

MEAN ERGODIC THEOREMS FOR ASYMPTOTIC ISOMETRY SEMIGROUPS IN BANACH SPACES

HIROMICHI MIYAKE (三宅 啓道)

1. INTRODUCTION

In 1948, Lorentz [12] introduced a notion of almost convergence for bounded sequences of real numbers: Let $\{x_n\}$ be a bounded sequence of real numbers. Then, $\{x_n\}$ is said to be almost convergent if $\mu_n(x_n) = \nu_n(x_n)$ for any Banach limits μ and ν . Day [7] defined a notion of almost convergence for bounded real-valued functions defined on an amenable semigroup.

On the other hand, von Neumann [15] introduced a notion of almost periodicity for bounded real-valued functions defined on an abstract group and proved the existence of the mean values for those functions. Later, Bochner and von Neumann [3] proved the existence of the mean values for vector-valued almost periodic functions defined on an abstract group with values in a complete locally convex space.

Motivated by the works of Lorentz and von Neumann, we [13] introduced notions of almost convergence in the sense of Lorentz and the mean values for vector-valued bounded functions defined on a left amenable semigroup with values in a locally convex space and also obtained characterizations of almost convergence for those functions in the case of commutative semigroups. By applying these notions and characterizations to transformation semigroups, we prove non-linear mean ergodic theorems for non-Lipschitzian asymptotically isometric semigroups on a compact convex subset of a general Banach space; see also [2], [19], [16], [17] and [14]. In this case, however, the mean value for such a semigroup is not always a common fixed point for it.

In this paper, we first introduce a notion of asymptotic isometry semigroups of continuous self-mappings of a closed convex subset C of a Banach space E , motivated by Hyers and Ulam [10] and discuss the action of such a semigroup S on the ω -limit set $\omega(x)$ of cluster points of the orbit of $x \in C$ under S by using Banach-Ulam's theorem and the structure theorem for the kernel of semigroups (Clifford's theorem). Next, we prove non-linear mean ergodic theorems for these

non-Lipschitzian semigroups in which the mean value for such a semigroup is a common fixed point for it in the case when a Banach space E is strictly convex and C is compact; see also [5], [11] and [1].

2. PRELIMINARIES

Throughout this paper, we denote by S a semigroup with identity and by E a locally convex topological vector space (or l.c.s.). We also denote by $\mathbb{R}_+$ and $\mathbb{N}_+$ the set of non-negative real numbers and the set of non-negative integers, respectively. Let $\langle E, F \rangle$ be the duality between vector spaces E and F . For each $y \in F$, we define a linear functional f_y on E by $f_y(x) = \langle x, y \rangle$. We denote by $\sigma(E, F)$ the weak topology on E generated by $\{f_y : y \in F\}$. E_σ denotes a l.c.s. E with the weak topology $\sigma(E, E')$. If X is a l.c.s., we denote by X' the topological dual of X . We also denote by $\langle \cdot, \cdot \rangle$ the canonical bilinear form between E and E' , that is, for $x \in E$ and $x' \in E'$, $\langle x, x' \rangle$ is the value of x' at x . If A is a subset of E , then the closure of A and the closure of convex hull of A is denoted by $\bar{A}$ and $\overline{\text{co}}A$, respectively.

We denote by $l^\infty(S)$ the Banach space of bounded real-valued functions defined on S . For each $s \in S$, we define operators $l(s)$ and $r(s)$ on $l^\infty(S)$ by

$$(l(s)f)(t) = f(st) \quad \text{and} \quad (r(s)f)(t) = f(ts)$$

for each $t \in S$ and $f \in l^\infty(S)$, respectively. A subspace X of $l^\infty(S)$ is said to be *translation invariant* if $l(s)X \subset X$ and $r(s)X \subset X$ for each $s \in S$. Let X be a subspace of $l^\infty(S)$ which contains constants. A linear functional μ on X is said to be a *mean* on X if $\|\mu\| = \mu(e) = 1$, where $e(s) = 1$ for each $s \in S$. We often write $\mu_s f(s)$ instead of $\mu(f)$ for each $f \in X$. For $s \in S$, we define a *point evaluation* δ_s by $\delta_s(f) = f(s)$ for each $f \in l^\infty(S)$. A convex combination of point evaluations is called a *finite mean*. As is well known, μ is a mean on X if and only if

