

Low dimensional Homotopy and homology for monoid presentations

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The homotopy theory introduced by Squier [9] gives a very general framework for geometrical approach to monoid presentations. It is considered to include the classical geometrical theory of group presentations ([6], [7]). In this note we report recent results on a relation between the homotopical and the homological finiteness conditions of monoids. The details will be given in [4].

1. Homotopy relations on the derivation graphs

Let Σ be a finite alphabet and Σ^* be the free monoid generated by Σ . Let E be a irreflexive symmetric relation on Σ^* . Let $=_E$ denote the congruence on Σ^* generated by E , that is, $=_E$ is the smallest compatible equivalence relation containing E . Let $M = \Sigma^*/=_E$ be the quotient monoid. We say M is presented by a presentation (Σ, E) and write as $M = M(\Sigma, E)$.

We define a graph $\Gamma = \Gamma(\Sigma, E)$ called the derivation graph of (Σ, E) as follows: The set of vertices is Σ^* , and for $e = (u, v) \in E$ and $x, y \in \Sigma^*$, $(x; u, v; y)$ is an edge from xuy to xvy . An equation $e = (u, v) \in E$ is identified with the naked edge $(1; u, v; 1)$. Since E is symmetric, if $e = (x; u, v; y)$ is an edge, then $(x; v, u; y)$ is also an edge, which is called the reverse of e and written e^{-1} .

Let $P(x, y)$ denote the set of (directed) paths from the source x to the target y in Γ and $P(\Gamma)$ the set of all the paths in Γ . The source and the target of a path p are denoted by $\sigma(p)$ and $\tau(p)$, respectively. For two paths p and q such that $\tau(p) = \sigma(q)$, the path $p \cdot q$ connecting p and q is defined in the obvious manner. For a path $p = e_1 \cdot e_2 \cdot \dots \cdot e_n$ its reverse $p^{-1} =$

$e_n^{-1} \cdot \dots \cdot e_2^{-1} \cdot e_1^{-1}$ is a path from $\tau(p)$ to $\sigma(p)$. A path p with $\sigma(p) = \tau(p)$ is called a closed path with base point $\sigma(p) = \tau(p)$. The set $P(x, x)$ of closed path with base point x is denoted simply by $P(x)$. A cycle is a simple closed path, that is, a closed path which does not pass the same edge twice. The trivial cycle i_x with base point x is the cycle of length 0 at x .

The free monoid Σ^* acts on $P(\Gamma)$ on the both sides as follows: Let $z, w \in \Sigma^*$. For an edge $e = (x; u, v; y)$ define

$$z \cdot e \cdot w = (zx; u, v; yw),$$

and for a path $p = e_1 \cdot e_2 \cdot \dots \cdot e_n$ define

$$z \cdot p \cdot w = (z \cdot e_1 \cdot w) \cdot (z \cdot e_2 \cdot w) \cdot \dots \cdot (z \cdot e_n \cdot w).$$

Two paths p and q with the same source and the same target are called parallel, and written as $p \parallel q$.

An equivalence relation $\sim$ on $P(\Gamma)$ is a homotopy relation (Squier, Otto and Kobayashi [9]), if it is contained in $\parallel$ and satisfies the following four conditions.

$$(H1) \quad \text{For any } x \in \Sigma^* \text{ and } e_1 = (u_1, v_1), e_2 = (u_2, v_2) \in E, \\ e_1 \cdot x u_2 \cdot v_1 x \cdot e_2 \sim u_1 x \cdot e_2 \cdot e_1 \cdot x v_2.$$

$$(H2) \quad \text{For any } p, q \in P(\Gamma) \text{ and } x, y \in \Sigma^*, \\ p \sim q \text{ implies } x \cdot p \cdot y \sim x \cdot q \cdot y.$$

$$(H3) \quad \text{For any } p, q, r, s \in P(\Gamma) \text{ with } \tau(r) = \sigma(p) = \sigma(q) \\ \text{and } \sigma(s) = \tau(p) = \tau(q),$$

