

MIXED PENTAGON EQUATION AND DOUBLE SHUFFLE RELATION

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ABSTRACT. This paper is a review of the paper [F4] where a geometric interpretation of the generalized (including the regularization relation) double shuffle relation for multiple L -values is given. In precise, it is shown that Enriquez' mixed pentagon equation implies the relations.

0. INTRODUCTION

Multiple L -values $L(k_1, \dots, k_m; \zeta_1, \dots, \zeta_m)$ are the complex numbers defined by the following series

$$(1) \quad L(k_1, \dots, k_m; \zeta_1, \dots, \zeta_m) := \sum_{0 < n_1 < \dots < n_m} \frac{\zeta_1^{n_1} \dots \zeta_m^{n_m}}{n_1^{k_1} \dots n_m^{k_m}}$$

for $m, k_1, \dots, k_m \in \mathbf{N}(= \mathbf{Z}_{>0})$ and $\zeta_1, \dots, \zeta_m \in \mu_N$ (the group of N -th roots of unity in $\mathbf{C}$). They converge if and only if $(k_m, \zeta_m) \neq (1, 1)$. Multiple zeta values are regarded as a special case for $N = 1$. These values have been discussed in several papers [AK, BK, G, R] etc. Multiple L -values appear as coefficients of the cyclotomic Drinfel'd associator Φ_{KZ}^N (5) in $U\mathfrak{F}_{N+1}$: the non-commutative formal power series ring with $N + 1$ variables A and $B(a)$ ($a \in \mathbf{Z}/N\mathbf{Z}$).

The mixed pentagon equation (4) is a geometric equation introduced by Enriquez [E]. The series Φ_{KZ}^N satisfies the equation, which yields non-trivial relations among multiple L -values. The generalised double shuffle relation (the double shuffle relation and the regularization relation) is a combinatorial relation among multiple L -values. It is formulated as (6) for $h = \Phi_{KZ}^N$. It is Zhao's remark [Z] that for specific N 's the generalized double shuffle relation does not provide all the possible relations among multiple L -values.

Our main theorem is an implication of the generalised double shuffle relation (6) from the mixed pentagon equation (4).

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Theorem 1. *Let $U\mathfrak{F}_{N+1}$ be the universal enveloping algebra of the free Lie algebra $\mathfrak{F}_{N+1}$ with variables A and $B(a)$ ($a \in \mathbf{Z}/N\mathbf{Z}$). Let h be a group-like element in $U\mathfrak{F}_{N+1}$ with $c_{B(0)}(h) = 0$ satisfying the mixed pentagon equation (4) with a group-like series $g \in U\mathfrak{F}_2$. Then h also satisfies the generalised double shuffle relation (6).*

The contents of the article are as follows: We recall the mixed pentagon equation in §1 and the generalised double shuffle relation in §2. In §3 we calculate the 0-th cohomologies of Chen's reduced bar complex for the Kummer coverings of the moduli spaces $\mathcal{M}_{0,4}$ and $\mathcal{M}_{0,5}$. Two variable cyclotomic multiple polylogarithms and their associated bar elements there are introduced in §4. By using them, we prove theorem 1 in §5.

1. MIXED PENTAGON EQUATION

This section is to recall Enriquez' mixed pentagon equation [E].

Let us fix notations: For $n \geq 2$, the Lie algebra $\mathfrak{t}_n$ of infinitesimal pure braids is the completed $\mathbf{Q}$ -Lie algebra with generators t^{ij} ($i \neq j$, $1 \leq i, j \leq n$) and relations $t^{ij} = t^{ji}$, $[t^{ij}, t^{ik} + t^{jk}] = 0$ and $[t^{ij}, t^{kl}] = 0$ for all distinct i, j, k, l . We note that $\mathfrak{t}_2$ is the 1-dimensional abelian Lie algebra generated by t^{12} . The element $z_n = \sum_{1 \leq i < j \leq n} t^{ij}$ is central in $\mathfrak{t}_n$. Put $\mathfrak{t}_n^0$ to be the Lie subalgebra of $\mathfrak{t}_n$ with the same generators except t^{1n} and the same relations as $\mathfrak{t}_n$. Then we have $\mathfrak{t}_n = \mathfrak{t}_n^0 \oplus \mathbf{Q} \cdot z_n$. Especially when $n = 3$, $\mathfrak{t}_3^0$ is a free Lie algebra $\mathfrak{F}_2$ of rank 2 with generators $A := t^{12}$ and $B = t^{23}$. For a partially defined map $f : \{1, \dots, m\} \rightarrow \{1, \dots, n\}$, the Lie algebra morphism $\mathfrak{t}_n \rightarrow \mathfrak{t}_m : x \mapsto x^f = x^{f^{-1}(1), \dots, f^{-1}(n)}$ is uniquely defined by $(t^{ij})^f = \sum_{i' \in f^{-1}(i), j' \in f^{-1}(j)} t^{i'j'}$.

For a pair $(\mu, g) \in \mathbf{Q} \times \exp \mathfrak{F}_2$ the *pentagon equation* is the following equation in $\exp \mathfrak{t}_4^0$

$$(2) \quad g^{1,2,34} g^{12,3,4} = g^{2,3,4} g^{1,23,4} g^{1,2,3}.$$

and *two hexagon equations* the following two equations in $\exp \mathfrak{F}_2 = \exp \mathfrak{t}_3^0$

$$(3) \quad g(A, B)g(B, A) = 1 \quad \text{and} \quad \exp\left\{\frac{\mu A}{2}\right\}g(C, A)\exp\left\{\frac{\mu C}{2}\right\}g(B, C)\exp\left\{\frac{\mu B}{2}\right\}g(A, B) = 1$$

with $C = -A - B$. These

By our notation, the equation (2) can be read as

$$g(t^{12}, t^{23} + t^{24})g(t^{13} + t^{23}, t^{34}) = g(t^{23}, t^{34})g(t^{12} + t^{13}, t^{24} + t^{34})g(t^{12}, t^{23}).$$

Remark 2. It is shown in [F2] that the two hexagon equations (3) are consequences of the pentagon equation (2).

Remark 3. The *Drinfel'd associator* $\Phi_{KZ} = \Phi_{KZ}(A, B) \in \mathbf{C}\langle\langle A, B \rangle\rangle$ is defined to be the quotient $\Phi_{KZ} = G_1(z)^{-1}G_0(z)$ where G_0 and G_1 are the solutions of the formal KZ equation

$$\frac{d}{dz}G(z) = \left(\frac{A}{z} + \frac{B}{z-1}\right)G(z)$$

such that $G_0(z) \approx z^A$ when $z \rightarrow 0$ and $G_1(z) \approx (1-z)^B$ when $z \rightarrow 1$ (cf.[Dr]). The series has the following expression

$$\Phi_{KZ} = 1 + \sum (-1)^m \zeta(k_1, \dots, k_m) A^{k_m-1} B \dots A^{k_1-1} B + (\text{regularized terms})$$

and the regularised terms are explicitly calculated to be linear combinations of multiple zeta values $\zeta(k_1, \dots, k_m) = L(k_1, \dots, k_m; 1, \dots, 1)$ in [F1] proposition 3.2.3 by Le-Murakami's method [LM]. It is shown in [Dr] that the pair $(2\pi\sqrt{-1}, \Phi_{KZ})$ satisfies the pentagon equation (2) and the hexagon equations (3).

