

An Example of a p -Quasihyponormal Operator

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Introduction. A bounded linear operator T on a Hilbert space $\mathcal{H}$ is called p -hyponormal if $(T^*T)^p \geq (TT^*)^p$ for $p > 0$, and T is called p -quasihyponormal if $T^*\{(T^*T)^p - (TT^*)^p\}T \geq 0$ for $p > 0$. T is called paranormal[8] if $\|Tx\|^2 \leq \|T^2x\|\|x\|$ for all $x \in \mathcal{H}$. It is well-known by Ando[3] that every p -hyponormal operator is paranormal. M. Lee and S. Lee showed that every p -quasihyponormal operator for $0 < p \leq 1$ is paranormal. It is well-known that every p -hyponormal operator $T = U|T|$ is q -hyponormal for all $q \in (0, p)$ by Heinz's inequality and its generalized Aluthge transform $T(s, t) = |T|^s U |T|^t$ for $s, t > 0$ is a q -hyponormal for some $q = q(s, t, p) > 0$. (See [1],[2],[6],[7] and [13]). But the assertions that p -quasihyponormal is q -quasihyponormal if $0 < q < p$ and the generalized Aluthge transform $T(s, t) = |T|^s U |T|^t$ for $s, t > 0$ of a p -quasihyponormal operator $T = U|T|$ is a q -quasihyponormal for some $q = q(s, t, p) > 0$ are not true.

In this paper, we give a p -quasihyponormal operator $T = U|T|$ such that (i) T is not q -quasihyponormal for all $q \in (0, p)$, (ii) $|T|^s U |T|^t$ for $s, t > 0$ is not q -quasihyponormal for all $q \in (0, \infty)$ and (iii) T is a p -quasihyponormal for a $p > 1$, but is not paranormal.

Lemma 1. (Hölder-McCarthy Inequality[9]) For any positive operator A and $x \in \mathcal{H}$,

- (1) $(A^r x, x) \leq \|x\|^{2(1-r)} (Ax, x)^r$ (if $0 < r \leq 1$),
- (2) $(A^r x, x) \geq \|x\|^{2(1-r)} (Ax, x)^r$ (if $r \geq 1$).

Using above lemma, M. Lee and H. Lee obtained the following.

Theorem 1. (M. Lee and H. Lee[10]) If T is a p -quasihyponormal operator such as $0 < p \leq 1$, then T is paranormal.

Here, we construct an example of p -quasihyponormal operator which satisfies the conditions(i)-(iii) in the introduction.

Let $\{\varepsilon_n; n \in \mathbb{Z}\}$ be the canonical orthonormal basis of $\ell^2(\mathbb{Z})$ and p_n the projection of $\ell^2(\mathbb{Z})$ to $\mathbb{C}\varepsilon_n$. Using the shift operator S on $\ell^2(\mathbb{Z})$ with $S\varepsilon_n = \varepsilon_{n+1}$ and positive 2×2 Hermitian matrices A and B , we define operators H and T on $\mathbb{C}^2 \otimes \ell^2(\mathbb{Z})$ by

$$H = \sum_{n < 0} A \otimes p_n + \sum_{n \geq 0} B \otimes p_n$$

and

$$T = (1 \otimes S)H.$$

$T = U|T|$, where $U = 1 \otimes S$ and $|T| = H$. Since $|T^*| = U|T|U^* = \sum_{n \leq 0} A \otimes p_n + \sum_{n > 0} B \otimes p_n$, it is easy to see that

$$T^*(|T|^{2p} - |T^*|^{2p})T = A(B^{2p} - A^{2p})A \otimes p_{-1}$$

for $p > 0$. Hence we have the following.

Lemma 2. T is p -quasihyponormal if and only if $A(B^{2p} - A^{2p})A \geq 0$.

In what follows we assume that A and B are of the form

$$\begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

respectively, here $\alpha > 0$. Let f be a function on the half interval $(0, \infty)$ defined by

$$f(p) = \left(\frac{9^p + 1}{2} \right)^{\frac{1}{2p}}.$$

Then it is strictly increasing.

Theorem 2. (1) T is p -quasihyponormal if and only if $\alpha \leq f(p)$.

(2) If $\alpha = f(p)$, then T is not q -quasihyponormal for $q \in (0, p)$, but q -quasihyponormal for $q \in [p, \infty)$. Hence T satisfies the condition(i).

Proof. (1) Since

$$B^{2p} = \frac{1}{2} \begin{pmatrix} 9^p + 1 & 9^p - 1 \\ 9^p - 1 & 9^p + 1 \end{pmatrix},$$

it is easy to see that T is p -quasihyponormal if and only if $(9^p + 1)/2 - \alpha^{2p} \geq 0$.

(2) It is immediate from (1). QED

Theorem 3. Let $T(s, t) = |T|^s U |T|^t$ for $s, t > 0$.

(1) If $T(s, t)$ is p -quasihyponormal, then $\alpha \leq f(s)$.

(2) If $\alpha = f(p)$ and $s \in (0, p)$, then $T(s, t)$ is not q -quasihyponormal for all $q > 0$. Hence T satisfies the condition(ii).

Proof. (1) Since

$$\begin{aligned} T(s, t)^* (|T(s, t)|^{2p} - |T(s, t)|^{2p}) T(s, t) \\ = A^{s+t} \{ (A^t B^{2s} A^t)^p - A^{2(s+t)p} \} A^{s+t} \otimes p_{-2} \\ + A^t B^s \{ B^{2(s+t)p} - (B^s A^{2t} B^s)^p \} B^s A^t \otimes p_{-1}, \end{aligned}$$

$T(s, t)$ is p -quasihyponormal if and only if

$$(A^t B^{2s} A^t)^p - A^{2(s+t)p} \geq 0, \text{ and } A^t B^s \{ B^{2(s+t)p} - (B^s A^{2t} B^s)^p \} B^s A^t \geq 0.$$

The former inequality implies that $\alpha \leq f(s)$.

(2) It is immediate from (1). QED

Theorem 4. T is paranormal if and only if $\alpha \leq \sqrt{5} = f(1)$.

Proof. It is well-known by Ando[3] that an operator S is a paranormal if and only if $S^{*2} S^2 - 2k S^* S + k^2 \geq 0$ for all $k \in \mathbb{R}$.

Since

$$\begin{aligned} T^{*2} T^2 - 2k T^* T + k^2 &= \sum_{n < -1} (A^2 - k)^2 \otimes p_n + (AB^2 A - 2k A^2 + k^2) \otimes p_{-1} \\ &\quad + \sum_{n \geq 0} (B^2 - k)^2 \otimes p_n. \end{aligned}$$

$$T \text{ is a paranormal} \Leftrightarrow AB^2 A - 2k A^2 + k^2 \geq 0 \quad \forall k \in \mathbb{R}$$

$$\Leftrightarrow 5\alpha^2 - 2k\alpha^2 + k^2 \geq 0 \quad \forall k \in \mathbb{R}$$

$$\Leftrightarrow \alpha^4 - 5\alpha^2 \leq 0$$

$$\Leftrightarrow \alpha \leq \sqrt{5} = f(1) \text{ (since } \alpha > 0 \text{). QED}$$

Remark. If $\alpha = f(p)$ for $p > 1$, then T is a p -quasihyponormal by Theorem 2, but T is not paranormal by Theorem 4. Hence T satisfies the condition(iii).

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