

ALGORITHMIC PROBLEMS OF AUTOMATA *

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In this paper, we shall introduce the concept “extended automata” and reduce the decision problem whether a finite semigroup is an amalgamation base for all semigroups or not to an algorithmic problem of automata.

1 Automata

Definition. A finite automaton $\mathcal{A} = (\Sigma, X, E, I, T)$: Σ is a finite set of states, $I (\subseteq \Sigma)$ is a set of initial states, $T (\subseteq \Sigma)$ is a set of terminal states, X is a set of finite letters, E is a subset of the product set $\Sigma \times X \times \Sigma$.

Each element of E is an edge of the form (σ, x, τ) and x is the label of the edge.

Definition A path P on $\mathcal{A}$ is a sequence of edges :

$P = (\sigma_0, x_1, \sigma_1), (\sigma_1, x_2, \sigma_2), \dots, (\sigma_{t-1}, x_t, \sigma_t)$ (t is a length of the path P)

The signature $Sg(P)$ of P is a word $x_1x_2 \dots x_t$.

$\sigma_0 \in I, \sigma_t \in T \Rightarrow Sg(P)$ is an acceptable word, which assigns a move from σ_0 to σ_t .

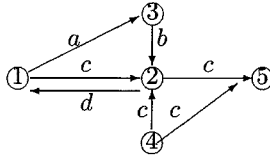
Definition. For a finite automaton $\mathcal{A} = (\Sigma, X, E, I, T)$, let $L (\subset X^*)$ be a set of all acceptable words. Simply, $L = \mathcal{L}(\mathcal{A})$ the language L is the accepted by $\mathcal{L}(\mathcal{A})$.

In general, L is called a regular language.

*This is an abstract and the paper will appear elsewhere.

Example.

$$\mathcal{A} = (\{1, 2, 3, 4, 5\}, \{a, b, c\}, E, \{1\}, \{5\})$$



$$\mathcal{L}(\mathcal{A}) = \{\{abd, cd\}^*cc, abc\}$$

Definition. Given a finite automaton $\mathcal{A} = (\Sigma, X, E, I, T)$. For a word $w \in X^*$, if there exist states $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ and paths P_1 from σ_0 to σ_1 , P_2 from σ_2 to σ_1 , P_3 from σ_2 to σ_3 with $w = Sg(P_1) = Sg(P_2) = Sg(P_3)$, then construct a new path with $Sp(P) = w$ from σ_0 to σ_3 . An extended path P with $Sg(P) = w'$ from σ to β is a path obtained by finitely many constructing new paths for subword w of w' in $\mathcal{A}$. i.e., if $w = w'w_1w_2w''$, $\sigma_0 \xrightarrow{w_1} \sigma_1 \xrightarrow{w_2} \sigma_2 \xleftarrow{w_2} \sigma_3 \xrightarrow{w_2} \sigma_4 \xleftarrow{w_1w_2} \sigma_5 \xrightarrow{w_1} \sigma_6 \xrightarrow{w_2} \sigma_7$ then $\sigma_0 \xrightarrow{w_1} \sigma_1 \xrightarrow{w_2} \sigma_4 \xleftarrow{w_1w_2} \sigma_5 \xrightarrow{w_1} \sigma_6 \xrightarrow{w_2} \sigma_7$ and $\sigma_0 \xrightarrow{w_1w_2} \sigma_7$

The extended regular language $\mathcal{L}^e(\mathcal{A})$ consists of all signatures of extended path from initial states to terminal states of $\mathcal{A}$.

Question. What a kind of language is an extended regular language of an automaton ?

We give a concrete description of extended automata.

Theorem. Given a finite automaton $\mathcal{A} = (\Sigma, X, E, I, T)$. For $\alpha, \beta \in \Sigma$, let $\mathcal{A}_{\sigma, \beta}$ be an automaton accepting all signatures of paths from σ to β .

For $\alpha, \beta \in \Sigma$, let $\mathcal{B}(\mathcal{A})_{\alpha, \beta}$ with a unique initial state α and a unique terminal state β be an automaton accepting all words in $\bigcup_{\gamma \in \Sigma} \mathcal{L}(\mathcal{A}_{\alpha, \gamma}) \cap \mathcal{L}(\mathcal{A}_{\beta, \gamma}) \cap \mathcal{L}(\mathcal{A}_{\gamma, \beta})$.

$n = 1$, $\mathcal{A}^{(1)}$ is an automaton obtained by pasting $\mathcal{B}(\mathcal{A})_{\sigma, \beta}$ to $\mathcal{A}$ by identifying α, β of $\mathcal{A}$ with α, β of $\mathcal{B}_{\alpha, \beta}$ for all $\alpha, \beta \in \Sigma$.

$n = k + 1$, $\mathcal{A}^{(n)}$ is an automaton obtained by pasting $\mathcal{B}(\mathcal{A}^{(k)})_{\sigma, \beta}$ to $\mathcal{A}^{(k)}$ by identifying α, β of $\mathcal{A}$ with α, β of $\mathcal{B}_{\alpha, \beta}$ for all $\alpha, \beta \in \Sigma$. with $\mathcal{A}_{\alpha, \beta}^{(k)}$ pasted for all $\alpha, \beta \in \Sigma$.

