REN}(\kappa) = -E^{REN}(D, \kappa) \otimes I + \kappa I \otimes H_b$$

$$+ \frac{1}{2m} \sum_{j=1}^N \sum_{\mu=1}^d \left(-2\kappa e \hbar i D_{\mu}^j \otimes A_{\mu}(\rho_j) + e^2 I \otimes A_{\mu}(\rho_j)^2 \right),$$

$$\widetilde{H}_{\vec{p}}(\kappa) = \tilde{E}(D, \kappa) \otimes I + \kappa I \otimes H_b,$$

where $E^{REN}(D, \kappa)$ and $\tilde{E}(D, \kappa)$ are pseudo differential operators on $L^2(\mathbb{R}^{dN})$ with symbols $E^{REN}(p, \kappa)$ and $\tilde{E}(p, \kappa)$ respectively.

Theorem 3.2 *Suppose $\vec{p} \in \tilde{P}$. Then $H_{\vec{p}}^{REN}(\kappa)$ and $\widetilde{H}_{\vec{p}}(\kappa)$ are essentially self-adjoint on any core of H_0 and bounded from below.*

Remark 3.3 *Write*

$$E(p, \kappa) = \frac{\hbar^2 \kappa^2}{2m} p^2 + \sum_{\mu=1}^d \sum_{i=1}^N \frac{\hbar^2 \kappa^2}{m} p_{\mu}^i \tilde{p}_{\mu}^i(\kappa) + \sum_{\mu=1}^d \sum_{i=1}^N \frac{\hbar^2 \kappa^2}{2m} \tilde{p}_{\mu}^i(\kappa)^2 + \square(\kappa). \quad (3.6)$$

Then the first and second terms on the right hand side of (3.6) diverge as $\kappa \rightarrow \infty$ for $p \neq 0$, but the rest terms not. Actually we see that

$$\lim_{\kappa \rightarrow \infty} \frac{\hbar^2 \kappa^2}{2m} \sum_{\mu=1}^d \sum_{i=1}^N \tilde{p}_{\mu}^i(\kappa)^2 = \frac{1}{2m} \left(\frac{e^2}{2m\kappa^2} \right) \left(\frac{d-1}{d} \right)^2 \sum_{\alpha=1}^d \sum_{k=1}^N \left(\sum_{j=1}^N \hbar p_{\alpha}^j \left\langle \frac{\hat{p}_j}{\sqrt{\omega^3}}, \frac{\hat{p}_k}{\sqrt{\omega}} \right\rangle_{L^2(\mathbb{R}^d)} \right)^2,$$

$$\equiv E^{\infty}(p).$$

Then, by (3.2), concerning an asymptotic behavior of $H_{\vec{p}}(\kappa)$ as $\kappa \rightarrow \infty$, we should subtract the first and second terms in the right hand side of (3.6) from the original Hamiltonian $H_{\vec{p}}(\kappa)$. However one can not say that $\tilde{p}_{\mu}^i(\kappa)^2$ is real and nonnegative for any $p \in \mathbb{R}^{dN}$. To

guarantee the nonnegative self-adjointness of the Hamiltonian $H_{\vec{p}}^{REN}(\kappa)$ with the divergence terms subtracted, we should define $\tilde{E}(p, \kappa)$ such as (3.5). In this sense, we may say that the operator $H_{\vec{p}}^{REN}(\kappa)$ has an interpretation of the Hamiltonian $H_{\vec{p}}(\kappa)$ with the infinite self-energy of the nonrelativistic particles subtracted.

We define

$$\mathcal{U}(\kappa) = \mathbf{F}^{-1} \left(\int_{\mathbb{R}^{dN}}^{\oplus} \mathcal{U}(\kappa, p) dp \right) \mathbf{F}.$$

Then we have the following theorem.

