

作用素値内積を用いた一般化シュワルツの不等式について
The generalized Schwarz inequality via operator-valued inner product

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1. INTRODUCTION

This report is based on [8].

Let $B(\mathcal{H})$ be the space of all bounded linear operators on a Hilbert space H , and I stands for the identity operator on H . An operator A in $B(\mathcal{H})$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x \rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For selfadjoint operators A and B , the order relation $A \geq B$ means that $A - B$ is positive.

The Cauchy-Schwarz inequality is one of the most useful and fundamental inequalities in functional analysis: For all vectors x and y in a Hilbert space H

$$(1.1) \quad |\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle.$$

We want to study a non-commutative version of the Cauchy-Schwarz inequality (1.1). Since the product AB of positive operators A and B is not always positive, we need to deform the Cauchy-Schwarz inequality (1.1) to be convenient. For example, the Cauchy-Schwarz inequality is transformed as follows: Dividing both sides in (1.1) by $\langle y, y \rangle (\neq 0)$

$$(1.2) \quad \overline{\langle x, y \rangle} \langle y, y \rangle^{-1} \langle x, y \rangle \leq \langle x, x \rangle$$

and taking square root of both sides in (1.1)

$$(1.3) \quad |\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle \langle y, y \rangle}.$$

Firstly, we consider the case of (1.2): Regarding a sesquilinear map $B(X, Y) = Y^*X$ for $X, Y \in B(\mathcal{H})$ as an operator-valued inner product, several operator versions for the Schwarz inequality are discussed by many researchers. For example, if $X, Y \in B(\mathcal{H})$, then the Schwarz inequality for operators

$$(1.4) \quad X^*Y(Y^*Y)^{-1}Y^*X \leq X^*X$$

holds. Indeed, since $Y(Y^*Y + \varepsilon I)^{-1}Y^* \leq I$ for all $\varepsilon > 0$ and $Y(Y^*Y + \varepsilon I)^{-1}Y^*$ is increasing for $\varepsilon \downarrow 0$, there exists the strong operator limit of $Y(Y^*Y + \varepsilon I)^{-1}Y^*$ as $\varepsilon \rightarrow 0$ and we define

$$X^*Y(Y^*Y)^{-1}Y^*X = \text{s-lim}_{\varepsilon \rightarrow 0} X^*Y(Y^*Y + \varepsilon I)^{-1}Y^*X$$

and write $X^*Y(Y^*Y)^{-1}Y^*X \in B(\mathcal{H})$. This formulation for matrices is firstly given by Marshall and Olkin in [10]. Let T be a positive operator and X, Y any two operators in $B(\mathcal{H})$. Replacing X and Y in (1.4) by $T^{1/2}X$ and $T^{1/2}Y$, respectively, we obtain $X^*TY(Y^*TY)^{-1}Y^*TX \in B(\mathcal{H})$ and

$$(1.5) \quad X^*TY(Y^*TY)^{-1}Y^*TX \leq X^*TX.$$

In [3], Bhatia and Davis showed some new operator versions of the Schwarz inequality for a positive linear map, which is a generalization of (1.5): A map Φ on $B(\mathcal{H})$ is called 2-positive if

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \geq 0 \quad \text{implies} \quad \begin{pmatrix} \Phi(A) & \Phi(B) \\ \Phi(C) & \Phi(D) \end{pmatrix} \geq 0.$$

It is known that if A is positive, X is positive invertible and B is any operators in $B(\mathcal{H})$, then

$$\begin{pmatrix} A & B \\ B^* & X \end{pmatrix} \geq 0 \iff A \geq BX^{-1}B^*.$$

From this it follows that if T is positive and Φ is a 2-positive linear map on $B(\mathcal{H})$, then

$$(1.6) \quad \Phi(X^*TY)\Phi(Y^*TY)^{-1}\Phi(Y^*TX) \leq \Phi(X^*TX)$$

for every $X, Y \in B(\mathcal{H})$.