$$p \sim q \text{ implies } r \cdot p \cdot s \sim r \cdot q \cdot s.$$

$$(H4) \quad \text{For any } e = (u, v) \in E, \\ e \cdot e^{-1} \sim i_u.$$

Let $\sim$ be a homotopy relation. The equivalence class of $p \in P(\Gamma)$ modulo the relation $\sim$ is written as $[p]_{\sim}$ or simply as $[p]$. The quotients $P(\Gamma)/\sim$ is denoted by $\pi_{\sim}(\Gamma) = \pi_{\sim}(\Sigma, E)$. The operation $\cdot$ of $P(\Gamma)$ induces the operation $\cdot$ of $\pi_{\sim}(\Gamma)$ by $[p] \cdot [q] = [p \cdot q]$. The action of Σ^* on $\pi_{\sim}(\Gamma)$ is also induced as $x \cdot [p] \cdot y = [x \cdot p \cdot y]$.

We can also consider the quotients $P(x, y)/\sim$ and $P(x)/\sim$, which are denoted by $\pi_{\sim}(x, y)$ and $\pi_{\sim}(x)$, respectively. Then,

$\pi_{\sim}(\Gamma) = \bigcup_{x,y \in \Sigma^*} \pi_{\sim}(x, y)$ forms a groupoid and $\pi_{\sim}(x)$ forms a group with operation $\cdot$. The group $\pi_{\sim}(x)$ is called the homotopy group at x with respect to $\sim$. In particular, $\pi_0(x)$ denotes the homotopy group at x with respect to $\sim_0$.

The parallelism $\parallel$ is the largest homotopy relation. Since homotopy relations are closed under intersection, for any subset B of $\parallel$, there is the smallest homotopy relation $\sim_B$ containing B , called the homotopy relation generated by B . The homotopy relation $\sim_0$ generated by the empty set is the smallest homotopy relation.

If $p \sim_0 q$, we say two paths p and q are strictly homotopic. If there is a finite subset B of $\parallel$ such that $\sim_B = \parallel$, (Σ, E) has finite homotopy type and B is called a finite homotopy base for it. If the empty set generates $\parallel$, that is, $\sim_0 = \parallel$, (Σ, E) is strictly aspherical.

2. Homotopy reduction systems

In this section (Σ, E) is a fixed monoid presentation and $\Gamma = \Gamma(\Sigma, E)$.

A subset B of $\parallel$ is a homotopy reduction system; B is a set of pairs of parallel paths, and an element (p, q) of B is called a rule and written $p \rightsquigarrow q$. For two paths p and q , we write $p \rightsquigarrow_B q$, if there are $x, y \in \Sigma^*$, $p_1, p_2 \in P(\Gamma)$ and $r \rightsquigarrow s \in B$ such that

$$p = p_1 \cdot (x \cdot r \cdot y) \cdot p_2, \quad q = p_1 \cdot (x \cdot s \cdot y) \cdot p_2.$$

The reflexive and transitive closure and the reflexive, symmetric transitive closure of $\rightsquigarrow_B$ is denoted by $\rightsquigarrow_B^*$ and $\Leftrightarrow_B^*$, respectively.

B is noetherian, if there is no infinite sequence of reductions:

$$p_1 \rightsquigarrow_B p_2 \rightsquigarrow_B \dots \rightsquigarrow_B p_i \rightsquigarrow_B \dots$$

B is confluent, if for any paths p, q and r such that $p \rightsquigarrow_B^* q$ and $p \rightsquigarrow_B^* r$, there is a path s such that $q \rightsquigarrow_B^* s$, $r \rightsquigarrow_B^* s$.

A path p is irreducible with respect to B , if there is no q such that $p \rightsquigarrow_B q$. If B is noetherian and confluent, then

for any path p there is a unique irreducible path $\hat{p}$ such that $p \rightsquigarrow_B^* \hat{p}$, which is called the canonical form of p with respect to $\rightsquigarrow_B$.