Theorem 3.4 ([6]) *Suppose that $\vec{p} \in \tilde{P}$. Then*

$$\begin{aligned} s - \lim_{\kappa \rightarrow \infty} \mathcal{U}(\kappa) &= \exp \left(\sum_{j=1}^N \frac{e\hbar}{m} D_{\mu}^j \otimes \left\{ a \left(\bigoplus_{r=1}^{d-1} \frac{e_{\mu}^r \hat{p}_j}{\sqrt{2\hbar c^3 \omega^3}} \right) - a^{\dagger} \left(\bigoplus_{r=1}^{d-1} \frac{e_{\mu}^r \hat{p}_j}{\sqrt{2\hbar c^3 \omega^3}} \right) \right\} \right), \\ &\equiv \mathcal{U}(\infty). \end{aligned}$$

We take scalar potentials V to be real-valued measurable functions on $\mathbb{R}^{dN}$ and put

$$C_{\kappa}(V) = \mathcal{U}^{-1}(\kappa)(V \otimes I)\mathcal{U}(\kappa), \quad C(V) = \mathcal{U}^{-1}(\infty)(V \otimes I)\mathcal{U}(\infty).$$

We introduce conditions **(V-1)** and **(V-2)** as follows.

(V-1) For sufficiently large $\kappa > 0$, $D(\tilde{E}(D, \kappa)) \subset D(V)$ and for $\lambda > 0$, $V(\tilde{E}(D, \kappa) + \lambda)^{-1}$ is bounded with

$$\lim_{\lambda \rightarrow \infty} \|V(\tilde{E}(D, \kappa) + \lambda)^{-1}\| = 0, \quad (3.7)$$

where the convergence is uniform in sufficiently large $\kappa > 0$.

(V-2) For $\lambda > 0$, $V(\tilde{E}(D, \kappa) + \lambda)^{-1}$ is strongly continuous in κ and

$$s - \lim_{\kappa \rightarrow \infty} V(\tilde{E}(D, \kappa) + \lambda)^{-1} = V(E^{\infty}(D) + \lambda)^{-1}.$$

The condition (3.7) yields that, by the Kato-Rellich theorem and commutativity of $\mathcal{U}(\kappa)$ and $(\tilde{E}(D, \kappa) + \lambda)^{-1}$, operators $\tilde{E}(D, \kappa) \otimes I + C_\kappa(V)$ are essentially self-adjoint on any core of $D(\tilde{E}(D, \kappa) \otimes I)$ and uniformly bounded from below in sufficiently large $\kappa > 0$. Moreover since $I \otimes H_b$ is nonnegative and commute with $\tilde{E}(D, \kappa) \otimes I$, one can see that

$$\widetilde{H}_{\tilde{\rho}}(V, \kappa) \equiv \tilde{E}(D, \kappa) \otimes I + C_\kappa(V) + \kappa I \otimes H_b$$

is essentially self-adjoint on any core of $D(\tilde{E}(D, \kappa) \otimes I + \kappa I \otimes H_b)$ and uniformly bounded from below in sufficiently large $\kappa > 0$. In particular, $D(H_0)$ is a core of $\widetilde{H}_{\tilde{\rho}}(V, \kappa)$. Put

$$H_{\tilde{\rho}}^{REN}(V, \kappa) \equiv H_{\tilde{\rho}}^{REN}(\kappa) + V \otimes I.$$

Theorem 3.5 *Let $\tilde{\rho} \in \tilde{P}$. Suppose that V satisfies (V-1) and (V-2). Then, for sufficiently large $\kappa > 0$, the operator $H_{\tilde{\rho}}^{REN}(V, \kappa)$ is essentially self-adjoint on $D(H_0)$ and bounded from below uniformly in sufficiently large $\kappa > 0$. Moreover the unitary operator $\mathcal{U}(\kappa)$ maps $D(H_0)$ onto itself and for $z \in \mathbb{C} \setminus \mathbb{R}$ or $z < 0$ with $|z|$ sufficiently large,*

$$(H_{\tilde{\rho}}^{REN}(V, \kappa) - z)^{-1} = \mathcal{U}(\kappa) (\widetilde{H}_{\tilde{\rho}}(V, \kappa) - z)^{-1} \mathcal{U}^{-1}(\kappa). \quad (3.8)$$

