

Formal Microlocalization

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Formal Specialization and Asymptotic Expansions (see [C])

First, let X be a real analytic manifold with $\mathcal{C}_X^\infty$ the sheaf of complex valued C^∞ -functions on X . Let U be a subanalytic open subset and Z a subanalytic closed subset of X . Recall that the Whitney functor, $\cdot \overset{w}{\otimes} \mathcal{C}_X^\infty$, defined by Kashiwara and Schapira [KS1], is characterized by:

$$\begin{aligned} \mathbf{D}_{\mathbb{R}-c}^b(\mathbb{C}_X) &\rightarrow \mathbf{D}^b(\mathcal{D}_X) \\ \mathbb{C}_U &\mapsto \mathcal{I}_{X, X \setminus U}^\infty, \text{ the subsheaf of } \mathcal{C}_X^\infty \text{ consisting of sections vanishing} \\ &\quad \text{at infinite order on } X \setminus U, \\ \mathbb{C}_Z &\mapsto \text{the sheaf of complex valued functions } C^\infty \text{ on } Z \text{ in the sens} \\ &\quad \text{of Whitney.} \end{aligned}$$

Recall that a C^∞ -function on a subset A of $\mathbb{R}^n$ in the sense of Whitney (see [W]) is a family $F = (F^k)_{k \in \mathbb{N}^n}$ of continuous functions on A such that:
 $\forall m \in \mathbb{N}, \forall k \in \mathbb{N}^n, |k| \leq m, \forall x \in A, \forall \varepsilon > 0$, there is a neighborhood U of x such that $\forall y, z \in U \cap A$

$$\left| F^k(z) - \sum_{|j+k| \leq m} \frac{(z-y)^j}{j!} F^{j+k}(y) \right| \leq \varepsilon \cdot \text{dist}(y, z)^{m-|k|} \quad (1)$$

Let M be a submanifold of X , and $\tilde{X}$ the normal deformation of X along M . We follow the notations of [KS2]:

$$\begin{array}{ccc} T_M X & \xhookrightarrow{s} & \tilde{X} \xleftarrow{\quad} \Omega = \{t > 0\} \\ \downarrow \tau & & \downarrow p \\ M & \xrightarrow{i} & X \end{array}$$

Definition 1. – Let $F \in \text{Ob}(\mathbf{D}_{\mathbb{R}-c}^b(\mathbb{C}_X))$. We set:

$$w\nu_M(F, \mathcal{C}_X^\infty) = s^{-1} R\mathcal{H}om_{\mathcal{D}_{\tilde{X}}}(\mathcal{D}_{\tilde{X} \rightarrow X}, (p^{-1}F)_{\tilde{\Omega}} \overset{w}{\otimes} \mathcal{C}_{\tilde{X}}^\infty), \quad (2)$$

and call it the Whitney specialization of F along M .

From now on, X will be a complex analytic manifold. We denote by $\overline{X}$ the complex conjugate of X , and by $X_{\mathbb{R}}$ the real underlying manifold. Recall the definition of the functor of formal cohomology: $F \overset{\vee}{\otimes} \mathcal{O}_X = R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, F \overset{\vee}{\otimes} \mathcal{C}_{X_{\mathbb{R}}}^{\infty})$.

Definition 2. – Let M be a submanifold of $X_{\mathbb{R}}$ and $F \in \text{Ob}(\mathbf{D}_{\mathbb{R}-c}^b(\mathbb{C}_X))$. We set:

$$w\nu_M(F, \mathcal{O}_X) = R\mathcal{H}om_{\tau^{-1}i^{-1}\mathcal{D}_{\overline{X}}}(\tau^{-1}i^{-1}\mathcal{O}_{\overline{X}}, w\nu_M(F, \mathcal{C}_{X_{\mathbb{R}}}^{\infty})), \quad (3)$$

and call it the formal specialization of F along M . If $F = \mathbb{C}_X$ or $F = \mathbb{C}_{X \setminus M}$, we denote it by $w\nu_M(\mathcal{O}_X)$ and $w^0\nu_M(\mathcal{O}_X)$, respectively.

Proposition 3. – Let $F \in \text{Ob}(\mathbf{D}_{\mathbb{R}-c}^b(\mathbb{C}_X))$ and $v \in T_M X$. Then:

$$H^k(w\nu_M(F, \mathcal{O}_X))_v \simeq \varinjlim R^k\Gamma(X; F_{\overline{U}} \overset{\vee}{\otimes} \mathcal{O}_X),$$

where U ranges through the family of subanalytic open subsets of X such that $v \notin C_M(X \setminus U)$.