The relation $\Leftrightarrow_B^*$ is contained in the homotopy relation $\sim_B$ generated by B . A system B is called complete, if it is noetherian and confluent and $\Leftrightarrow_B^* = \sim_B$. If B is complete, then

$$p \sim_B q \Leftrightarrow \hat{p} = \hat{q}.$$

A system B is reduced, if for any $(p, q) \in B$, p is irreducible under $B - \{(p, q)\}$ and q is irreducible under B . B is simple, if any $(p, q) \in B$ cannot be written as

$$p = p' \cdot xry \cdot p'', \quad q = p' \cdot xsy \cdot p''$$

with paths p' , p'' and $x, y \in \Sigma^*$ such that at least one of p' , p'' , x and y is non-trivial. These p' , p'' , x and y are the coats of (p, q) .

Two reduction systems B_1 and B_2 are equivalent, if $\sim_{B_1} = \sim_{B_2}$.

Theorem 2.1 For any complete homotopy reduction system B generating $\parallel$, there is a reduced simple complete system $\bar{B}$ generating B such that $|\bar{B}| \leq |B|$.

Let B be a reduction system. Let r and s be two subpaths of a path p ; $p = q \cdot xry \cdot t = q' \cdot x'sy' \cdot t'$ with paths q, t, q', t' and $x, y, x', y' \in \Sigma^*$. r is left to s in p if the length of q is shorter than that of q' . A sequence of reductions is left-most, if in each step of the reductions a rule is applied to a left-most applicable subpath. If the system B is simple and reduced, a left-most applicable subpath is unique, and so a sequence of the left-most reductions starting with given path is unique.

Let p and q be parallel paths in $\Gamma = \Gamma(\Sigma, E)$. A sequence of two-way reductions from p to q is a sequence

$$h: p_0 \Leftrightarrow_B p_2 \Leftrightarrow_B \dots \Leftrightarrow_B p_n$$

with $q_0 = p$ and $q_n = q$. $P_i \Leftrightarrow P_{i+1}$ is a one-step reductions with respect to $B \cup B^{-1}$, $B^{-1} = \{b^{-1} | b \in B\}$, where b^{-1} is the

reverse reduction of b . The path p is the source and q is the target of h and are denoted by $\sigma(h)$ and $\tau(h)$, respectively. h is a (homotopy) reduction cycle, if $\sigma(h) = \tau(h)$.

Let h be a sequence of (two-way) reductions between parallel paths p and q . Let $x, y \in \Sigma^*$ and s and t be paths such that $\tau(s) = x\sigma(p)y$ and $\sigma(t) = x\tau(p)y$. Then $s \cdot xpy \cdot t$ and $s \cdot xqy \cdot t$ are parallel path and we have a sequence of reductions $s \cdot x \cdot h \cdot y \cdot t$ between them, which coincides with h on the subpath p and does not affect the outside. Let g be another sequence of reductions from q to r , then $h \star g$ denotes the conjunction of them; $h \star g$ is the reduction sequence from p to r . We give less priority to the operation $\star$ over $\cdot$ and $\circ$, thus, for example, $(s \cdot x \cdot h \cdot y \cdot t) \star g$ is written as $s \cdot xhy \cdot t \star g$. The reverse sequence of reductions of h is denoted by h^{-1} .

3. Critical pairs

Two paths p and q are coaxal, if there are $x, x', y, y' \in \Sigma^*$ such that $x \cdot p \cdot x' = y \cdot q \cdot y'$. The foursome (x, x', y, y') is an adjuster for p and q . An adjuster with the length $|xx'yy'|$ minimal is unique and called the minimal adjuster for p and q . If (x, x', y, y') is minimal, x or x' and y or y' are empty words.