$$\inf_{s \in S} f(s) \leq \mu(f) \leq \sup_{s \in S} f(s)$$

for each $f \in X$; see Day [7] and Takahashi [20] for more details. Let X be also translation invariant. Then, a mean μ on X is said to be *left* (or *right*) *invariant* if $\mu(l(s)f) = \mu(f)$ (or $\mu(r(s)f) = \mu(f)$) for each $s \in S$ and $f \in X$. A mean μ on X is said to be *invariant* if μ is both left and right invariant. If there exists a left (or right) invariant mean on X , then X is said to be *left* (or *right*) *amenable*. If X is also left and right amenable, then X is said to be *amenable*. We know from Day [7] that if S is commutative, then X is amenable. Let $\{\mu_\alpha\}$ be a net of means on X . Then $\{\mu_\alpha\}$ is said to be *asymptotically invariant* (or *strongly*

regular) if for each $s \in S$, both $l(s)'\mu_\alpha - \mu_\alpha$ and $r(s)'\mu_\alpha - \mu_\alpha$ converge to 0 in the weak topology $\sigma(X', X)$ (or the norm topology), where $l(s)'$ and $r(s)'$ are the adjoint operators of $l(s)$ and $r(s)$, respectively. Such nets were first studied by Day [7].

We denote by $l^\infty(S, E)$ the vector space of vector-valued functions defined on S with values in E such that for each $f \in l^\infty(S, E)$, $f(S) = \{f(s) : s \in S\}$ is bounded. Let $\mathfrak{U}$ is a neighborhood base of 0 in E and let $M(V) = \{f \in l^\infty(S, E) : f(S) \subset V\}$ for each $V \in \mathfrak{U}$. A family $\mathfrak{B} = \{M(V) : V \in \mathfrak{U}\}$ is a filter base in $l^\infty(S, E)$. Then, $l^\infty(S, E)$ is a l.c.s. with the topology $\mathfrak{T}$ of uniform convergence on S that has a neighborhood base $\mathfrak{B}$ of 0. For each $s \in S$, we define the operators $L(s)$ and $R(s)$ on $l^\infty(S, E)$ by

$$(L(s)f)(t) = f(st) \quad \text{and} \quad (R(s)f)(t) = f(ts)$$

for each $t \in S$ and $f \in l^\infty(S, E)$, respectively. Let $f \in l^\infty(S, E)$. We denote by $\mathcal{RO}(f)$ the right orbit of f , that is, the set $\{R(s)f \in l^\infty(S, E) : s \in S\}$ of right translates of f . Similarly, we also denote by $\mathcal{LO}(f)$ the left orbit of f , that is, the set $\{L(s)f \in l^\infty(S, E) : s \in S\}$ of left translates of f . A subspace Ξ of $l^\infty(S, E)$ is said to be *translation invariant* if $L(s)\Xi \subset \Xi$ and $R(s)\Xi \subset \Xi$ for each $s \in S$. Let Ξ be a subspace of $l^\infty(S, E)$ which contains constant functions. For each $s \in S$, we define a (vector-valued) point evaluation Δ_s by $\Delta_s(f) = f(s)$ for each $f \in l^\infty(S, E)$. A convex combination of vector-valued point evaluations is said to be a (vector-valued) *finite mean*. A mapping M of Ξ into E is called a *vector-valued mean* on Ξ if M is contained in the closure of convex hull of $\{\Delta_s : s \in S\}$ in the product space $(E_\sigma)^\Xi$. Then, a vector-valued mean M on Ξ is a linear continuous mapping of Ξ into E such that (i) $Mp = p$ for each constant function p in Ξ , and (ii) $M(f)$ is contained in the closure of convex hull of $f(S)$ for each $f \in \Xi$. We denote by Φ_Ξ the set of vector-valued means on Ξ . Let Ξ be also translation invariant. Then, a vector-valued mean M on Ξ is said to be *left* (or *right*) *invariant* if $M(L(s)f) = M(f)$ (or $M(R(s)f) = M(f)$) for each $s \in S$ and $f \in \Xi$, respectively. A vector-valued mean M on Ξ is said to be *invariant* if M is both left and right invariant.