For $n \geq 2$ and $N \geq 1$, the Lie algebra $\mathfrak{t}_{n,N}$ is the completed $\mathbf{Q}$ -Lie algebra with generators t^{1i} ($2 \leq i \leq n$), $t(a)^{ij}$ ($i \neq j$, $2 \leq i, j \leq n$, $a \in \mathbf{Z}/N\mathbf{Z}$) and relations $t(a)^{ij} = t(-a)^{ji}$, $[t(a)^{ij}, t(a+b)^{ik} + t(b)^{jk}] = 0$, $[t^{1i} + t^{1j} + \sum_{c \in \mathbf{Z}/N\mathbf{Z}} t(c)^{ij}, t(a)^{ij}] = 0$, $[t^{1i}, t^{1j} + \sum_{c \in \mathbf{Z}/N\mathbf{Z}} t(c)^{ij}] = 0$, $[t^{1i}, t(a)^{jk}] = 0$ and $[t(a)^{ij}, t(b)^{kl}] = 0$ for all $a, b \in \mathbf{Z}/N\mathbf{Z}$ and all distinct i, j, k, l ($2 \leq i, j, k, l \leq n$). We note that $\mathfrak{t}_{n,1}$ is equal to $\mathfrak{t}_n$ for $n \geq 2$. We have a natural injection $\mathfrak{t}_{n-1,N} \hookrightarrow \mathfrak{t}_{n,N}$. The Lie subalgebra $\mathfrak{f}_{n,N}$ of $\mathfrak{t}_{n,N}$ generated by t^{1n} and $t(a)^{in}$ ($2 \leq i \leq n-1$, $a \in \mathbf{Z}/N\mathbf{Z}$) is free of rank $(n-2)N+1$ and forms an ideal of $\mathfrak{t}_{n,N}$. Actually it shows that $\mathfrak{t}_{n,N}$ is a semi-direct product of $\mathfrak{f}_{n,N}$ and $\mathfrak{t}_{n-1,N}$. The element $z_{n,N} = \sum_{1 \leq i < j \leq n} t^{ij}$ with $t^{ij} = \sum_{a \in \mathbf{Z}/N\mathbf{Z}} t(a)^{ij}$ ($2 \leq i < j \leq n$) is central in $\mathfrak{t}_{n,N}$. Put $\mathfrak{t}_{n,N}^0$ to be the Lie subalgebra of $\mathfrak{t}_{n,N}$ with the same generators except t^{1n} . Then we have $\mathfrak{t}_{n,N} = \mathfrak{t}_{n,N}^0 \oplus \mathbf{Q} \cdot z_{n,N}$. Occasionally we regard $\mathfrak{t}_{n,N}^0$ as the quotient $\mathfrak{t}_{n,N}/\mathbf{Q} \cdot z_{n,N}$. Especially when $n=3$, $\mathfrak{t}_{3,N}^0$ is free Lie algebra $\mathfrak{F}_{N+1}$ of rank $N+1$ with generators $A := t^{12}$ and $B(a) = t(a)^{23}$ ($a \in \mathbf{Z}/N\mathbf{Z}$).

For a partially defined map $f : \{1, \dots, m\} \rightarrow \{1, \dots, n\}$ such that $f(1) = 1$, the Lie algebra morphism $\mathfrak{t}_{n,N} \rightarrow \mathfrak{t}_{m,N} : x \mapsto x^f = x^{f^{-1}(1), \dots, f^{-1}(n)}$ is uniquely defined by $(t(a)^{ij})^f = \sum_{i' \in f^{-1}(i), j' \in f^{-1}(j)} t(a)^{i'j'}$ ($i \neq j$, $2 \leq i, j \leq n$) and $(t^{1j})^f = \sum_{j' \in f^{-1}(j)} t^{1j'} + \frac{1}{2} \sum_{j', j'' \in f^{-1}(j)} \sum_{c \in \mathbf{Z}/N\mathbf{Z}} t(c)^{j'j''} + \sum_{i' \neq 1 \in f^{-1}(1), j' \in f^{-1}(j)} \sum_{c \in \mathbf{Z}/N\mathbf{Z}} t(c)^{i'j'}$ ($2 \leq j \leq n$). Again for a partially defined map $g : \{2, \dots, m\} \rightarrow \{1, \dots, n\}$, the Lie algebra morphism $\mathfrak{t}_n \rightarrow \mathfrak{t}_{m,N} : x \mapsto x^g = x^{g^{-1}(1), \dots, g^{-1}(n)}$ is uniquely defined by $(t^{ij})^g = \sum_{i' \in g^{-1}(i), j' \in g^{-1}(j)} t(0)^{i'j'}$ ($i \neq j$, $1 \leq i, j \leq n$).

For a pair $(g, h) \in \exp \mathfrak{F}_2 \times \exp \mathfrak{F}_{N+1}$, the *mixed pentagon equation* means the following equation in $\exp \mathfrak{t}_{4,N}^0$

$$(4) \quad h^{1,2,34} h^{12,3,4} = g^{2,3,4} h^{1,23,4} h^{1,2,3}.$$

By our notation, each term in the equation (4) can be read as

$$h^{1,2,34} = h(t^{12}, t^{23}(0) + t^{24}(0), t^{23}(1) + t^{24}(1), \dots, t^{23}(N-1) + t^{24}(N-1)),$$

$$h^{12,3,4} = h(t^{13} + \sum_c t^{23}(c), t^{34}(0), t^{34}(1), \dots, t^{34}(N-1)),$$

$$g^{2,3,4} = g(t^{23}(0), t^{34}(0)),$$

$$h^{1,23,4} = h(t^{12} + t^{13} + \sum_c t^{23}(c), t^{24}(0) + t^{34}(0), \dots, t^{24}(N-1) + t^{34}(N-1)),$$

$$h^{1,2,3} = h(t^{12}, t^{23}(0), t^{23}(1), \dots, t^{23}(N-1)).$$

Remark 4. In [E], the cyclotomic analogue $\Phi_{KZ}^N \in \exp \mathfrak{F}_{N+1}(\mathbf{C})$ of the Drinfel'd associator is introduced to be the renormalised holonomy from 0 to 1 of the KZ-like differential equation

$$\frac{d}{dz} H(z) = \left(\frac{A}{z} + \sum_{a \in \mathbf{Z}/N\mathbf{Z}} \frac{B(a)}{z - \zeta_N^a} \right) H(z)$$

with $\zeta_N = \exp\{\frac{2\pi\sqrt{-1}}{N}\}$, i.e., $\Phi_{KZ}^N = H_1^{-1} H_0$ where H_0 and H_1 are the solutions such that $H_0(z) \approx z^A$ when $z \rightarrow 0$ and $H_1(z) \approx (1-z)^{B(0)}$ when $z \rightarrow 1$ (cf.[E]). There appear multiple L -values (1) in each of its coefficient;

$$(5) \quad \Phi_{KZ}^N = 1 + \sum (-1)^m L(k_1, \dots, k_m; \xi_1, \dots, \xi_m) A^{k_m-1} B(a_m) \cdots A^{k_1-1} B(a_1) \\ + (\text{regularized terms})$$

with $\xi_1 = \zeta_N^{a_2-a_1}$, $\dots$, $\xi_{m-1} = \zeta_N^{a_m-a_{m-1}}$ and $\xi_m = \zeta_N^{-a_m}$, where the regularised terms can be explicitly calculated to combinations of multiple L -values by the method of Le-Murakami [LM]. In [E] it is shown that the triple $(2\pi\sqrt{-1}, \Phi_{KZ}, \Phi_{KZ}^N)$ satisfies the mixed pentagon equation (4). This is achieved by considering monodromy in the pentagon formed by the divisors $y = 0$, $x = 1$, the exceptional divisor of the blowing-up at $(1, 1)$, $y = 1$ and $x = 0$ in $\mathcal{M}_{0,5}^{(N)}$ (see §3).