Proof: Since $\mathcal{U}(\kappa)$ maps $D(I \otimes H_b)$ onto itself (see Theorem 3.1) and $-\Delta \otimes I$ commutes with $\mathcal{U}(\kappa)$ on $D(-\Delta \otimes I)$, $\mathcal{U}(\kappa)$ maps $D(H_0)$ onto itself. Put

$$S_0^\infty(\mathbb{R}^{dN}) = \{f \in L^2(\mathbb{R}^{dN}) | \hat{f} \in C_0^\infty(\mathbb{R}^{dN})\}.$$

At first, by Theorem 3.1, we see that for $\Phi \in S_0^\infty(\mathbb{R}^{dN}) \hat{\otimes} D(H_b)$,

$$H_{\tilde{\rho}}^{REN}(V, \kappa)\Phi = \mathcal{U}(\kappa) \widetilde{H}_{\tilde{\rho}}(V, \kappa) \mathcal{U}^{-1}(\kappa) \Phi. \quad (3.9)$$

By a limiting argument we can extend (3.9) to $\Phi \in D(H_0)$. Since $D(H_0)$ is a core of $\widetilde{H}_{\tilde{\rho}}(V, \kappa)$ and $\mathcal{U}(\kappa)$ maps $D(H_0)$ onto itself, the right hand side of (3.9) is essentially self-adjoint on $D(H_0)$. So is the left hand side of (3.9). (3.8) can be easily shown. $\square$

We want to consider a scaling limit of $H_{\vec{\rho}}^{REN}(V, \kappa)$ as $\kappa \rightarrow \infty$. Let V satisfy **(V-1)**. Then since $D(C(V)) \supset D(-\Delta) \hat{\otimes} D(H_b)$, one can define, for $\Phi \in \mathcal{F}(\mathcal{W})$ and $\Psi \in D(H_b)$, a symmetric operator $E_{\Phi, \Psi}(C(V))$ with $D(E_{\Phi, \Psi}(C(V))) = D(-\Delta)$ by

$$\langle f, E_{\Phi, \Psi}(C(V))g \rangle_{L^2(\mathbb{R}^{dN})} = \langle f \otimes \Phi, C(V)(g \otimes \Psi) \rangle_{\mathcal{F}}, \quad f \in L^2(\mathbb{R}^{dN}), g \in D(-\Delta).$$

In particular, we call $E_{\Omega, \Omega}(C(V)) \equiv E_{\Omega}(C(V))$ “the partial expectation of $C(V)$ with respect to Ω ”.

Theorem 3.6 *Let $\vec{\rho} \in \tilde{P}$. Suppose that V satisfies the conditions **(V-1)** and **(V-2)**. Then for $z \in \mathbb{C} \setminus \mathbb{R}$ or $z < 0$ with $|z|$ sufficiently large,*

$$s - \lim_{\kappa \rightarrow \infty} (H_{\vec{\rho}}^{REN}(V, \kappa) - z)^{-1} = \mathcal{U}(\infty) \left\{ (E^{\infty}(D) + E_{\Omega}(C(V)) - z)^{-1} \otimes P_0 \right\} \mathcal{U}^{-1}(\infty), \quad (3.10)$$

where P_0 is the projection from $\mathcal{F}(\mathcal{W})$ to the one dimensional subspace $\{\alpha\Omega | \alpha \in \mathbb{C}\}$.