We also denote by $l_c^\infty(S, E)$ the subspace of $l^\infty(S, E)$ such that for each $f \in l_c^\infty(S, E)$, $f(S)$ is relatively weakly compact in E . Let X be a subspace of $l^\infty(S)$ containing constants such that for each $f \in l_c^\infty(S, E)$ and $x' \in E'$, a function $s \mapsto \langle f(s), x' \rangle$ is contained in X . Such an X is called *admissible*. Let $\mu \in X'$. Then, for each $f \in l_c^\infty(S, E)$, we define a linear functional $\tau(\mu)f$ on E' by

$$\tau(\mu)f : x' \mapsto \mu \langle f(\cdot), x' \rangle.$$

It follows from the bipolar theorem that $\tau(\mu)f$ is contained in E . Then, a mapping τ of X' onto $\Phi_{l_c^\infty(S,E)}$ is linear and continuous where X' is equipped with the weak topology $\sigma(X', X)$. Indeed, for each mean μ on X , $\tau(\mu)$ is a vector-valued mean on $l_c^\infty(S, E)$ (generated by μ). Conversely, every vector-valued mean on $l_c^\infty(S, E)$ is also a vector-valued mean in the sense of Goldberg and Irwin [9], that is, for each $M \in \Phi_{l_c^\infty(S,E)}$, there exists a mean μ on X such that $\tau(\mu) = M$. Note that $\Phi_{l_c^\infty(S,E)}$ is compact and convex in $(E_\sigma)^{l_c^\infty(S,E)}$; see also Day [7], Takahashi [19, 20] and Kada and Takahashi [11]. Let X be also translation invariant and amenable. If μ is a left (or right) invariant mean on X , then $\tau(\mu)$ is also left (or right) invariant. Conversely, if M is a left (or right) invariant vector-valued mean on $l_c^\infty(S, E)$, then there exists a left (or right) invariant mean μ on X such that $\tau(\mu) = M$.

Let C be a closed convex subset of a l.c.s. E and let $\mathfrak{F}$ be the semigroup of self-mappings of C under operator multiplication. If T is a semigroup homomorphism of S into $\mathfrak{F}$, then T is said to be a *representation* of S as self-mappings of C . Let $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of S as self-mappings of C such that for each $x \in C$, the orbit $\mathcal{O}(x) = \{T(s)x : s \in S\}$ of x under $\mathcal{S}$ is relatively weakly compact in C and let X be a subspace of $l^\infty(S)$ containing constants such that for each $x \in C$ and $x' \in E'$, a function $s \mapsto \langle T(s)x, x' \rangle$ is contained in X . Such an X is called *admissible* with respect to $\mathcal{S}$. If no confusion will occur, then X is simply called admissible. Let $\mu \in X'$. Then, there exists a unique point x_0 of E such that $\mu\langle T(\cdot)x, x' \rangle = \langle x_0, x' \rangle$ for each $x' \in E'$. We denote such a point x_0 by $T(\mu)x$. Note that if μ is a mean on X , then for each $x \in C$, $T(\mu)x$ is contained in the closure of convex hull of the orbit $\mathcal{O}(x)$ of x under $\mathcal{S}$; see Takahashi [19, 20].

3. ON ALMOST CONVERGENCE FOR VECTOR-VALUED FUNCTIONS

In this section, we recall a notion of almost convergence for those functions and summarize its characterizations for the sake of completeness; see also Miyake [13].