Remark 5. In [EF] it is proved that the mixed pentagon equation (4) implies the distribution relation for a specific case and that the octagon equation follows from the mixed pentagon equation and the special action condition for $N = 2$.

2. DOUBLE SHUFFLE RELATION

This section is to recall the generalised double shuffle relation in Racinet's setting [R].

Let us fix notations: Let $\mathfrak{F}_{Y_N}$ be the completed graded Lie $\mathbf{Q}$ -algebra generated by $Y_{n,a}$ ($n \geq 1$ and $a \in \mathbf{Z}/N\mathbf{Z}$) with $\deg Y_{n,a} = n$. Put $U\mathfrak{F}_{Y_N}$ its universal enveloping algebra: the non-commutative formal series ring with free variables $Y_{n,a}$ ($n \geq 1$ and $a \in \mathbf{Z}/N\mathbf{Z}$). Let $\pi_Y : U\mathfrak{F}_{N+1} \rightarrow U\mathfrak{F}_{Y_N}$ be the $\mathbf{Q}$ -linear map between non-commutative formal power series rings that sends all the words ending in A to zero and the word $A^{n_m-1}B(a_m) \cdots A^{n_1-1}B(a_1)$ ($n_1, \dots, n_m \geq 1$ and $a_1, \dots, a_m \in \mathbf{Z}/N\mathbf{Z}$) to

$$(-1)^m Y_{n_m, -a_m} Y_{n_{m-1}, a_m - a_{m-1}} \cdots Y_{n_1, a_2 - a_1}.$$

Define the coproduct Δ_* of $U\mathfrak{F}_{Y_N}$ by $\Delta_* Y_{n,a} = \sum_{k+l=n, b+c=a} Y_{k,b} \otimes Y_{l,c}$ ($n \geq 0$ and $a \in \mathbf{Z}/N\mathbf{Z}$) with $Y_{0,a} := 1$ if $a = 0$ and 0 if $a \neq 0$. For $h = \sum_{W: \text{word}} c_W(h) W \in U\mathfrak{F}_{N+1}$, define the series shuffle regularization $h_* = h_{\text{corr}} \cdot \pi_Y(h)$ with the correction term

$$h_{\text{corr}} = \exp \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n} c_{A^{n-1}B(0)}(h) Y_{1,0}^n \right).$$

For a series $h \in \exp \mathfrak{F}_{N+1}$ the *generalised double shuffle relation* stands for the following relation in $U\mathfrak{F}_{Y_N}$

$$(6) \quad \Delta_*(h_*) = h_* \hat{\otimes} h_*.$$

Remark 6. The series Φ_{KZ}^N (5) satisfies the generalised double shuffle relation (6) because regularised multiple L -values satisfy the double shuffle relation.

3. BAR CONSTRUCTIONS

This section gives a review of the notion of the reduced bar construction and calculates it for $\mathcal{M}_{0,4}^{(N)}$ and $\mathcal{M}_{0,5}^{(N)}$.

We recall the notion of Chen's reduced bar construction [C]. Let $(A^\bullet = \bigoplus_{q=0}^{\infty} A^q, d)$ be a differential graded algebra (DGA). The reduced bar complex $\bar{B}^\bullet(A)$ is the tensor algebra $\bigoplus_{r=0}^{\infty} (\bar{A}^\bullet)^{\otimes r}$ with $\bar{A}^\bullet = \bigoplus_{i=0}^{\infty} \bar{A}^i$ where $\bar{A}^0 = A^1/dA^0$ and $\bar{A}^i = A^{i+1}$ ($i > 0$). We denote $a_1 \otimes \cdots \otimes a_r$ ($a_i \in \bar{A}^\bullet$) by $[a_1 | \cdots | a_r]$. The degree of elements in $\bar{B}^\bullet(A)$ is given by the total degree of $\bar{A}^\bullet$. Put $Ja = (-1)^{p-1}a$ for $a \in \bar{A}^p$. Define

$$d'[a_1 | \cdots | a_k] = \sum_{i=1}^k (-1)^i [Ja_1 | \cdots | Ja_{i-1} | da_i | a_{i+1} | \cdots | a_k]$$

and

$$d''[a_1 | \cdots | a_k] = \sum_{i=1}^k (-1)^{i-1} [Ja_1 | \cdots | Ja_{i-1} | Ja_i \cdot a_{i+1} | a_{i+2} | \cdots | a_k].$$

Then $d' + d''$ forms a differential. The differential and the shuffle product (loc.cit.) give $\bar{B}^\bullet(A)$ a structure of commutative DGA. Actually it also forms a Hopf algebra, whose coproduct Δ is given by

$$\Delta([a_1 | \cdots | a_r]) = \sum_{s=0}^r [a_1 | \cdots | a_s] \otimes [a_{s+1} | \cdots | a_r].$$

For a smooth complex manifold $\mathcal{M}$, $\Omega^\bullet(\mathcal{M})$ means the de Rham complex of smooth differential forms on $\mathcal{M}$ with values in $\mathbf{C}$. We denote the 0-th cohomology of the reduced bar complex $\bar{B}^\bullet(\Omega(\mathcal{M}))$ with respect to the differential by $H^0 \bar{B}(\mathcal{M})$.

Let $\mathcal{M}_{0,4}$ be the moduli space $\{(x_1, \dots, x_4) \in (\mathbf{P}_\mathbf{C}^1)^4 | x_i \neq x_j (i \neq j)\} / PGL_2(\mathbf{C})$ of 4 different points in $\mathbf{P}^1$. It is identified with $\{z \in \mathbf{P}_\mathbf{C}^1 | z \neq 0, 1, \infty\}$ by sending $[(0, z, 1, \infty)]$ to z . Denote its Kummer N -covering

$$\mathbf{G}_m \backslash \mu_N = \{z \in \mathbf{P}_\mathbf{C}^1 | z^N \neq 0, 1, \infty\}$$

by $\mathcal{M}_{0,4}^{(N)}$. The space $H^0 \bar{B}(\mathcal{M}_{0,4}^{(N)})$ is generated by

$$\omega_0 := d \log(z) \text{ and } \omega_\zeta := d \log(z - \zeta) \quad (\zeta \in \mu_N).$$

We have an identification $H^0 \bar{B}(\mathcal{M}_{0,4}^{(N)})$ with the graded $\mathbf{C}$ -linear dual of $U\mathfrak{F}_{N+1}$,

$$H^0 \bar{B}(\mathcal{M}_{0,4}^{(N)}) \simeq U\mathfrak{F}_{N+1}^* \otimes \mathbf{C},$$

by $\text{Exp } \Omega_4^{(N)} := \sum X_{i_m} \cdots X_{i_1} \otimes [\omega_{i_m} | \cdots | \omega_{i_1}] \in U\mathfrak{F}_{N+1} \hat{\otimes}_\mathbf{Q} H^0 \bar{B}(\mathcal{M}_{0,4}^{(N)})$.