In the framework of an operator-valued inner product, the formulation of the Schwarz operator inequality is very important, but the left-hand sides of the Schwarz inequalities (1.4) and (1.6) for operators are expressed as the strong-operator limits unless Y^*Y and $\Phi(Y^*TY)$ are invertible. This fact is a cause of difficulty in application. Thus, we consider the case of (1.3). For this, we recall the geometric operator mean, also see [2, 11]. Let A and B be two positive operators in $B(\mathcal{H})$. The geometric operator mean $A \sharp B$ of A and B is defined by

$$A \sharp B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{\frac{1}{2}} A^{\frac{1}{2}}$$

if A is invertible. The geometric operator mean has the monotonicity:

$$0 \leq A \leq C \quad \text{and} \quad 0 \leq B \leq D \quad \text{implies} \quad A \sharp B \leq C \sharp D$$

and the subadditivity:

$$A \sharp B + C \sharp D \leq (A + C) \sharp (B + D).$$

By monotonicity, we can uniquely extend the definition of $A \sharp B$ for all positive operators A and B by setting

$$A \sharp B = \text{s-lim}_{\varepsilon \rightarrow 0} (A + \varepsilon I) \sharp (B + \varepsilon I).$$

In this case, the geometric operator mean $A \sharp B$ for positive operators A and B always exists in $B(\mathcal{H})$ and it has all the desirable properties as geometric mean such as monotonicity, continuity from above, transformer inequality, subadditivity and self-duality so on.

J.I. Fujii in [5] studied another version of the Schwarz operator inequality in terms of the geometric operator mean, which is a main tool of our research:

Theorem A. *Let Φ be a 2-positive map on $B(\mathcal{H})$. Then*

$$(1.7) \quad |\Phi(Y^*X)| \leq \Phi(X^*X) \sharp U^*\Phi(Y^*Y)U$$

for every $X, Y \in B(\mathcal{H})$, where U is a partial isometry in the polar decomposition of $\Phi(Y^*X) = U|\Phi(Y^*X)|$.

In this paper, by virtue of the Cauchy-Schwarz operator inequality due to J.I. Fujii, we show weighted mixed Schwarz operator inequalities in terms of the geometric operator mean. As applications, we show the covariance-variance operator inequality via the geometric operator mean which differs from Bhatia-Davis's one. By our formulation, we

show a Robertson type inequality associated to a unital completely positive linear maps on $B(\mathcal{H})$.

2. WEIGHTED MIXED SCHWARZ OPERATOR INEQUALITIES

First of all, we discuss weighted mixed Schwarz operator inequalities in terms of the geometric operator mean.

For $T \in B(\mathcal{H})$, $T = W|T|$ is the polar decomposition of T where W is a partial isometry and $|T| = (T^*T)^{1/2}$ with the kernel condition $\ker(W) = \ker(|T|)$. Note that WW^* is the projection onto the range of $|T^*|$, and W^*W is the projection onto the range of $|T|$. Then it follows from [9] that

$$(2.1) \quad W|T|^q W^* = |T^*|^q \quad \text{for any } q > 0.$$

Furuta in [9] showed the following weighted mixed Schwarz inequalities:

Theorem B (Weighted mixed Schwarz inequalities). *For any operator T in $B(\mathcal{H})$,*

$$|\langle Tx, y \rangle|^2 \leq \langle |T|^{2\alpha} x, x \rangle \langle |T^*|^{2\beta} y, y \rangle$$

*holds for any $x, y \in H$ and for any real number α, β with $\alpha + \beta = 1$. Moreover, for $1 < \alpha < 1$, the equality holds if and only if $|T|^{2\alpha} x$ and T^*y are linearly dependent if and only if Tx and $|T^*|^{2\beta} y$ are linearly dependent. For $\alpha = 1$, the equality holds if and only if Tx and y are linearly dependent. For $\alpha = 0$, the equality holds if and only if x and T^*y are linearly dependent.*

By Theorem A, we have the following weighted mixed Schwarz operator inequality, which is an operator version of Theorem B:

Theorem 2.1 (Weighted mixed Schwarz operator inequality). *Let Φ be a 2-positive map on $B(\mathcal{H})$ and T an operator in $B(\mathcal{H})$. If $X, Y \in B(\mathcal{H})$, then*

$$(2.2) \quad |\Phi(Y^*TX)| \leq \Phi(X^*|T|^{2\alpha}X) \sharp U^*\Phi(Y^*|T^*|^{2\beta}Y)U$$

*for any $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$, where $\Phi(Y^*TX) = U|\Phi(Y^*TX)|$ is the polar decomposition of $\Phi(Y^*TX)$. In particular, in the case of $\alpha = 0, 1$*

$$|\Phi(Y^*TX)| \leq \Phi(X^*W^*WX) \sharp U^*\Phi(Y^*|T^*|^{2\beta}Y)U$$

and

$$|\Phi(Y^*TX)| \leq \Phi(X^*|T|^2X) \sharp U^*\Phi(Y^*WW^*Y)U,$$

where $T = W|T|$ is the polar decomposition of T .