Proof: By **(V-1)** and **(V-2)**, we see that

(V-1)' For sufficiently large $\kappa > 0$, $D(\tilde{E}(D, \kappa)) \subset D(C_{\kappa}(V))$ and for $\lambda > 0$,

$C_{\kappa}(V)(\tilde{E}(D, \kappa) + \lambda)^{-1}$ is bounded with

$$\lim_{\lambda \rightarrow \infty} \|C_{\kappa}(V)(\tilde{E}(D, \kappa) + \lambda)^{-1}\| = 0,$$

where the convergence is uniform in sufficiently large $\kappa > 0$.

(V-2)' For $\lambda > 0$, $C_{\kappa}(V)(\tilde{E}(D, \kappa) + \lambda)^{-1}$ is strongly continuous in κ and

$$s - \lim_{\kappa \rightarrow \infty} C_{\kappa}(V)(\tilde{E}(D, \kappa) + \lambda)^{-1} = C(V)(E^{\infty}(D) + \lambda)^{-1}.$$

From **(V-1)'**, **(V-2)'** and iterating the second resolvent formula with respect to the pair $(\widetilde{H}_{\vec{\rho}}(\kappa), \widetilde{H}_{\vec{\rho}}(V, \kappa))$, it follows that

$$s - \lim_{\kappa \rightarrow \infty} (\widetilde{H}_{\vec{\rho}}(V, \kappa) - z)^{-1} = (E^{\infty}(D) \otimes I + (I \otimes P_0)C(V)(I \otimes P_0) - z)^{-1} I \otimes P_0.$$

Since

$$(I \otimes P_0)C(V)(I \otimes P_0) = E_\Omega(C(V)),$$

we see that

$$s - \lim_{\kappa \rightarrow \infty} (\widetilde{H}_{\vec{p}}(V, \kappa) - z)^{-1} = (E^\infty(D) + E_\Omega(C(V)) - z)^{-1} \otimes P_0.$$

Thus by Theorems 3.4 and 3.5, we get (3.10). $\square$

We want to see $E_\Omega(C(V))$ more explicitly. For $\vec{p} \in \tilde{P}$, let $\tilde{\Delta}^\infty = (\tilde{\Delta}_{ij}^\infty)_{1 \leq i, j \leq d}$, where

$$\tilde{\Delta}_{ij}^\infty = \frac{1}{2} \frac{d-1}{d} \left(\frac{\hbar}{mc} \right)^2 \frac{e^2}{\hbar c} \int_{\mathbb{R}^d} dk \frac{\hat{\rho}_i(k) \hat{\rho}_j(k)}{\omega(k)^3}.$$

Let $\mathbf{I}_{d \times d}$ denote $d \times d$ -identity matrix. Since $\Delta^\infty \equiv \tilde{\Delta}^\infty \otimes \mathbf{I}_{d \times d}$ is a nonnegative symmetric matrix, there exist unitary matrices $\mathbf{T}$ so that

$$\mathbf{T} \Delta^\infty \mathbf{T}^{-1} = \begin{pmatrix} \lambda_1 \mathbf{I}_{d \times d} & & & \\ & \lambda_2 \mathbf{I}_{d \times d} & & \\ & & \ddots & \\ & & & \lambda_N \mathbf{I}_{d \times d} \end{pmatrix}, \quad (3.11)$$

where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0$.

Theorem 3.7 Suppose $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_K > 0$, $\lambda_{K+1} = \dots = \lambda_N = 0$ and fix a unitary operator $\mathbf{T}$ in (3.11). Let $x = (x_1, \dots, x_N)$, $x_j \in \mathbb{R}^d$, $j = 1, \dots, N$ and V satisfy

$$\int_{\mathbb{R}^{dK}} dy_1 \dots dy_K |V| \circ \mathbf{T}^{-1}(y_1, \dots, y_K, (\mathbf{T}x)_{K+1}, \dots, (\mathbf{T}x)_N) \exp \left(-\frac{\sum_{j=1}^K |(\mathbf{T}x)_j - y_j|^2}{2\lambda_1 \dots \lambda_K} \right) < \infty. \quad (3.12)$$