Here the sum is taken over $m \geq 0$ and $i_1, \dots, i_m \in \{0\} \cup \mu_N$ and $X_0 = A$ and $X_\zeta = B(a)$ when $\zeta = \zeta_N^a$. It is easy to see that the identification is compatible with Hopf algebra structures. We note that the product $l_1 \cdot l_2 \in H^0 \bar{B}(\mathcal{M}_{0,4}^{(N)})$ for $l_1, l_2 \in H^0 \bar{B}(\mathcal{M}_{0,4}^{(N)})$ is given by $l_1 \cdot l_2(f) := \sum_i l_1(f_1^{(i)}) l_2(f_2^{(i)})$ for $f \in U\mathfrak{F}_{N+1} \otimes \mathbf{C}$ with $\Delta(f) = \sum_i f_1^{(i)} \otimes f_2^{(i)}$. Occasionally we regard $H^0 \bar{B}(\mathcal{M}_{0,4}^{(N)})$ as the regular function ring of $F_{N+1}(\mathbf{C}) = \{g \in U\mathfrak{F}_{N+1} \otimes \mathbf{C} | g : \text{group-like}\} = \{g \in U\mathfrak{F}_{N+1} \otimes \mathbf{C} | g(0) = 1, \Delta(g) = g \otimes g\}$.

Let $\mathcal{M}_{0,5}$ be the moduli space $\{(x_1, \dots, x_5) \in (\mathbf{P}_\mathbf{C}^1)^5 | x_i \neq x_j (i \neq j)\} / PGL_2(\mathbf{C})$ of 5 different points in $\mathbf{P}^1$. It is identified with $\{(x, y) \in \mathbf{G}_m^2 | x \neq 1, y \neq 1, xy \neq 1\}$ by sending $[(0, xy, y, 1, \infty)]$ to (x, y) . Denote its Kummer N^2 -covering

$$\{(x, y) \in \mathbf{G}_m^2 | x^N \neq 1, y^N \neq 1, (xy)^N \neq 1\}$$

by $\mathcal{M}_{0,5}^{(N)}$. It is identified with $W_N/\mathbf{C}^\times$ by $(x, y) \mapsto (xy, y, 1)$ where

$$W_N = \{(z_2, z_3, z_4) \in \mathbf{G}_m \mid z_i^N \neq z_j^N (i \neq j)\}.$$

The space $H^0 \bar{B}(\mathcal{M}_{0,5}^{(N)})$ is a subspace of the tensor coalgebra generated by

$$\omega_{1,i} := d \log z_i \text{ and } \omega_{i,j}(a) := d \log(z_i - \zeta_N^a z_j) \quad (2 \leq i, j \leq 4, a \in \mathbf{Z}/N).$$

Proposition 7. *We have an identification*

$$H^0 \bar{B}(\mathcal{M}_{0,5}^{(N)}) \simeq (U\mathbf{t}_{4,N}^0)^* \otimes \mathbf{C}.$$

Proof . By [K], $H^0 \bar{B}(W_N)$ can be calculated to be the 0-th cohomology $H^0 \bar{B}^\bullet(S)$ of the reduced bar complex of the Orlik-Solomon algebra $S^\bullet$. The algebra $S^\bullet$ is the (trivial-)differential graded $\mathbf{C}$ -algebra $S^\bullet = \bigoplus_{q=0}^\infty S^q$ defined by generators

$$\omega_{1,i} = d \log z_i \text{ and } \omega_{i,j}(a) = d \log(z_i - \zeta_N^a z_j) \quad (2 \leq i, j \leq 4, a \in \mathbf{Z}/N\mathbf{Z})$$

in degree 1 and relations

$$\omega_{i,j}(a) = \omega_{j,i}(-a), \quad \omega_{ij}(a) \wedge \{\omega_{ik}(a+b) + \omega_{jk}(b)\} = 0,$$

$$\{\omega_{1i} + \omega_{1j} + \sum_{c \in \mathbf{Z}/N\mathbf{Z}} \omega(c)_{ij}\} \wedge \omega(a)_{ij} = 0,$$

$$\omega_{1i} \wedge \{\omega_{1j} + \sum_{c \in \mathbf{Z}/N\mathbf{Z}} \omega(c)_{ij}\} = 0,$$

$$\omega_{1i} \wedge \omega(a)_{jk} = 0 \quad \text{and} \quad \omega(a)_{ij} \wedge \omega(b)_{kl} = 0$$

for all $a, b \in \mathbf{Z}/N\mathbf{Z}$ and all distinct i, j, k, l ($2 \leq i, j, k, l \leq n$). By direct calculation, the element

$$\sum_{i=2}^4 t_{1i} \otimes \omega_{1i} + \sum_{2 \leq i < j \leq 4, a \in \mathbf{Z}/N\mathbf{Z}} t_{ij}(a) \otimes \omega_{ij}(a) \in (\mathbf{t}_{4,N})^{\deg=1} \otimes S^1$$

yields a Hopf algebra identification of $H^0 \bar{B}(W_N)$ with $(U\mathbf{t}_{4,N})^* \otimes \mathbf{C}$ since both are quadratic.

By the long exact sequence of cohomologies induced from the $\mathbf{G}_m$ -bundle $W_N \rightarrow \mathcal{M}_{0,5}^{(N)} = W_N/\mathbf{C}^\times$, we get

$$0 \rightarrow H^1(\mathcal{M}_{0,5}^{(N)}) \rightarrow H^1(W_N) \rightarrow H^1(\mathbf{G}_m) \rightarrow 0$$

and

$$H^i(\mathcal{M}_{0,5}^{(N)}) \simeq H^i(W_N) \quad (i \geq 2).$$

It yields the identification of the subspace $H^0 \bar{B}(\mathcal{M}_{0,5}^{(N)})$ of $H^0 \bar{B}(W_N)$ with $(U\mathbf{t}_{4,N}^0)^* \otimes \mathbf{C}$. $\square$

The above identification is induced from

$$\text{Exp } \Omega_5^{(N)} := \sum t_{J_m} \cdots t_{J_1} \otimes [\omega_{J_m} | \cdots | \omega_{J_1}] \in U\mathfrak{t}_{4,N}^0 \hat{\otimes}_{\mathbf{Q}} H^0 \bar{B}(\mathcal{M}_{0,5}^{(N)})$$

where the sum is taken over $m \geq 0$ and $J_1, \dots, J_m \in \{(1, i) | 2 \leq i \leq 4\} \cup \{(i, j, a) | 2 \leq i < j \leq 4, a \in \mathbf{Z}/N\mathbf{Z}\}$.