Actually we obtain the asymptotic expansions. Let U be an open subanalytic relatively compact regular contractible subset of X and $f \in \mathcal{O}(U)$. We say that f admits an asymptotic expansion in U along M if it verifies one of the three equivalent following propositions:

- (i) $\nu_M f \in \Gamma(V(U); w\nu_M(\mathcal{O}_X))$,
- (ii) For all proper subsector U' of U , $f|_{U'}$ is extendible to a C^{∞} -function on X ,
- (iii) In a local coordinates system $(z) = (x, y) \in \mathbb{R}^{n-p} \times \mathbb{R}^p$, where M is defined by $\{x = 0\}$, there is formal series $\sum_k a_k(y)x^k$ with coefficients C^{∞} in a neighborhood of $\overline{U} \cap M$ in M such that for all proper subsectors U' of U , and for all multi-indices $N \in \mathbb{N}^{n-p}$, there is a constant $C > 0$, such that

$$\forall z \in U', \quad \left| f(z) - \sum_{k < N} a_k(y)x^k \right| \leq C|x^N|.$$

Then considering the distinguished triangle :

$$w^0\nu_M(\mathcal{O}_X) \rightarrow w\nu_M(\mathcal{O}_X) \rightarrow \mathbb{C}_M \overset{\vee}{\otimes} \mathcal{O}_X \xrightarrow{+1}$$

with $X = \mathbb{C}$ and $M = \{0\}$ we reobtain a result of Malgrange [Mal] and Sibuya [Si].

Unfortunately we have yet construct a formal specialization along an analytic subset. But using the method we hope to find the strongly asymptotically developable function of Majima [Maj]. In that case, the equivalence of (ii) and (iii) is already proved by Zurro [Z].

Formal Microlocalization

Let us consider the inverse Fourier-Sato transform of the formal specialization. We denote by p_1 and p_2 the first and second projections from $T_M X \times_M T_M^* X$. Let $P = \{(x, y) \in T_M X \times_M T_M^* X / \langle x, y \rangle \geq 0\}$.

$$\begin{array}{ccc}
 P & \xrightarrow{\quad} & T_M X \times_M T_M^* X \\
 & \searrow p_1 & \swarrow p_2 \\
 T_M X & & T_M^* X \\
 & \searrow \tau & \swarrow \pi \\
 & M &
 \end{array}$$

Definition 4. – We set:

$$w\mu_M(F, \mathcal{O}_X) = (w\nu_M(F, \mathcal{O}_X))^\vee = R p_{2!}(p_1^! w\nu_M(F, \mathcal{O}_X))_P \quad (4)$$

and call it the formal microlocalization of F along M .

Proposition 5. – Let $F \in \text{Ob}(\mathbf{D}_{\mathbb{R}-c}^b(\mathbb{C}_X))$ and $p \in T_M^* X$. Then:

$$H^k(w\mu_M(F, \mathcal{O}_X))_p \simeq \varinjlim_U H^{k+l}(F_U \overset{\circ}{\otimes} \mathcal{O}_X)_{\pi(p)} \quad (5)$$

where U ranges through the family of subanalytic open subsets of X such that $p \in \text{int}(C_M(U)^{oa})$, the interior of the polar set to U and l is the codimension of M in X .

We denote by Δ the diagonal of $X \times X$. Let q_1 and q_2 the first and second projections defined on $X \times X$, Let us identify $T_\Delta^*(X \times X)$ with T^*X by the first projection from $T^*(X \times X) = T^*X \times T^*X$.

Definition 6. – Let $F \in \text{Ob}(\mathbf{D}_{\mathbb{R}-c}^b(\mathbb{C}_X))$. We set:

$$F \overset{\circ}{\otimes}_\mu \mathcal{O}_X = R\mathcal{H}om_{\pi^{-1}\mathcal{D}_{X \times X}}(\pi^{-1}\mathcal{D}_{X \times X \xrightarrow{q_1} X}, w\mu_\Delta(q_2^{-1}F, \mathcal{O}_{X \times X})). \quad (6)$$