Moreover we suppose that the left hand side of (3.12) is locally bounded. Then the partial expectation $E_\Omega(C(V))$ is given by a multiplication operator V_{eff} ;

$$\begin{aligned} V_{eff}(x) &= (2\pi\lambda_1 \dots \lambda_K)^{-\frac{d}{2}} \int_{\mathbb{R}^{dK}} dy_1 \dots dy_K V \circ \mathbf{T}^{-1}(y_1, \dots, y_K, (\mathbf{T}x)_{K+1}, \dots, (\mathbf{T}x)_N) \\ &\quad \times \exp \left(-\frac{\sum_{j=1}^K |(\mathbf{T}x)_j - y_j|^2}{2\lambda_1 \dots \lambda_K} \right). \end{aligned}$$

In particular, in the case where $\tilde{\Delta}^\infty$ is non-degenerate, V_{eff} is given by

$$V_{eff}(x) = (2\pi \det \tilde{\Delta}^\infty)^{-\frac{d}{2}} \int_{\mathbb{R}^{dN}} V(y) \exp\left(-\frac{|x-y|^2}{2 \det \tilde{\Delta}^\infty}\right) dy.$$

Proof: Suppose $V \in \mathcal{S}(\mathbb{R}^{dN})$, which is the set of the rapidly decreasing infinitely continuously differentiable functions on $\mathbb{R}^{dN}$. Then the direct calculation shows that for $f, g \in L^2(\mathbb{R}^{dN})$

$$\langle f, E_\Omega(C(V))g \rangle_{L^2(\mathbb{R}^{dN})} = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dk \bar{f}(x) g(x) e^{ikx} \hat{V}(k) e^{-\frac{1}{2} \sum_{\mu=1}^d \sum_{i,j=1}^N \Delta_{ij}^\infty k_\mu^i k_\mu^j}.$$

Hence we have

$$\langle f, E_\Omega(C(V))g \rangle_{L^2(\mathbb{R}^{dN})} = \langle f, V_{eff}g \rangle_{L^2(\mathbb{R}^{dN})}. \quad (3.13)$$

We next consider the case where V is bounded. In this case we can approximate V by a sequence $\{V_n\}_{n=1}^\infty$, $V_n \in \mathcal{S}(\mathbb{R}^{dN})$, such that

$$\|V - V_n\|_\infty \rightarrow 0 \quad (n \rightarrow \infty),$$

where $\|\cdot\|_\infty$ denotes the sup norm. Then we have

$$E_\Omega(C(V_n)) \rightarrow E_\Omega(C(V)) \quad (n \rightarrow \infty),$$

strongly. Moreover $(V_n)_{eff}(x) \rightarrow V_{eff}(x)$ for all $x \in \mathbb{R}^{dN}$. Thus for $f, g \in L^2(\mathbb{R}^{dN})$, (3.13) follows for such V . Finally, let V satisfy (3.12). Define

$$V_n = \begin{cases} V(x) & |V(x)| \leq n, \\ n & |V(x)| > n. \end{cases}$$

Hence for $f \in L^2(\mathbb{R}^{dN})$ and $g \in D(-\Delta)$, we have

$$\langle f, E_\Omega(C(V_n))g \rangle_{L^2(\mathbb{R}^{dN})} \rightarrow \langle f, E_\Omega(C(V))g \rangle_{L^2(\mathbb{R}^{dN})} \quad (n \rightarrow \infty).$$

On the other hand, since the left hand side of (3.12) is locally bounded, we can see that for $f \in C_0^\infty(\mathbb{R}^{dN})$ and $g \in D(-\Delta)$,

$$\langle f, (V_n)_{eff}g \rangle_{L^2(\mathbb{R}^{dN})} \rightarrow \langle f, V_{eff}g \rangle_{L^2(\mathbb{R}^{dN})} \quad (n \rightarrow \infty),$$

which completes the proof. □

Remark 3.8 In Theorem 3.7, in the case where $\tilde{\Delta}^\infty$ is non-degenerate, since the left hand side of (3.12) is continuous in $x \in \mathbb{R}^{dN}$, it is necessarily locally bounded.