Especially the identification between degree 1 terms is given by

$$\begin{aligned} \Omega_5^{(N)} &= \sum_{i=2}^4 t_{1i} d \log z_i + \sum_{2 \leq i < j \leq 4} \sum_{a \in \mathbf{Z}/N\mathbf{Z}} t_{i,j}(a) d \log(z_i - \zeta_N^a z_j) \\ &\in \mathfrak{t}_{4,N}^0 \otimes H_{DR}^1(\mathcal{M}_{0,5}^{(N)}). \end{aligned}$$

In terms of the coordinate (x, y) ,

$$\begin{aligned} \Omega_5^{(N)} &= t_{12} d \log(xy) + t_{13} d \log y + \sum_a t_{23}(a) d \log y(x - \zeta_N^a) \\ &\quad + \sum_a t_{24}(a) d \log(xy - \zeta_N^a) + \sum_a t_{34}(a) d \log(y - \zeta_N^a) \\ &= t_{12} d \log x + \sum_a t_{23}(a) d \log(x - \zeta_N^a) + (t_{12} + t_{13} + t_{23}) d \log y \\ &\quad + \sum_a t_{34}(a) d \log(y - \zeta_N^a) + \sum_a t_{24}(a) d \log(xy - \zeta_N^a). \end{aligned}$$

It is easy to see that the identification is compatible with Hopf algebra structures. We note again that the product $l_1 \cdot l_2 \in H^0 \bar{B}(\mathcal{M}_{0,5}^{(N)})$ for $l_1, l_2 \in H^0 \bar{B}(\mathcal{M}_{0,5}^{(N)})$ is given by $l_1 \cdot l_2(f) := \sum_i l_1(f_1^{(i)}) l_2(f_2^{(i)})$ for $f \in U\mathfrak{t}_{4,N}^0 \otimes \mathbf{C}$ with $\Delta(f) = \sum_i f_1^{(i)} \otimes f_2^{(i)}$ (Δ : the coproduct of $U\mathfrak{t}_{4,N}^0$). Occasionally we also regard $H^0 \bar{B}(\mathcal{M}_{0,5}^{(N)})$ as the regular function ring of $K_4^N(\mathbf{C}) = \{g \in U\mathfrak{t}_{4,N}^0 \otimes \mathbf{C} | g : \text{group-like}\}$.

By a generalization of Chen's theory [C] to the case of tangential basepoints, especially for $\mathcal{M} = \mathcal{M}_{0,4}^{(N)}$ or $\mathcal{M}_{0,5}^{(N)}$, we have an isomorphism

$$\rho : H^0 \bar{B}(\mathcal{M}) \simeq I_o(\mathcal{M})$$

as algebras over $\mathbf{C}$ which sends $\sum_{I=(i_m, \dots, i_1)} c_I [\omega_{i_m} | \cdots | \omega_{i_1}]$ ($c_I \in \mathbf{C}$) to $\sum_I c_I \text{It} \int_o \omega_{i_m} \circ \cdots \circ \omega_{i_1}$. Here $\sum_I c_I \text{It} \int_o \omega_{i_m} \circ \cdots \circ \omega_{i_1}$ means the iterated integral defined by

$$(7) \sum_I c_I \int_{0 < t_1 < \cdots < t_{m-1} < t_m < 1} \omega_{i_m}(\gamma(t_m)) \cdot \omega_{i_{m-1}}(\gamma(t_{m-1})) \cdots \omega_{i_1}(\gamma(t_1))$$

for all analytic paths $\gamma : (0, 1) \rightarrow \mathcal{M}(\mathbf{C})$ starting from the tangential basepoint o (defined by $\frac{d}{dz}$ for $\mathcal{M} = \mathcal{M}_{0,4}^{(N)}$ and defined by $\frac{d}{dx}$ and $\frac{d}{dy}$ for $\mathcal{M} = \mathcal{M}_{0,5}^{(N)}$) at the origin in $\mathcal{M}$ (for its treatment see also [De]§15)

and $I_o(\mathcal{M})$ stands for the $\mathbf{C}$ -algebra generated by all such homotopy invariant iterated integrals with $m \geq 1$ and $\omega_{i_1}, \dots, \omega_{i_m} \in H_{DR}^1(\mathcal{M})$.

4. TWO VARIABLE CYCLOTOMIC MULTIPLE POLYLOGARITHMS

We introduce cyclotomic multiple polylogarithms, $Li_{\mathbf{a}}(\bar{\zeta}(z))$ and $Li_{\mathbf{a}, \mathbf{b}}(\bar{\zeta}(x), \bar{\eta}(y))$, and their associated bar elements, $l_{\mathbf{a}}^{\bar{\zeta}}$ and $l_{\mathbf{a}, \mathbf{b}}^{\bar{\zeta}(x), \bar{\eta}(y)}$, which play important roles to prove our main theorems.

For a pair $(\mathbf{a}, \bar{\zeta})$ with $\mathbf{a} = (a_1, \dots, a_k) \in \mathbf{Z}_{>0}^k$ and $\bar{\zeta} = (\zeta_1, \dots, \zeta_k)$ with $\zeta_i \in \mu_N$: the group of roots of unity in $\mathbf{C}$ ($1 \leq i \leq k$), its weight and its depth are defined to be $wt(\mathbf{a}, \bar{\zeta}) = a_1 + \dots + a_k$ and $dp(\mathbf{a}, \bar{\zeta}) = k$ respectively. Put $\bar{\zeta}(x) = (\zeta_1, \dots, \zeta_{k-1}, \zeta_k x)$. Put $z \in \mathbf{C}$ with $|z| < 1$. Consider the following complex analytic function, *one variable cyclotomic multiple polylogarithm*

$$Li_{\mathbf{a}}(\bar{\zeta}(z)) := \sum_{0 < m_1 < \dots < m_k} \frac{\zeta_1^{m_1} \dots \zeta_{k-1}^{m_{k-1}} (\zeta_k z)^{m_k}}{m_1^{a_1} \dots m_{k-1}^{a_{k-1}} m_k^{a_k}}.$$

It satisfies the following differential equation

$$\frac{d}{dz} Li_{\mathbf{a}}(\bar{\zeta}(z)) = \begin{cases} \frac{1}{z} Li_{(a_1, \dots, a_{k-1}, a_k-1)}(\bar{\zeta}(z)) & \text{if } a_k \neq 1, \\ \frac{1}{\zeta_k^{-1} - z} Li_{(a_1, \dots, a_{k-1})}(\zeta_1, \dots, \zeta_{k-2}, \zeta_{k-1} z) & \text{if } a_k = 1, k \neq 1, \\ \frac{1}{\zeta_1^{-1} - z} & \text{if } a_k = 1, k = 1. \end{cases}$$

It gives an iterated integral starting from o , which lies on $I_o(\mathcal{M}_{0,4}^{(N)})$. Actually by the map ρ it corresponds to an element of the $\mathbf{Q}$ -structure $U\mathfrak{F}_{N+1}^*$ of $V(\mathcal{M}_{0,4}^{(N)})$ denoted by $l_{\mathbf{a}}^{\bar{\zeta}}$. It is expressed as

$$l_{\mathbf{a}}^{\bar{\zeta}} = (-1)^k [\underbrace{|\omega_0| \dots |\omega_0|}_{a_k-1} |\omega_{\zeta_k^{-1}}| \underbrace{|\omega_0| \dots |\omega_0|}_{a_{k-1}-1} |\omega_{\zeta_k^{-1} \zeta_{k-1}^{-1}}| |\omega_0| \dots |\omega_0| \omega_{\zeta_k^{-1} \dots \zeta_1^{-1}}^{-1}].$$

By the standard identification $\mu \simeq \mathbf{Z}/N\mathbf{Z}$ sending $\zeta_N = \exp\{\frac{2\pi\sqrt{-1}}{N}\} \mapsto 1$, for a series $\varphi = \sum_{W: \text{word}} c_W(\varphi) W$ it is calculated by

$$l_{\mathbf{a}}^{\bar{\zeta}}(\varphi) = (-1)^k c_{A^{a_k-1} B(-e_k) A^{a_{k-1}-1} B(-e_k - e_{k-1}) \dots A^{a_1-1} B(-e_k - \dots - e_1)}(\varphi)$$

with $\zeta_i = \zeta_N^{e_i}$ ($e_i \in \mathbf{Z}/N\mathbf{Z}$).