Let $p_1 \rightsquigarrow q_1, p_2 \rightsquigarrow q_2 \in B$. Suppose $p_1 = p'_1 \cdot t_1, p_2 = t_2 \cdot p'_2$ with coaxal t_1 and t_2 and nontrivial p'_1 and p'_2 . In this situation we say that $p_1 \rightsquigarrow q_1$ overlaps with $p_2 \rightsquigarrow q_2$ on the left. Let (x, x', y, y') be the minimal adjuster for t_1 and t_2 . Then we have $x \cdot p_1 \cdot x' \cdot y \cdot p'_2 \cdot y' = x \cdot p'_1 \cdot x' \cdot y \cdot p_2 \cdot y' (= p)$ and $p \rightsquigarrow_B x \cdot q_1 \cdot x' \cdot y \cdot p'_2 \cdot y', p \rightsquigarrow_B x \cdot p'_1 \cdot x' \cdot y \cdot q_2 \cdot y'$. We call the pair of paths

$$(x \cdot q_1 \cdot x' \cdot y \cdot p'_2 \cdot y', x \cdot p'_1 \cdot x' \cdot y \cdot q_2 \cdot y')$$

a critical pair of overlapping type. Next suppose $p_1 = p'_1 \cdot t_1 \cdot p''_1$ and t_1 and p_2 are coaxal. Let (x, x', y, y') be the minimal adjuster for them. Then we have $x \cdot p_1 \cdot x' = x \cdot p'_1 \cdot x' \cdot y \cdot p_2 \cdot y' \cdot x \cdot p''_1 \cdot x' (= p)$ and $p \rightsquigarrow_B x \cdot q_1 \cdot x', p \rightsquigarrow_B x \cdot p'_1 \cdot x' \cdot y \cdot q_2 \cdot y' \cdot x \cdot p''_1 \cdot x'$. We call

$$(x \cdot q_1 \cdot x', x \cdot p'_1 \cdot x' \cdot y \cdot q_2 \cdot y' \cdot x \cdot p''_1 \cdot x')$$

a critical pair of inclusion type. If B is reduced, there is no critical pair of inclusion type.

A critical pair (r, s) is resolvable, if there is a path t such that $r \leadsto_B^* t$ and $s \leadsto_B^* t$. Since locally confluent noetherian system are confluent ([1], [3]), we have

Theorem 3.1. A noetherian reduction system is confluent, if and only if all the critical pairs are resolvable.

We consider the following subsets of $\parallel$:

$$B_0 = \{ e \cdot e^{-1} \leadsto i_u \mid e = (u, v) \in E \}$$

and

$$B_\emptyset = \{ u_1 x \cdot e_2 \cdot e_1 \cdot x v_2 \leadsto e_1 \cdot x u_2 \cdot v_1 x \cdot e_2 \mid \\ x \in \Sigma^*, e_1 = (u_1, v_1), e_2 = (u_2, v_2) \in E \}.$$

Let B be a homotopy reduction system for (Σ, E) generating the parallelism $\parallel$. We suppose that $B \cap (B_0 \cup B_\emptyset) = \emptyset$. Let $\bar{C}$ be the set of critical pairs associated with $\bar{B} = B_0 \cup B_\emptyset \cup B$.

Let $b_1 = (p, q) \in B_\emptyset$, where

$$p = u_1 x e_2 \cdot e_1 x v_2, q = e_1 x u_2 \cdot v_1 x e_2,$$

with $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2) \in E$. Let $b_2 = (s, t)$ be a rule in B which overlaps with b_1 on the left, that is, $s =$

$w e_1 z \cdot s_2$ with $w, z \in \Sigma^*$ such that z is a prefix of $x v_2$ or $x v_2$ is a prefix of z . If z is a prefix of x , we say that they overlap weakly, otherwise we say that they overlap strongly.