Proof. We only prove the case of $0 < \alpha < 1$. It follows that

$$\begin{aligned} |\Phi(Y^*TX)| &= |\Phi(Y^*W|T|X)| = |\Phi(Y^*W|T|^\beta|T|^\alpha X)| && \text{by } \alpha + \beta = 1 \\ &\leq \Phi(X^*|T|^{2\alpha}X) \sharp U^*\Phi(Y^*W|T|^{2\beta}W^*Y)U && \text{by Theorem A} \\ &= \Phi(X^*|T|^{2\alpha}X) \sharp U^*\Phi(Y^*|T^*|^{2\beta}Y)U && \text{by (2.1)} \end{aligned}$$

and so we have the desired inequality (2.2). $\square$

Next, we consider the equality condition in (1.7) of Theorem A. To show it, we need some preliminaries. First of all, we recall the Moore-Penrose inverse: For a given operator $A \in B(\mathcal{H})$ having a closed range, it is well known that the equations $AGA = A$, $GAG = G$, $(AG)^* = AG$ and $(GA)^* = GA$ have a unique common solution for $G \in B(\mathcal{H})$, denoted by $G = A^\dagger$ and called the Moore-Penrose inverse of A . In [6], J.I.Fujii showed a relation between the geometric operator mean and the Moore-Penrose inverse:

$$(2.3) \quad A \sharp B \leq A^{1/2} \left((A^{1/2})^\dagger B (A^{1/2})^\dagger \right)^{1/2} A^{1/2}.$$

We show that the equality in (2.3) holds under a kernel condition:

Lemma 2.2. *Let A and B be positive operators in $B(\mathcal{H})$. If A has a closed range and $\ker A \subset \ker B$, then*

$$A \sharp B = A^{1/2} \left((A^{1/2})^\dagger B (A^{1/2})^\dagger \right)^{1/2} A^{1/2}.$$

Lemma 2.3. *Let A and B be positive operators in $B(\mathcal{H})$. If A has a closed range and $\ker A \subset \ker(BA^\dagger B)$, then $A \sharp BA^\dagger B = R(A)BR(A)$, where $R(A)$ is the range projection of A . In addition, if $\ker A \subset \ker B$, then $A \sharp BA^\dagger B = B$.*

Lemma 2.4. *Let A, B and C be positive operators in $B(\mathcal{H})$. If A has a closed range and $\ker A \subset \ker B \cap \ker C$, then $A \sharp B = A \sharp C$ implies $B = C$.*

We show the following equality condition in (1.7) of Theorem A:

Theorem 2.5. *Let Φ be a 2-positive map on $B(\mathcal{H})$. For every $X, Y \in B(\mathcal{H})$, let U be a partial isometry in the polar decomposition of $\Phi(Y^*X) = U|\Phi(Y^*X)|$. If $\Phi(X^*X)$ has a closed range, then the equality in (1.7) of Theorem A holds if and only if $U^*\Phi(Y^*Y)U = |\Phi(Y^*X)|\Phi(X^*X)^\dagger|\Phi(Y^*X)|$.*

We note that in the case that Φ is the identity map in Theorem 2.5, we see that if $\Phi(X^*X)$ has a closed range, then the equality condition $U^*\Phi(Y^*Y)U = |\Phi(Y^*X)|\Phi(X^*X)^\dagger|\Phi(Y^*X)|$ holds if and only if there exists $W \in B(\mathcal{H})$ such that $YU = XW$, that is, $\{YU, X\}$ is linearly dependent.

As an application, we have the following equality condition of Theorem 2.1:

Theorem 2.6. *Let Φ be a 2-positive map on $B(\mathcal{H})$ and T an operator in $B(\mathcal{H})$. For every $X, Y \in B(\mathcal{H})$, let U be a partial isometry in the polar decomposition of $\Phi(Y^*TX) = U|\Phi(Y^*TX)|$ and $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$. If $\Phi(X^*|T|^{2\alpha}X)$ has a closed range, then the equality in (2.2) of Theorem 2.1 holds if and only if $U^*\Phi(Y^*|T^*|^{2\beta}Y)U = |\Phi(Y^*TX)|\Phi(X^*|T|^{2\alpha}X)^\dagger|\Phi(Y^*TX)|$.*