For $\mathbf{a} = (a_1, \dots, a_k) \in \mathbf{Z}_{>0}^k$, $\mathbf{b} = (b_1, \dots, b_l) \in \mathbf{Z}_{>0}^l$, $\bar{\zeta} = (\zeta_1, \dots, \zeta_k)$, $\bar{\eta} = (\eta_1, \dots, \eta_l)$ with $\zeta_i, \eta_j \in \mu_N$ and $x, y \in \mathbf{C}$ with $|x| < 1$ and $|y| < 1$, consider the following complex function, the *two variables multiple polylogarithm*

$$Li_{\mathbf{a}, \mathbf{b}}(\bar{\zeta}(x), \bar{\eta}(y)) := \sum_{\substack{0 < m_1 < \dots < m_k \\ < n_1 < \dots < n_l}} \frac{\zeta_1^{m_1} \dots \zeta_{k-1}^{m_{k-1}} (\zeta_k x)^{m_k} \cdot \eta_1^{n_1} \dots \eta_{l-1}^{n_{l-1}} (\eta_l y)^{n_l}}{m_1^{a_1} \dots m_{k-1}^{a_{k-1}} m_k^{a_k} \cdot n_1^{b_1} \dots n_{l-1}^{b_{l-1}} n_l^{b_l}}.$$

It satisfies the following differential equations.

$$\begin{aligned} \frac{d}{dx} Li_{\mathbf{a}, \mathbf{b}}(\bar{\zeta}(x), \bar{\eta}(y)) \\ = \begin{cases} \frac{1}{x} Li_{(a_1, \dots, a_{k-1}, a_k-1), \mathbf{b}}(\bar{\zeta}(x), \bar{\eta}(y)) & \text{if } a_k \neq 1, \\ \frac{1}{\zeta_k^{-1}-x} Li_{(a_1, \dots, a_{k-1}), \mathbf{b}}(\zeta_1, \dots, \zeta_{k-2}, \zeta_{k-1}x, \bar{\eta}(y)) - \left(\frac{1}{x} + \frac{1}{\zeta_k^{-1}-x}\right) \cdot \\ \quad Li_{(a_1, \dots, a_{k-1}, b_1), (b_2, \dots, b_l)}(\zeta_1, \dots, \zeta_{k-1}, \zeta_k \eta_1 x, \eta_2, \dots, \eta_{l-1}, \eta_l y) & \text{if } a_k = 1, k \neq 1, l \neq 1, \\ \frac{1}{\zeta_1^{-1}-x} Li_{\mathbf{b}}(\eta(y)) - \left(\frac{1}{x} + \frac{1}{\zeta_1^{-1}-x}\right) Li_{(b_1), (b_2, \dots, b_l)}(\zeta_1 \eta_1 x, \eta_2, \dots, \eta_{l-1}, \eta_l y) & \text{if } a_k = 1, k = 1, l \neq 1, \\ \frac{1}{\zeta_k^{-1}-x} Li_{(a_1, \dots, a_{k-1}), b_1}(\zeta_1, \dots, \zeta_{k-1}x, \eta_1 y) - \left(\frac{1}{x} + \frac{1}{\zeta_k^{-1}-x}\right) \cdot \\ \quad Li_{(a_1, \dots, a_{k-1}, b_1)}(\zeta_1, \dots, \zeta_{k-1}, \zeta_k \eta_1 xy) & \text{if } a_k = 1, k \neq 1, l = 1, \\ \frac{1}{\zeta_1^{-1}-x} Li_{b_1}(\eta_1 y) - \left(\frac{1}{x} + \frac{1}{\zeta_1^{-1}-x}\right) Li_{b_1}(\zeta_1 \eta_1 xy) & \text{if } a_k = 1, k = 1, l = 1, \end{cases} \\ \frac{d}{dy} Li_{\mathbf{a}, \mathbf{b}}(\bar{\zeta}(x), \bar{\eta}(y)) \\ = \begin{cases} \frac{1}{y} Li_{\mathbf{a}, (b_1, \dots, b_{l-1}, b_l-1)}(\bar{\zeta}(x), \bar{\eta}(y)) & \text{if } b_l \neq 1, \\ \frac{1}{\eta_l^{-1}-y} Li_{\mathbf{a}, (b_1, \dots, b_{l-1})}(\bar{\zeta}(x), \eta_1, \dots, \eta_{l-2}, \eta_{l-1}y) & \text{if } b_l = 1, l \neq 1, \\ \frac{1}{\eta_1^{-1}-y} Li_{\mathbf{a}}(\bar{\zeta}(\eta_1 xy)) & \text{if } b_l = 1, l = 1. \end{cases} \end{aligned}$$

By analytic continuation, the functions $Li_{\mathbf{a}, \mathbf{b}}(\bar{\zeta}(x), \bar{\eta}(y))$, $Li_{\mathbf{b}, \mathbf{a}}(\bar{\eta}(y), \bar{\zeta}(x))$, $Li_{\mathbf{a}}(\bar{\zeta}(x))$, $Li_{\mathbf{a}}(\bar{\zeta}(y))$ and $Li_{\mathbf{a}}(\bar{\zeta}(xy))$ give iterated integrals starting from o , which lie on $I_o(\mathcal{M}_{0,5}^{(N)})$. They correspond to elements of the $\mathbf{Q}$ -structure $(Ut_{4,N}^0)^*$ of $V(\mathcal{M}_{0,5}^{(N)})$ by the map ρ denoted by $l_{\mathbf{a}, \mathbf{b}}^{\bar{\zeta}(x), \bar{\eta}(y)}$, $l_{\mathbf{b}, \mathbf{a}}^{\bar{\eta}(y), \bar{\zeta}(x)}$, $l_{\mathbf{a}}^{\bar{\zeta}(x)}$, $l_{\mathbf{a}}^{\bar{\eta}(y)}$ and $l_{\mathbf{a}}^{\bar{\zeta}(xy)}$ respectively. Note that they are expressed as

$$\sum_{I=(i_m, \dots, i_1)} c_I [\omega_{i_m} | \dots | \omega_{i_1}]$$

for some $m \in \mathbf{N}$ with $c_I \in \mathbf{Q}$ and $\omega_{i_j} \in \{\frac{dx}{x}, \frac{dx}{\zeta-x}, \frac{dy}{y}, \frac{dy}{\zeta-y}, \frac{xdy+ydx}{\zeta-xy} (\zeta \in \mu_N)\}$.

5. PROOF OF MAIN THEOREMS

This section gives a proof of theorem 1.