If b_1 overlaps weakly with b_2 on the left, then $x = z x'$ for some $x' \in \Sigma^*$ and $w p \cdot s_2 x' v_2 = w u_1 x e_2 \cdot s x' v_2$. Then there are one-step reductions

$$k_1: w p \cdot s_2 x' v_2 \leadsto_{B_\emptyset} w q \cdot s_2 x' v_2$$

and

$$k_2: w u_1 z x' e_2 \cdot s x' v_2 \leadsto_B w u_1 x e_2 \cdot t x' v_2,$$

and we have a critical pair $(w q \cdot s_2 x' v_2, w u_1 x e_2 \cdot t x' v_2)$. This critical pair is always resolvable because we have the following reductions:

$$j_1': w q \cdot s_2 x' v_2 = w e_1 x u_2 \cdot w v_1 x e_2 \cdot s_2 x' v_2$$

$$\begin{aligned} &\sim_{B_0} we_1xu_2 \cdot s_2x'u_2 \cdot \tau(s)x'e_2 = sx'u_2 \cdot \tau(s)x'e_2 \\ &\sim_B tx'u_2 \cdot \tau(t)x'e_2, \end{aligned}$$

and

$$j'_2: wu_1xe_2 \cdot tx'v_2 \sim_{B_0} tx'u_2 \cdot \tau(t)x'e_2.$$

If b_1 weakly overlaps with b_2 on the right, that is, $s = s_1 \cdot ze_2w$ and w is a suffix of x ; $x = x'z$. Then we have $u_1x's_1 \cdot pw = u_1x's \cdot e_1xv_2w$ and we have a critical pair $(u_1x't \cdot e_1xv_2w, u_1x's_1 \cdot qw)$. This critical pair is also resolvable in a similar way.

Theorem 3.2. If E and B is finite, then there are only a finite number of strong overlappings between B_0 and B , and there are only a finite number of critical pairs of inclusion type for $\bar{B}$.

Theorem 3.4. Assume that E is non-special and $\bar{B} = B_0 \cup B_0 \cup B$ is noetherian. Then $\bar{B}$ is complete, if and only if the following four types of critical pairs are all resolvable: critical pairs coming from (1) overlapping between rules from B_0 and B , (2) overlapping between rules inside B , (3) strong overlapping between rules from B_0 and B , and (4) inclusion between rules from $\bar{B}$ and B .

Corollary 3.5. Assume that both E and B is finite and $\bar{B} = B_0 \cup B_0 \cup B$ is noetherian, then it is decidable whether $\bar{B}$ is complete.

4. The (coated) left canonical reduction system

An equation $(u, v) \in E$ is special, if $u = 1$ or $v = 1$. The system E is special, if all the equations in it are special. E is non-special, if every equation in it is not special.

The simplest complete system called the left canonical reduction system is given in the following

Theorem 4.1. If (Σ, E) is non-special, then $B = B_0 \cup B_0$

is a complete reduction system such that $\sim_B = \sim_0$.

Thus, for a non-special presentation, any path p has the unique canonical form $\hat{p}$ with $p \sim_0 \hat{p}$.

Corollary 4.2. For parallel paths p and q , $p \sim_0 q$ if and only if $\hat{p} = \hat{q}$. In particular, for a closed path p at $x \in \Sigma^*$, $p \sim_0 i_x$ if and only if $\hat{p} = i_x$.

Corollary 4.3. Let (Σ, E) be a finite non-special presentation. For given paths p and q it is decidable whether $p \sim_0 q$.

Let $x, y \in \Sigma^*$. We say that x overlaps with y on the left if there is $u \neq 1$ such that $x = x_1 u$, $y = u y_1$ for some x_1, y_1 with $x_1 y_1 \neq 1$. Let $\text{OVL}(x, y)$ be the set of all such words u . Since it is convenient to include the perfect overlapping, we define the set $\overline{\text{OVL}}(x, y)$ by

$$\overline{\text{OVL}}(x, y) = \{u \in \Sigma^+ \mid x = x_1 u, y = u y_1, x_1, y_1 \in \Sigma^*\}.$$

Let (Σ, E) be a non-special presentation. For $e = (u, v) \in E$, let $\lambda = \lambda(e)$ (resp. $\rho = \rho(e)$) be the maximal common prefix (resp. suffix) of u and v . We call λ (resp. ρ) the left (resp. right) coat of e .