3. VARIANCE-COVARIANCE INEQUALITY

We recall the notion of the covariance and the variance of operators defined by Fujii, Furuta, Nakamoto and Takahasi [7]. In 1954, the noncommutative probability theory is founded by H. Umegaki as an application of the theory of von Neumann algebra in [12]. An operator $A \in B(\mathcal{H})$ plays the role of a random variable, that is, for every unit vector $x \in \mathcal{H}$, the functional $\langle Ax, x \rangle$ on the operator algebra may be thought as an expectation at a state x (with $\|x\| = 1$). The covariance of operators A and B at a state x is introduced by

$$(3.1) \quad \text{cov}_x(A, B) = \langle A^*Bx, x \rangle - \langle A^*x, x \rangle \langle Bx, x \rangle,$$

and the variance of A at a state x by

$$\text{var}_x(A) = \langle A^*Ax, x \rangle - |\langle Ax, x \rangle|^2.$$

The following variance-covariance inequality is an application of the Cauchy-Schwarz inequality:

$$(3.2) \quad |\text{cov}_x(A, B)| \leq \sqrt{\text{var}_x(A)\text{var}_x(B)}.$$

In [3], Bhatia and Davis studied a noncommutative analogue of variance and covariance in statistics, which is a generalization of the covariance (3.1) at a state: Let Φ be a unital completely positive linear map on $B(\mathcal{H})$. The covariance $\text{cov}(A, B)$ between two operators A and B is defined by

$$\text{cov}(A, B) = \Phi(A^*B) - \Phi(A)^*\Phi(B).$$

The variance of A is defined by

$$\text{var}(A) = \text{cov}(A, A) = \Phi(A^*A) - \Phi(A)^*\Phi(A).$$

Since Φ is completely positive, then the variance of A is positive, i.e., $\text{var}(A) \geq 0$. Bhatia and Davis showed the following counterpart of the variance-covariance inequality in the context of noncommutative probability, which is a generalization of the variance-covariance inequality (3.2): For all $A, B \in B(\mathcal{H})$,

$$\text{cov}(A, B)\text{var}(B)^{-1}\text{cov}(A, B)^* \in B(\mathcal{H})$$

and

$$\text{cov}(A, B)\text{var}(B)^{-1}\text{cov}(A, B)^* \leq \text{var}(A).$$

By virtue of the geometric operator mean, we show the following variance-covariance inequality:

Theorem 3.1. *Let Φ be a unital completely positive linear map on $B(\mathcal{H})$ and A, B two operators in $B(\mathcal{H})$. Then*

$$(3.3) \quad |\text{cov}(A, B)| \leq U^*\text{var}(A)U \sharp \text{var}(B),$$

where $\text{cov}(A, B) = U|\text{cov}(A, B)|$ is the polar decomposition of $\text{cov}(A, B)$.

Proof. It follows from [3, Theorem 1] that the 2×2 operator matrix

$$\begin{pmatrix} \text{var}(A) & \text{cov}(A, B) \\ \text{cov}(A, B)^* & \text{var}(B) \end{pmatrix}$$

is positive. Then we have

$$\begin{aligned} 0 &\leq \begin{pmatrix} U^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \text{var}(A) & \text{cov}(A, B) \\ \text{cov}(A, B)^* & \text{var}(B) \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} U^*\text{var}(A)U & U^*U|\text{cov}(A, B)| \\ |\text{cov}(A, B)|U^*U & \text{var}(B) \end{pmatrix} = \begin{pmatrix} U^*\text{var}(A)U & |\text{cov}(A, B)| \\ |\text{cov}(A, B)| & \text{var}(B) \end{pmatrix}. \end{aligned}$$

Since $A \sharp B = \max\{X \geq 0 : \begin{pmatrix} A & X \\ X & B \end{pmatrix} \geq 0\}$, we have the desired inequality (3.3). □

4. COMUTATION RELATION AND COVARIANCE

In this section, we discuss the near relation of the variance-covariance inequality with the Heisenberg uncertainty principle in quantum physics. In [4], Enomoto pointed out that the variance-covariance inequality (3.2) is exactly the generalized Schrödinger inequality: Let A and B be (not necessarily bounded) selfadjoint operators on a Hilbert space $\mathcal{H}$. Let $D(AB)$ and $D(BA)$ be the domain of AB and BA , respectively. Let $\{A, B\}$ and $[A, B]$ be the Jordan product $AB + BA$ and the commutator $AB - BA$, respectively. Then

$$|\text{cov}_x(A, B)|^2 = \left(\frac{1}{2} \langle \{A, B\}x, x \rangle - \langle Ax, x \rangle \langle Bx, x \rangle \right)^2 + \left(\frac{1}{2i} \langle [A, B]x, x \rangle \right)^2$$

for every unit vector $x \in D(AB) \cap D(BA)$. In particular, the following Robertson type inequality holds:

$$\sqrt{\text{var}_x(A)\text{var}_x(B)} \geq \frac{1}{2} |\langle [A, B]x, x \rangle|$$

and the following Schrödinger type inequality holds:

$$\text{var}_x(A)\text{var}_x(B) \geq \left| \frac{1}{2} \langle \{A, B\}x, x \rangle - \langle Ax, x \rangle \langle Bx, x \rangle \right|^2 + \frac{1}{4} |\langle [A, B]x, x \rangle|^2.$$