Proof of theorem 1. Let $\mathbf{a} = (a_1, \dots, a_k) \in \mathbf{Z}_{>0}^k$, $\mathbf{b} = (b_1, \dots, b_l) \in \mathbf{Z}_{>0}^l$, $\bar{\zeta} = (\zeta_1, \dots, \zeta_k)$ and $\bar{\eta} = (\eta_1, \dots, \eta_l)$ with $\zeta_i, \eta_j \in \mu_N \subset \mathbf{C}$ ($1 \leq i \leq k$ and $1 \leq j \leq l$). Put $\bar{\zeta}(x) = (\zeta_1, \dots, \zeta_{k-1}, \zeta_k x)$ and

$\bar{\eta}(y) = (\eta_1, \dots, \eta_{l-1}, \eta_l y)$. Recall that multiple polylogarithms satisfy the following analytic identity, the series shuffle formula in $I_o(\mathcal{M}_{0,5}^{(N)})$:

$$Li_{\mathbf{a}}(\bar{\zeta}(x)) \cdot Li_{\mathbf{b}}(\bar{\eta}(y)) = \sum_{\sigma \in Sh^{\leq}(k,l)} Li_{\sigma(\mathbf{a},\mathbf{b})}^{\sigma(\bar{\zeta}(x),\bar{\eta}(y))}.$$

Here $Sh^{\leq}(k,l) := \cup_{N=1}^{\infty} \{\sigma : \{1, \dots, k+l\} \rightarrow \{1, \dots, N\} \mid \sigma \text{ is onto, } \sigma(1) < \dots < \sigma(k), \sigma(k+1) < \dots < \sigma(k+l)\}$, $\sigma(\mathbf{a}, \mathbf{b}) := (c_1, \dots, c_N)$ with

$$c_i = \begin{cases} a_s + b_{t-k} & \text{if } \sigma^{-1}(i) = \{s, t\} \text{ with } s < t, \\ a_s & \text{if } \sigma^{-1}(i) = \{s\} \text{ with } s \leq k, \\ b_{s-k} & \text{if } \sigma^{-1}(i) = \{s\} \text{ with } s > k, \end{cases}$$

and $\sigma(\bar{\zeta}(x), \bar{\eta}(y)) := (z_1, \dots, z_N)$ with

$$z_i = \begin{cases} x_s y_{t-k} & \text{if } \sigma^{-1}(i) = \{s, t\} \text{ with } s < t, \\ x_s & \text{if } \sigma^{-1}(i) = \{s\} \text{ with } s \leq k, \\ y_{s-k} & \text{if } \sigma^{-1}(i) = \{s\} \text{ with } s > k, \end{cases}$$

for $x_i = \zeta_i$ ($i \neq k$), $\zeta_k x$ ($i = k$) and $y_j = \eta_j$ ($j \neq l$), $\eta_l y$ ($j = l$). Since ρ is an embedding of algebras, the above analytic identity immediately implies the algebraic identity, the series shuffle formula in the $\mathbf{Q}$ -structure $(U\mathfrak{t}_{4,N}^0)^*$ of $V(\mathcal{M}_{0,5}^{(N)})$

$$(8) \quad l_{\mathbf{a}}^{\bar{\zeta}(x)} \cdot l_{\mathbf{b}}^{\bar{\eta}(y)} = \sum_{\sigma \in Sh^{\leq}(k,l)} l_{\sigma(\mathbf{a},\mathbf{b})}^{\sigma(\bar{\zeta}(x),\bar{\eta}(y))}.$$

Let (g, h) be a pair in theorem 1. By the group-likeness of h , i.e. $h \in \exp \mathfrak{F}_{N+1}$, the product $h^{1,23,4} h^{1,2,3}$ is group-like, i.e. belongs to $\exp \mathfrak{t}_{4,N}^0$. Hence $\Delta(h^{1,23,4} h^{1,2,3}) = (h^{1,23,4} h^{1,2,3}) \hat{\otimes} (h^{1,23,4} h^{1,2,3})$, where Δ is the standard coproduct of $U\mathfrak{t}_{4,N}^0$. Therefore

$$\begin{aligned} l_{\mathbf{a}}^{\bar{\zeta}(x)} \cdot l_{\mathbf{b}}^{\bar{\eta}(y)}(h^{1,23,4} h^{1,2,3}) &= (l_{\mathbf{a}}^{\bar{\zeta}(x)} \hat{\otimes} l_{\mathbf{b}}^{\bar{\eta}(y)})(\Delta(h^{1,23,4} h^{1,2,3})) \\ &= l_{\mathbf{a}}^{\bar{\zeta}(x)}(h^{1,23,4} h^{1,2,3}) \cdot l_{\mathbf{b}}^{\bar{\eta}(y)}(h^{1,23,4} h^{1,2,3}). \end{aligned}$$

Evaluation of the equation (8) at the group-like element $h^{1,23,4} h^{1,2,3}$ gives the series shuffle formula

$$(9) \quad l_{\mathbf{a}}^{\bar{\zeta}}(h) \cdot l_{\mathbf{b}}^{\bar{\eta}}(h) = \sum_{\sigma \in Sh^{\leq}(k,l)} l_{\sigma(\mathbf{a},\mathbf{b})}^{\sigma(\bar{\zeta},\bar{\eta})}(h)$$

for admissible pairs ¹ $(\mathbf{a}, \bar{\zeta})$ and $(\mathbf{b}, \bar{\eta})$ by the results in [F4] because the group-likeness and (4) for h implies $c_0(h) = 1$ and $c_A(h) = 0$.

¹A pair $(\mathbf{a}, \bar{\zeta})$ with $\mathbf{a} = (a_1, \dots, a_k)$ and $\bar{\zeta} = (\zeta_1, \dots, \zeta_k)$ is called *admissible* if $(a_k, \zeta_k) \neq (1, 1)$.

By putting $l_1^{1,S}(h) := -T$ and $l_{\mathbf{a}}^{\bar{\zeta},S}(h) := l_{\mathbf{a}}^{\bar{\zeta}}(h)$ for all admissible pairs $(\mathbf{a}, \bar{\zeta})$, the series regularized value $l_{\mathbf{a}}^{\bar{\zeta},S}(h)$ in $\mathbf{Q}[T]$ (T : a parameter which stands for $\log z$, cf. [R]) for a non-admissible pair $(\mathbf{a}, \bar{\zeta})$ is uniquely determined in such a way (cf. [AK]) that the above series shuffle formulae remain valid for $l_{\mathbf{a}}^{\bar{\zeta},S}(h)$ with all pairs $(\mathbf{a}, \bar{\zeta})$.

Define the integral regularized value $l_{\mathbf{a}}^{\bar{\zeta},I}(h)$ in $\mathbf{Q}[T]$ for all pairs $(\mathbf{a}, \bar{\zeta})$ by $l_{\mathbf{a}}^{\bar{\zeta},I}(h) = l_{\mathbf{a}}^{\bar{\zeta}}(e^{TB(0)}h)$. Equivalently $l_{\mathbf{a}}^{\bar{\zeta},I}(h)$ for any pair $(\mathbf{a}, \bar{\zeta})$ can be uniquely defined in such a way that the iterated integral shuffle formulae (loc.cit) remain valid for all pairs $(\mathbf{a}, \bar{\zeta})$ with $l_1^{1,I}(h) := -T$ and $l_{\mathbf{a}}^{\bar{\zeta},I}(h) := l_{\mathbf{a}}^{\bar{\zeta}}(h)$ for all admissible pairs $(\mathbf{a}, \bar{\zeta})$ because they hold for admissible pairs by the group-likeness of h (cf. loc.cit).