Definition 1. Let S be left amenable and let $f \in l_c^\infty(S, E)$. Then, f is said to be *almost convergent* in the sense of Lorentz if

$$\tau(\mu)f = \tau(\nu)f$$

for any left invariant means μ and ν on $l^\infty(S)$. Note that f is almost convergent in the sense of Lorentz if and only if $M(f) = N(f)$ for any left invariant vector-valued means M and N on $l_c^\infty(S, E)$.

Theorem 1. Let S be left amenable and let $f \in l_c^\infty(S, E)$. Then, the following are equivalent:

- (i) f is almost convergent in the sense of Lorentz;
- (ii) the closure of convex hull of $\mathcal{RO}(f)$ contains exactly one constant function with value p in the topology of weakly pointwise convergence on S .

In this case, we call such a value p the mean value of f ; see also von Neumann [15], Bochner and von Neumann [3] and Miyake and Takahashi [14]. Let $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of S as self-mappings of a weakly compact convex subset C of a locally convex space E . We define a mapping $\phi_{\mathcal{S}}$ of C into $l_c^\infty(S, E)$ by $\phi_{\mathcal{S}}(x)(s) = T(s)x$ for each $x \in C$ and $s \in S$. Then, $\mathcal{S}$ is said to be almost convergent in the sense of Lorentz if for each $x \in C$, $\phi_{\mathcal{S}}(x)$ has the mean value p_x . Such a point p_x is also said to be the mean value of x under $\mathcal{S}$.

Theorem 2. Let S be commutative, let $f \in l_c^\infty(S, E)$ and let X be a closed, translation invariant and admissible subspace of $l^\infty(S)$ containing constants. Then, the following are equivalent:

- (i) f is almost convergent in the sense of Lorentz;
- (ii) there exists a strongly regular net $\{\lambda_\alpha\}$ of finite means such that $\{\tau(\lambda_\alpha).f\}$ converges in the topology τ_{wu} of weakly uniform convergence on S ;
- (iii) for each strongly regular net $\{\mu_\alpha\}$ of means on X , $\{\tau(\mu_\alpha).f\}$ converges in the topology τ_{wu} .

4. MEAN ERGODIC THEOREMS FOR ASYMPTOTIC ISOMETRY SEMIGROUPS

By applying a notion and a characterization (Theorem 2) of almost convergence in the sense of Lorentz for vector-valued bounded functions defined on a commutative semigroup with values in a locally convex space to transformation semigroups, we prove mean ergodic theorems for non-Lipschitzian asymptotic isometry semigroups in strictly convex Banach spaces. The following theorems are crucial for proving our results.

Theorem 3 (Banach-Ulam, [21]). A compact metric space cannot be isometric with a proper subset of itself.

Let I be a subset of S . Then, I is said to be a left (or right) ideal of S if for each $\sigma \in S$ and $\tau \in I$, $\sigma\tau \in I$ (or $\tau\sigma \in I$), respectively. If I is a left and right ideal of S , then I is said to be a two-sided ideal of S . The intersection of the two-sided ideals of S is called the kernel of S and denoted by $K(S)$. If $K(S)$ is non-empty, it is the smallest two-sided ideal of S . The structure theorem for the kernel of semigroups

is known in the case when semigroups have minimal left and minimal right ideals.

Theorem 4 (Clifford, [6]). *Let S be a compact semitopological semigroup. Then $K(S)$ is non-empty. If L is a minimal left ideal of S and R is a minimal right ideal of S , then L and R are contained in $K(S)$ and $L \cap R$ contains a unique idempotent e , that is, $ee = e$. In this case, $L \cap R$ is a compact topological group with e as identity.*

Definition 2. Let S be commutative and let $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of S as continuous self-mappings of a closed convex subset C of a Banach space. Then, $\mathcal{S}$ is said to be an *asymptotic isometry semigroup* on C if there exists a net $\{\epsilon(s)\}$ of non-negative real numbers converging to 0 such that for each $x, y \in C$ and $s \in S$,

$$|||T(s)x - T(s)y|| - ||x - y||| \leq \epsilon(s).$$

The following lemmas are immediately deduced from Banach-Ulam's theorem and Clifford's theorem, respectively.