Let $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ be equations from E . Let ρ (resp. λ) be the right (resp. left) coat of e_1 (resp. e_2); $u_1 = u'_1 \rho$, $v_1 = v'_1 \rho$, $u_2 = \lambda u'_2$, $v_2 = \lambda v'_2$. For a word $\tau \in \overline{\text{OVL}}(\rho, \lambda)$; $\rho = \rho' \tau$, $\lambda = \tau \lambda'$, we consider a pair $b_\tau = (u'_1 \rho' \cdot e_2 \cdot e_1 \lambda' v'_2, e_1 \cdot \lambda' u'_2 \cdot v'_1 \rho' \cdot e_2)$ of parallel paths in Γ . Let B_c be the set of all these pairs b_τ of paths in Γ .

An equation of the form (v, u) (or (u, v)) with $u \in \Sigma^*$ and $v \in u \Sigma^* \cap \Sigma^* u$ is called subspecial. A system E is non-subspecial if it contains no subspecial equation.

Theorem 4.4. If E is non-subspecial, then the reduction system $B = B_0 \cup B_\emptyset \cup B_c$ is complete.

The B in Theorem 4.1 is called the coated left canonical reduction system. If this B_c generates $\parallel$, then E is called coated aspherical.

5. Low-dimensional homology

Let a monoid M be given by a presentation (Σ, E) . For $x \in \Sigma^*$, $\bar{x}$ denotes the image of x in M . Let R be an orientation of E , that is, R is an asymmetric relation such that $E = R \cup R^{-1}$, where $R^{-1} = \{(v, u) | (u, v) \in R\}$. The oriented R is called a rewriting system for M . Let $A = \mathbb{Z}M$ be the monoid algebra of M over $\mathbb{Z}$. The additive group $\mathbb{Z}$ is considered to be a right A -module by the trivial action:

$$1 \cdot x = 1$$

for $x \in M$.

We say that M satisfies the (right) homological finiteness condition FP_n , if there is a resolution of $\mathbb{Z}$:

$$F_n \rightarrow F_{n-1} \rightarrow \dots \rightarrow F_0 \rightarrow \mathbb{Z}$$

such that F_i are finitely generated free right A -modules for $i = 0, \dots, n$.

The algebra A itself is considered to be the free cyclic right A -module which we denote by F_0 . There is a natural surjection $\varepsilon: F_0 \rightarrow \mathbb{Z}$, which is an A -homomorphism defined by $\varepsilon(1) = 1$.

Let F_1 be the free right A -module generated by the set Σ . We write $\langle a \rangle$ for the generator of F_1 corresponding to $a \in \Sigma$.

For a word $x = a_1 \dots a_n \in \Sigma^*$, we define an element $\langle x \rangle$ of F_1 by

$$\langle x \rangle = \langle a_1 \rangle \cdot \overline{a_2 \dots a_n} + \langle a_2 \rangle \cdot \overline{a_3 \dots a_n} + \dots + \langle a_n \rangle.$$

Let F_2 be the free right A -module generated by the set R . By $\langle r \rangle$ we denote the corresponding generator to $r \in R$.