We show a Robertson type inequality associated to a unital completely positive linear map on $B(\mathcal{H})$:

Theorem 4.1 (Robertson type inequality). *Let Φ be a unital completely positive linear map on $B(\mathcal{H})$. Then for every selfadjoint operators $A, B \in B(\mathcal{H})$, there exists an isometry $V \in B(\mathcal{H})$ such that*

$$U^* \text{var}(A) U \sharp \text{var}(B) \geq V^* \left(\frac{\Phi([A, B]) - [\Phi(A), \Phi(B)]}{2i} \right)_+ V,$$

where $\text{cov}(A, B) = U|\text{cov}(A, B)|$ is the polar decomposition of $\text{cov}(A, B)$ and X_+ is the positive part of a selfadjoint operator $X \in B(\mathcal{H})$.

Proof. It follows from [1, Proposition 2.1] that there exists an isometry $V \in B(\mathcal{H})$ such that $\text{Re}(-i \text{cov}(A, B))_+ \leq V|-i \text{cov}(A, B)|V^*$ and so

$$V^* \text{Re}(\text{cov}(A, B))_+ V \leq |\text{cov}(A, B)|.$$

Since $\text{Im}(\text{cov}(A, B)) = \frac{1}{2i} (\Phi(AB - BA) - (\Phi(A)\Phi(B) - \Phi(B)\Phi(A)))$, we have

$$\begin{aligned} |\text{cov}(A, B)| &= |-i \text{cov}(A, B)| \\ &\geq V^* \text{Re}(-i \text{cov}(A, B))_+ V \\ &= V^* \text{Im}(\text{cov}(A, B))_+ V \\ &= V^* \left(\frac{\Phi([A, B]) - [\Phi(A), \Phi(B)]}{2i} \right)_+ V. \end{aligned}$$

Hence we have the desired inequality by Theorem 3.1. $\square$

Under the restricted condition, we have a Schrödinger type inequality associated to a unital completely positive linear map on $B(\mathcal{H})$:

Corollary 4.2 (Schrödinger type inequality). *Let Φ be a unital completely positive linear map on $B(\mathcal{H})$ and $A, B \in B(\mathcal{H})$ two selfadjoint operators. If $\Phi(AB) - \Phi(A)\Phi(B)$ is normal, then*

$$\begin{aligned} & U^* \text{var}(A) U \sharp \text{var}(B) \\ & \geq \left(\frac{1}{4} (\Phi(\{A, B\}) - \{\Phi(A), \Phi(B)\})^2 + \left(\frac{\Phi([A, B]) - [\Phi(A), \Phi(B)]}{2i} \right)^2 \right)^{\frac{1}{2}} \\ & \geq \frac{1}{2} |\Phi([A, B]) - [\Phi(A), \Phi(B)]|, \end{aligned}$$

where $\text{cov}(A, B) = U|\text{cov}(A, B)|$ is the polar decomposition of $\text{cov}(A, B)$.

Proof. For every selfadjoint $A, B \in B(\mathcal{H})$, we have

$$\begin{aligned} |\text{cov}(A, B)| &= |\text{Re}(\text{cov}(A, B)) + i \text{Im}(\text{cov}(A, B))| \\ &= \left(\frac{1}{4} (\Phi(\{A, B\}) - \{\Phi(A), \Phi(B)\})^2 + \left(\frac{\Phi([A, B]) - [\Phi(A), \Phi(B)]}{2i} \right)^2 + \frac{1}{2} (X^*X - XX^*) \right)^{\frac{1}{2}}, \end{aligned}$$

where $X = \Phi(AB) - \Phi(A)\Phi(B)$. Since X is normal, it follows that $X^*X = XX^*$ and so we have the desired inequality by Theorem 3.1. $\square$

Acknowledgements. The author is partially supported by JSPS KAKENHI Grant Number JP16K05253.

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