We call V_{eff} “the effective potential with respect to V ”. We give a typical example of scalar potentials V and ultraviolet cut-off functions $\vec{\rho}$.

Example 3.9 Let

$$\tilde{\Delta}_{ij}^\infty = \delta_{ij} \frac{1}{2} \frac{d-1}{d} \left(\frac{\hbar}{mc} \right)^2 \frac{e^2}{\hbar c} \int_{\mathbb{R}^d} dk \frac{\hat{\rho}_i(k)^2}{\omega(k)^3}.$$

Then there exist positive constants δ_1 and δ_2 such that for sufficiently large $\kappa > 0$

$$\delta_1 |p|^2 \leq \tilde{E}(p, \kappa) \leq \delta_2 |p|^2. \quad (3.14)$$

Let $d = 3$ and V be the Coulomb potential;

$$V(x_1, \dots, x_N) = - \sum_{j=1}^N \frac{\alpha_j}{|x_j|} + \sum_{i \neq j} \frac{\beta_{ij}}{|x_i - x_j|}, \quad \alpha_j \geq 0, \beta_{ij} \geq 0.$$

Then V is the Kato class potential ([10], Theorem X.16). Namely for any $\epsilon > 0$, there exists $b \geq 0$ such that $D(V) \supset D(-\Delta)$ and

$$\|V\Phi\|_{L^2(\mathbb{R}^{3N})} \leq \epsilon \|-\Delta\Phi\|_{L^2(\mathbb{R}^{3N})} + b \|\Phi\|_{L^2(\mathbb{R}^{3N})}. \quad (3.15)$$

Together with (3.14) and (3.15), one can see that V satisfies (V-1), (V-2) and for any $t > 0$

$$\int_{\mathbb{R}^{3d}} |V|(y) e^{-t|x-y|^2} dy < \infty.$$

Then the scaling limit of the Pauli-Fierz Hamiltonian with the Coulomb potential exists and has the effective potential given by

$$V_{eff}(x) = (2\pi\gamma)^{-\frac{3}{2}} \int_{\mathbb{R}^{3N}} V(y) e^{-\frac{|x-y|^2}{2\gamma}} dy,$$

$$\gamma = \left\{ \frac{1}{3} \left(\frac{\hbar}{mc} \right)^2 \frac{e^2}{\hbar c} \right\}^N \prod_{j=1}^N \left(\int_{\mathbb{R}^3} dk \frac{\hat{\rho}_j^2(k)}{\omega(k)^3} \right).$$

4 CONCLUDING REMARK

As is seen in Theorem 3.7, the effective potential V_{eff} is characterized by the matrix-valued functional $\tilde{\Delta}^\infty = \tilde{\Delta}^\infty(\vec{\rho})$, which has the following mathematical meaning; putting

$$\mathcal{U}(\infty)(x_i \otimes I)\mathcal{U}^{-1}(\infty) - x_i \otimes I \equiv \Delta x_i, \quad i = 1, \dots, N,$$

we see that the partial expectation of $\Delta x_i \Delta x_j$ with respect to Ω is as follows;

$$E_\Omega[(\Delta x_i \Delta x_j)] = \tilde{\Delta}_{ij}^\infty(\vec{\rho})I.$$

In one-nonrelativistic particle case, the author in [5] show that the partial expectation $E_\Omega[(\Delta x)^2]$ with respect to Ω may be interpreted as the mean square fluctuation in position of one-nonrelativistic particle ([2]). In this sense, $\tilde{\Delta}_{ij}^\infty(\vec{\rho})$ may also be interpreted as correlation of fluctuations in position of the i -th and the j -th nonrelativistic particles under the action of quantized radiation fields.

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