Let B be a homotopy base for (Σ, E) . Let F_3 be the free right A -module generated by B . Again $\langle b \rangle$ denote the generator corresponding to $b \in B$. For $e = (u, v) \in E$ define

$$\langle e \rangle = \begin{cases} \langle r \rangle & \text{if } r = (u, v) \in R \\ -\langle r \rangle & \text{if } r = (v, u) \in R. \end{cases}$$

Let $p = x_1 e_1 y_1 \cdot \dots \cdot x_n e_n y_n$ be a path in $\Gamma = \Gamma(\Sigma, E)$, where $x_i,$

$y_i \in \Sigma^*$ and e_i are edges. Then, define an element $\langle p \rangle$ of F_2 by

$$\langle p \rangle = \langle e_1 \rangle \cdot \overline{y_1} + \dots + \langle e_n \rangle \cdot \overline{y_n}.$$

Now, we define A-homomorphisms $\partial_1: F_1 \rightarrow F_0$, $\partial_2: F_2 \rightarrow F_1$ and $\partial_3: F_3 \rightarrow F_2$ by

$$\partial_1(\langle a \rangle) = \overline{a} - \overline{1}$$

for $a \in \Sigma$,

$$\partial_2(\langle r \rangle) = \langle u \rangle - \langle v \rangle$$

for a rule $r = (u, v) \in R$, and

$$\partial_3(\langle b \rangle) = \langle p \rangle - \langle q \rangle$$

for $b = (p, q) \in B$. Easily we find

$$\varepsilon \cdot \partial_1 = \partial_1 \cdot \partial_2 = \partial_2 \cdot \partial_3 = 0,$$

and hence we have a complex:

$$F_3 \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} \mathbb{Z} \rightarrow 0. \quad (5.1)$$

Theorem 5.1 ([2], [5], [8]). The complex (5.1) is exact.

Theorem 5.2. If a finitely presented monoid M has finite homotopy type, then it satisfies right and left FP_3 . If M has a strictly aspherical presentation, then there is a free resolution over $\mathbb{Z}$:

$$0 \rightarrow F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} \mathbb{Z} \rightarrow 0,$$

and M has cohomological dimension at most 2.

Now suppose that B is a homotopy base and $\overline{B} = B_0 \cup B_\emptyset \cup B$ is a simple reduced complete reduction system. Let $\overline{C}$ be the set of critical pairs associated with $\overline{B}$, and let $C (\subset \overline{C})$ be the set of critical pairs of type (1) - (3) in Theorem 3.4 (since $\overline{B}$ is reduced we need not consider the critical pairs of type (4)).

Let $c = (p, q) \in C$ be a critical pair which comes from path r by one-step reductions $h_1: r \rightsquigarrow_B p$ and $h_2: r \rightsquigarrow_B q$. Since B is complete, there are sequences of reductions from p to $\hat{r}$ and q to $\hat{r}$, where r is the canonical form of r . Let h_3 and h_4 be the left-most reductions from p and q to $\hat{r}$ respectively. Then, we have a cycle of two-way reductions:

$$H(c) = h_1 \star h_3 \star h_4^{-1} \star h_2^{-1},$$

which is called the reduction cycle associated with c .

Now we define a free A -module F_4 and a boundary map $\partial_4: F_4 \rightarrow F_3$ as follows. Let F_4 be the free right A -module generated by C , and $\langle c \rangle$ denotes the generator corresponding to $c \in C$. The A -homomorphism ∂_4 is defined by

$$\partial_4(\langle c \rangle) = \langle H(c) \rangle$$

for $c \in C$. We can show that $\partial_3 \circ \partial_4 = 0$, and we have a complex:

$$F_4 \xrightarrow{\partial_4} F_3 \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} \mathbb{Z} \rightarrow 0. \quad (5.2)$$

Theorem 5.3. If $\bar{B} = B_0 \cup B_\emptyset \cup B$ is a complete reduction system, then the complex (5.2) is exact.

Theorem 5.4. If M is given by a finite presentation which admits a finite homotopy base B such that $\bar{B} = B_0 \cup B_\emptyset \cup B$ (or $\bar{B} = B_0 \cup B_r \cup B$) is a complete system, then M satisfies FP_4 .

Corollary 5.5. If M is given by a finite non-subspecial presentation that is coated aspherical, then M satisfies FP_4 .

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