Proposition 7. – Let M be a submanifold of $X_{\mathbb{R}}$ and k the immersion of $T_M^* X$ in T^*X . Then:

$$\mathbb{C}_M \overset{\circ}{\otimes}_\mu \mathcal{O}_X \simeq k_* w\mu_M(\mathbb{C}_X, \mathcal{O}_X) \quad (7)$$

Proposition 8. – Let $F \in \text{Ob}(\mathbf{D}_{\mathbb{R}-c}^b(\mathbb{C}_X))$ and $p \in T^*X$. Then:

$$H^k(F \overset{\circ}{\otimes}_\mu \mathcal{O}_X)_p \simeq \varinjlim_{U, V} H^k(Rq_{1!}(q_2^!(F_U))_V \overset{\circ}{\otimes} \mathcal{O}_X)_{\pi(p)} \quad (8)$$

where U ranges through the family of neighborhood of $\pi(p)$ in X and V ranges through the family of subanalytic open subsets of $X \times X$ such that $(p^a, p) \in \text{int}(C_\Delta(V)^{oa})$.

Proposition 9. – *Let $F \in \text{Ob}(\mathbf{D}_{\mathbb{R}-c}^b(\mathbb{C}_X))$. Then:*

$$R\pi_*(F \overset{w}{\otimes}_{\mu} \mathcal{O}_X) \simeq F \overset{w}{\otimes} \mathcal{O}_X, \quad (9)$$

$$R\pi_!(F \overset{w}{\otimes}_{\mu} \mathcal{O}_X) \simeq F \otimes \mathcal{O}_X, \quad (10)$$

and we have the distinguished triangle:

$$F \otimes \mathcal{O}_X \rightarrow F \overset{w}{\otimes} \mathcal{O}_X \rightarrow R\pi_*(F \overset{w}{\otimes}_{\mu} \mathcal{O}_X|_{T^*X}) \xrightarrow{+1}. \quad (11)$$

Let X and Y be two complex manifolds. We denote by p_X and p_Y the first and second projection from $T^*(X \times Y)$ to T^*X and T^*Y . Let $\mathcal{M} \in \mathbf{D}^b(\pi^{-1}\mathcal{D}_{X \times Y})$, $\mathcal{N} \in \mathbf{D}^b(\pi^{-1}\mathcal{D}_Y)$, $F \in \mathbf{D}_{\mathbb{R}-c}^b(\mathbb{C}_X)$ and $K \in \mathbf{D}_{\mathbb{R}-c}^b(\mathbb{C}_{X \times Y})$. We denote :

$$\begin{aligned} \mathcal{M} \circ_{\mathcal{D}_Y} \mathcal{N} &= Rp_{X!}(\mathcal{M} \otimes_{\mathcal{D}_Y} p_Y^{*-1}\mathcal{N}) \\ K \circ F &= Rq_{Y!}(K \otimes q_Y^{-1}F). \end{aligned}$$

Using the morphism constructed in the last chapter of [KS1]:

$$\text{Thom}(F, \mathcal{O}_X) \otimes_{\mathcal{O}_X} (F \otimes G) \overset{w}{\otimes} \mathcal{O}_X \rightarrow G \overset{w}{\otimes} \mathcal{O}_X,$$

we obtain a morphism for integral transformation with the functor $T_{\mu\text{hom}}$ defined in [A].

Theorem 10. – *Let $F \in \mathbf{D}_{\mathbb{R}-c}^b(\mathbb{C}_X)$ and $K \in \mathbf{D}_{\mathbb{R}-c}^b(\mathbb{C}_{X \times Y})$. Then we have a natural morphism:*

$$T_{\mu\text{hom}}(K, \mathcal{O}_{X \times Y}^{(0, d_Y)}[d_Y])^a \circ_{\mathcal{D}_Y} ((K \circ F) \overset{w}{\otimes}_{\mu} \mathcal{O}_Y) \rightarrow F \overset{w}{\otimes}_{\mu} \mathcal{O}_X. \quad (12)$$

From this theorem, with $X = Y$ and $K = \mathbb{C}_{\Delta}$, we get a natural morphism:

$$(\mathcal{E}_X^{\mathbb{R}, f})^a \otimes (F \overset{w}{\otimes}_{\mu} \mathcal{O}_X) \rightarrow F \overset{w}{\otimes}_{\mu} \mathcal{O}_X.$$

In particular, $H^j(F \overset{w}{\otimes}_{\mu} \mathcal{O}_X)_p$ has a structure of $(\mathcal{E}_X)_p$ -module for any p in T^*X .

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