Lemma 1. *Let S be commutative, let $\mathcal{S}$ be a representation of S as continuous self-mappings of a compact convex subset C of a Banach space and let $x \in C$. Then the closure $\overline{\mathcal{S}}$ of $\mathcal{S}$ is a compact left semitopological semigroup in the product topology of C^C . If $\mathcal{S}$ is an asymptotic isometry semigroup on C , then the kernel $K(\overline{\mathcal{S}})$ of $\overline{\mathcal{S}}$ is a non-empty, commutative, compact topological semigroup of isometries of C . Moreover, $K(\overline{\mathcal{S}})$ acting on $\omega(x)$ is contained in a compact topological group G , with identity mapping as identity, of isometries of $\omega(x)$, where $\omega(x)$ is the set of cluster points of the orbit $\mathcal{O}(x)$ of x under $\mathcal{S}$.*

Lemma 2. *Let S be commutative, let $\mathcal{S}$ be a representation of S as continuous self-mappings of a compact convex subset C of a Banach space and let $x \in C$. If $\mathcal{S}$ is an asymptotic isometry semigroup on C , then $K(\overline{\mathcal{S}})$ is a commutative, compact topological group, with identity mapping as identity, of isometries of $\omega(x)$ and $\overline{\mathcal{S}} = K(\overline{\mathcal{S}})$ acts on $\omega(x)$. Moreover, $\omega(x)$ is a minimal set with respect to $\mathcal{S}$, that is, for each $y \in \omega(x)$, the orbit $\mathcal{O}(y)$ of y under $\mathcal{S}$ is dense in $\omega(x)$.*

From the works of Bruck [5] and Atsushiba and Takahashi [1], the above lemmas imply the following result.

Lemma 3. *Let S be commutative, let $\mathcal{S}$ be a representation of S as continuous self-mappings of a compact convex subset C of a strictly convex Banach space and let $x \in C$. If $\mathcal{S}$ is an asymptotic isometry semigroup on C , then $K(\overline{\mathcal{S}})$ is an affine isometry group acting on $\overline{C}\omega(x)$.*

It follows from Lemma 3 and Markov-Kakutani's fixed point theorem that for each $x \in C$, $\overline{\text{co}} \omega(x)$ contains a unique common fixed point for $\mathcal{S}$. By using Theorem 2, we can prove mean ergodic theorems for asymptotic isometry semigroups in strictly convex Banach spaces.

Theorem 5. *Let S be commutative, let $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of S as continuous self-mappings of a compact convex subset C of a strictly convex Banach space, let X be a closed, translation invariant and admissible subspace of $l^\infty(S)$ containing constants and let $\{\mu_\alpha\}$ be a strongly regular net of means on X . If $\mathcal{S}$ is an asymptotic isometry semigroup on C , then $\mathcal{S}$ is almost convergent in the sense of Lorentz, that is, for each $x \in C$, $\{T(\mu_\alpha)T(h)x\}$ converges to the mean value p_x of x under $\mathcal{S}$ in C uniformly in $h \in S$. In this case,*

$$\{p_x\} = \bigcap_{s \in S} \overline{\text{co}}\{T(t+s)x : t \in S\} \cap F(\mathcal{S}) = \{T(\mu)x\}$$

for each invariant mean μ on X , where $F(\mathcal{S})$ is the set of common fixed points for $\mathcal{S}$.

For example, the following corollaries are the case when a semigroup S is the set of the non-negative integers or real numbers.

Corollary 1. *Let C be a compact convex subset of a strictly convex Banach space, let T be a continuous self-mapping of C and let $\{\epsilon_n\}$ be a sequence of non-negative real numbers converging to 0 such that for each $x, y \in C$ and $n \in \mathbb{N}_+$,*

$$|||T^n x - T^n y|| - ||x - y||| \leq \epsilon_n.$$

Then, for each $x \in C$, the Cesàro means

$$\frac{1}{n} \sum_{i=0}^{n-1} T^{i+h} x$$

converge to a fixed point of T in C uniformly in $h \in \mathbb{N}_+$.