Let $\mathbb{L}$ be the $\mathbf{Q}$ -linear map from $\mathbf{Q}[T]$ to itself defined via the generating function:

$$\mathbb{L}(\exp Tu) = \sum_{n=0}^{\infty} \mathbb{L}(T^n) \frac{u^n}{n!} = \exp \left\{ - \sum_{n=1}^{\infty} l_n^{1,I}(h) \frac{u^n}{n} \right\}.$$

Proposition 8. *Let h be an element as in theorem 1. Then the regularization relation holds, i.e. $l_{\mathbf{a}}^{\bar{\zeta},S}(h) = \mathbb{L}(l_{\mathbf{a}}^{\bar{\zeta},I}(h))$ for all pairs $(\mathbf{a}, \bar{\zeta})$.*

Proof . We may assume that $(\mathbf{a}, \bar{\zeta})$ is non-admissible because the proposition is trivial if it is admissible. Put $1^n = (\underbrace{1, 1, \dots, 1}_n)$. When

$\mathbf{a} = 1^n$ and $\bar{\zeta} = \bar{1}^n$, the proof is given by the same argument to [F3] as follows: By the series shuffle formulae,

$$\sum_{k=0}^m (-1)^k l_{k+1}^{\bar{1},S}(h) \cdot l_{1^{m-k}}^{\bar{1},S}(h) = (m+1) l_{1^{m+1}}^{\bar{1},S}(h)$$

for $m \geq 0$. Here we put $l_{\emptyset}^{\emptyset,S}(h) = 1$. This means

$$\sum_{k,l \geq 0} (-1)^k l_{k+1}^{\bar{1},S}(h) \cdot l_{1^l}^{\bar{1},S}(h) u^{k+l} = \sum_{m \geq 0} (m+1) l_{1^{m+1}}^{\bar{1},S}(h) u^m.$$

Put $f(u) = \sum_{n \geq 0} l_{1^n}^{\bar{1},S}(h) u^n$. Then the above equality can be read as

$$\sum_{k \geq 0} (-1)^k l_{k+1}^{\bar{1},S}(h) u^k = \frac{d}{du} \log f(u).$$

Integrating and adjusting constant terms gives

$$\sum_{n \geq 0} l_{1^n}^{\bar{1},S}(h) u^n = \exp \left\{ - \sum_{n \geq 1} (-1)^n l_n^{\bar{1},S}(h) \frac{u^n}{n} \right\} = \exp \left\{ - \sum_{n \geq 1} (-1)^n l_n^{\bar{1},I}(h) \frac{u^n}{n} \right\}$$

because $l_n^{\bar{1},S}(h) = l_n^{\bar{1},I}(h) = l_n^1(h)$ for $n > 1$ and $l_1^{\bar{1},S}(h) = l_1^{\bar{1},I}(h) = -T$. Since $l_{1^m}^{\bar{1},I}(h) = \frac{(-T)^m}{m!}$, we get $l_{1^m}^{\bar{1},S}(h) = \mathbb{L}(l_{1^m}^{\bar{1},I}(h))$.

When $(\mathbf{a}, \bar{\zeta})$ is of the form $(\mathbf{a}'1^l, \bar{\zeta}'1^l)$ with $(\mathbf{a}', \bar{\zeta}')$ admissible, the proof is given by the following induction on l . By (8),

$$l_{\mathbf{a}'}^{\bar{\zeta}'(x)}(h') \cdot l_{1^l}^{\bar{1}^l(y)}(h') = \sum_{\sigma \in Sh \leq (k,l)} l_{\sigma(\mathbf{a}', 1^l)}^{\sigma(\bar{\zeta}'(x), \bar{1}^l(y))}(h')$$

for $h' = e^{T\{t^{23}(0)+t^{24}(0)+t^{34}(0)\}} h^{1,23,4} h^{1,2,3}$ with $k = dp(\mathbf{a}')$. The group-likeness and (4) for h implies $c_0(h) = 1$ and $c_A(h) = 0$ and the group-likeness and our assumption $c_{B(0)}(h) = 0$ implies $c_{B(0)^n}(h) = 0$ for $n \in \mathbf{Z}_{>0}$. Hence by the results in [F4]

$$l_{\mathbf{a}'}^{\bar{\zeta}'}(h) \cdot l_{1^l}^{\bar{1}^l,I}(h) = \sum_{\sigma \in Sh \leq (k,l)} l_{\sigma(\mathbf{a}', 1^l)}^{\sigma(\bar{\zeta}', \bar{1}^l), I}(h).$$

Then by our induction assumption, taking the image by the map $\mathbb{L}$ gives

$$l_{\mathbf{a}'}^{\bar{\zeta}'}(h) \cdot l_{1^l}^{\bar{1}^l,S}(h) = \mathbb{L}(l_{\mathbf{a}'1^l}^{\bar{\zeta}'\bar{1}^l,I}(h)) + \sum_{\sigma \neq id \in Sh \leq (k,l)} l_{\sigma(\mathbf{a}', 1^l)}^{\sigma(\bar{\zeta}', \bar{1}^l), S}(h).$$

Since $l_{\mathbf{a}'}^{\bar{\zeta}',S}(h)$ and $l_{1^l}^{\bar{1}^l,S}(h)$ satisfy the series shuffle formula, $\mathbb{L}(l_{\mathbf{a}'}^{\bar{\zeta}',I}(h))$ must be equal to $l_{\mathbf{a}'}^{\bar{\zeta}',S}(h)$, which concludes proposition 8. $\square$

Embed $U\mathfrak{F}_{Y_N}$ into $U\mathfrak{F}_{N+1}$ by sending $Y_{m,a}$ to $-A^{m-1}B(-a)$. Then by the above proposition,

$$\begin{aligned} l_{\mathbf{a}}^{\bar{\zeta},S}(h) &= \mathbb{L}(l_{\mathbf{a}}^{\bar{\zeta},I}(h)) = \mathbb{L}(l_{\mathbf{a}}^{\bar{\zeta}}(e^{TB(0)}h)) = l_{\mathbf{a}}^{\bar{\zeta}}(\mathbb{L}(e^{TB(0)}\pi_Y(h))) \\ &= l_{\mathbf{a}}^{\bar{\zeta}}(\exp \left\{ - \sum_{n=1}^{\infty} l_n^{1,I}(h) \frac{B(0)^n}{n} \right\} \cdot \pi_Y(h)) \\ &= l_{\mathbf{a}}^{\bar{\zeta}}(\exp \left\{ -TY_{1,0} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} c_{A^{n-1}B(0)}(h) Y_{1,0}^n \right\} \cdot \pi_Y(h)) = l_{\mathbf{a}}^{\bar{\zeta}}(e^{-TY_{1,0}} h_*) \end{aligned}$$

for all $(\mathbf{a}, \bar{\zeta})$ because $l_1^1(h) = 0$. As for the third equality we use $(\mathbb{L} \otimes_{\mathbf{Q}} id) \circ (id \otimes_{\mathbf{Q}} l_{\mathbf{a}}^{\bar{\zeta}}) = (id \otimes_{\mathbf{Q}} l_{\mathbf{a}}^{\bar{\zeta}}) \circ (\mathbb{L} \otimes_{\mathbf{Q}} id)$ on $\mathbf{Q}[T] \otimes_{\mathbf{Q}} U\mathfrak{F}_{N+1}$. All $l_{\mathbf{a}}^{\bar{\zeta},S}(h)$'s satisfy the series shuffle formulae (9), so the $l_{\mathbf{a}}^{\bar{\zeta}}(e^{-TY_{1,0}} h_*)$'s do also. By putting $T = 0$, we get that $l_{\mathbf{a}}^{\bar{\zeta}}(h_*)$'s also satisfy the series shuffle formulae for all $\mathbf{a}$. Therefore $\Delta_*(h_*) = h_* \hat{\otimes} h_*$. This completes the proof of theorem 1. $\square$

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