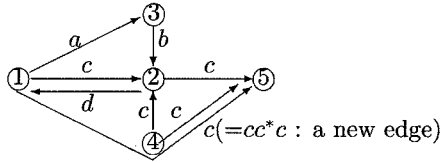
$$\mathcal{A}^e = \bigcup_{n=1}^{\infty} \mathcal{A}^{(n)}. \text{ (Possibly, an infinite automaton)}$$

Then

$$\text{the extended regular language } \mathcal{L}^e(\mathcal{A}) = \mathcal{L}(\mathcal{A}^e).$$

Example.

$$\mathcal{A} = (\{1, 2, 3, 4, 5\}, \{a, b, c\}, E, \{1\}, \{5\})$$



$$\mathcal{L}^e(\mathcal{A}) = \mathcal{L}(\mathcal{A}^e) = \{\{abd, cd\}^*cc, abc, \{abd, cd\}^*c\}$$

2 Decision problems

Extended automata are applicable to decision problems of semigroup amalgamations.

Result 1[5, Theorem 2.1]. Let S be a finite semigroup and $\mathcal{I}(S^1)$ the injective hull of the left S -set S^1 .

Then S has the representation extension property if and only if for any right S -set X_S , the canonical map $: X \rightarrow X \otimes_S \mathcal{I}(S^1)$ ($x \mapsto x \otimes 1$) is injective.

Result 2[6, the main theorem]. The decision problem whether a finite semigroup has the representation extension property or not is decidable.

Lemma. Let X be a right S -set and Y a left S -set. Then $x \otimes y = x' \otimes y'$ in $X \otimes Y$ if and only if there exist $s_1, \dots, s_n, t_1, \dots, t_n \in S^1$, $x_1, \dots, x_n \in X$ and $y_2, \dots, y_n \in Y$ such that

$$\begin{array}{ll} x &= x_1 s_1, & s_1 y &= t_1 y_2 \\ x_1 t_1 &= x_2 s_2, & s_2 y_2 &= t_2 y_3 \\ &\vdots & &\vdots \\ x_{n-1} t_{n-1} &= x_n s_n, & s_n y_n &= t_n y' \\ x_n t_n &= x' \end{array} \quad (1)$$

Then we call the system of equations (1) a scheme of length n X and Y joining (x, y) to (x', y') .

proposition. Let S be a finite regular semigroup. Then S is left absolutely flat if and only if for a $\mathcal{R}$ -class of S , a right S -set X and a left S -set Y , $x \otimes y = x' \otimes y'$ in $(xS \cup x'S) \otimes_S Y$

for all $x, x' \in X$ and all $y, y' \in Y$ such that there exists $s_i, t_i \in S$ and $x_i \in X, y_i \in Y$ such that

$$\begin{array}{rcl}
 x & = & x_1 s_1, & s_1 y & = & t_1 y_2 \\
 x_1 t_1 & = & x_2 s_2, & s_2 y_2 & = & t_2 y_3 \\
 & \vdots & & & & \vdots \\
 x_{n-1} t_{n-1} & = & x_n s_n, & s_n y_n & = & t_n y' \\
 x_n t_n & = & x' & & &
 \end{array} \tag{2}$$

and there exists some $1 \leq j \leq n$ such that

$$xS \subseteq x_1 t_1 S \subseteq \cdots \subseteq x_{j-1} t_{j-1} S \subseteq x_j t_j S^1 \supseteq x_{j+1} t_{j+1} S \supseteq \cdots \supseteq x_{n-1} t_{n-1} S \supseteq x' S.$$

Hereafter we call such a set of equations (2) a upward-convex scheme joining from (x, y) to (x', y') over X and Y .

Problem 1. The decision problem whether a finite semigroup is left absolutely flat or not.

an idea for a positive solution of the problem: There are a correspondence between schemes on tensor products of S -sets and paths from initial states and terminal states on finite automata.

The problem is reduced to decision problem whether or not the language of an automaton $\mathcal{A}$ are contained in the language of the extended automaton of a smaller automaton than $\mathcal{A}$ (by deleting some states!).

Result 3. [5, Theorem 2.2] and [4, Theorem 6.11]. A semigroup S is an amalgamation base for all semigroups if and only if for each a right S -set X , a left S -set Y and a S -biset N which is the injective hull of the S -biset S^1 ,

the map $: X \otimes Y \longrightarrow X \otimes N \otimes Y$ ($x \otimes y \longrightarrow x \otimes 1 \otimes y$) is injective.

Problem 2. The decision problem whether a finite semigroup is an amalgamation base for all semigroups or not.

an idea for a positive solution of the problem: There are a correspondence between schemes on tensor products of S -sets and paths from initial states and terminal states on finite automata.

The problem is reduced to decision problem whether or not the language of an automaton $\mathcal{A}$ are contained properly in the language of the extended automaton of a smaller finite automaton than $\mathcal{A}$ (by deleting some states!).

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