Corollary 2. *Let C be a compact convex subset of a strictly convex Banach space, let $\mathcal{S} = \{T(t) : t \in \mathbb{R}_+\}$ be a one-parameter semigroup of continuous self-mappings of C and let $\{\epsilon(t)\}$ be a net of non-negative real numbers converging to 0 such that for each $x, y \in C$ and $t \in \mathbb{R}_+$,*

$$|||T(t)x - T(t)y|| - ||x - y||| \leq \epsilon(t).$$

Then, for each $x \in C$, the Bohr means

$$\frac{1}{t} \int_0^t T(t+h)x \, dt$$

converge to a common fixed point for $\mathcal{S}$ in C uniformly in $h \in \mathbb{R}_+$ as $t \rightarrow +\infty$.

REFERENCES

- [1] S. Atsushiba and W. Takahashi, *A nonlinear strong ergodic theorem for nonexpansive mappings with compact domain*, Math. Japonica, **52** (2000), 183-195.
- [2] J. B. Baillon, *Un théorème de type ergodique pour les contractions non linéaires dans un espace de Hilbert*, C. R. Acad. Sci. Paris Sér. A-B, **280** (1975), 1511-1514.
- [3] S. Bochner and J. von Neumann, *Almost periodic functions in groups. II*, Trans. Amer. Math. Soc., **37** (1935), 21-50.
- [4] R. E. Bruck, *On the almost-convergence of iterates of a nonexpansive mapping in Hilbert space and the structure of the weak ω -limit set*, Israel J. Math., **29** (1978), 1-16.
- [5] R. E. Bruck, *A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces*, Israel J. Math., **32** (1979), 107-116.
- [6] A. H. Clifford, *Semigroups containing minimal ideals*, Amer. J. Math., **70** (1948), 521-526.
- [7] M. M. Day, *Amenable semigroup*, Illinois J. Math., **1** (1957), 509-544.
- [8] W. F. Eberlein, *Abstract ergodic theorems and weak almost periodic functions*, Trans. Amer. Math. Soc., **67** (1949), 217-240.
- [9] S. Goldberg and P. Irwin, *Weakly almost periodic vector-valued functions*, Dissertationes Math. (Rozprawy Mat.), **157** (1979), 1-42.
- [10] D. H. Hyers and S. M. Ulam, *On approximate isometries*, Bull. Amer. Math. Soc., **51** (1945), 288-292.
- [11] O. Kada and W. Takahashi, *Strong convergence and nonlinear ergodic theorems for commutative semigroups of nonexpansive mappings*, Nonlinear Anal., **28** (1997), 495-511.
- [12] G. G. Lorentz, *A contribution to the theory of divergent sequences*, Acta. Math., **80** (1948), 167-190.
- [13] H. Miyake, *On almost convergence for vector-valued functions and its application*, in Nonlinear Analysis and Convex Analysis (S. Akashi ed.), RIMS Kôkyûroku **1755** (2011), 68-75.
- [14] H. Miyake and W. Takahashi, *Vector-valued weakly almost periodic functions and mean ergodic theorems in Banach spaces*, J. Nonlinear Convex Anal., **9** (2008), 255-272.
- [15] J. von Neumann, *Almost periodic functions in a group, I*, Trans. Amer. Math. Soc., **36** (1934), 445-492.
- [16] G. Rodé, *An ergodic theorem for semigroups of nonexpansive mappings in a Hilbert space*, J. Math. Anal. Appl., **85** (1982), 172-178.
- [17] W. M. Ruess and W. H. Summers, *Weak almost periodicity and the strong ergodic limit theorem for contraction semigroups*, Israel J. Math., **64** (1988), 139-157.
- [18] H. H. Schaefer, *Topological Vector Spaces*, Springer-Verlag, New York, 1971.
- [19] W. Takahashi, *A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space*, Proc. Amer. Math. Soc., **81** (1981), 253-256.
- [20] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000.
- [21] S. M. Ulam, *The Scottish book of Lwów*, mimeographed.