

An Equilibrium Theorem for Subdifferential

by

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1 Introduction

我々は ある制約条件のもとで目的関数 f を最小化する数理計画問題に対して、特に凸関数 $f: X \rightarrow \mathbb{R}$ における凸最適化問題を考える。このとき、この問題を考えることは 制約集合と目的関数により表された関数の劣微分が X の共役空間 X^* の null vector θ^* を含む問題を考えることに置き換えられる。従ってそのような問題を考えることは、凸最適化問題を考える上で重要な問題であり、ここでは $\theta^* \in \partial f(x)$ なる解 x の存在について考察する。

2 Equilibrium Theorem

この論文での 主定理を示すにおいて Ky Fan's Inequality が重要な役割を果たす。

Theorem 2.1 (Ky Fan's Inequality) *Let K be a w -compact convex subset of a Banach space X and $\varphi: X \times X \rightarrow \mathbb{R}$ be a function satisfying :*

- (i) $\forall y \in K, x \rightarrow \varphi(x, y)$ is w -lower semicontinuous ;
- (ii) $\forall x \in K, y \rightarrow \varphi(x, y)$ is concave ;
- (iii) $\varphi(y, y) \leq 0$, for all $y \in K$.

Then, there exists $\bar{x} \in K$ such that $\forall y \in K, \varphi(\bar{x}, y) \leq 0$.

ここで、 $\mathfrak{L}(X, X^*)$ は X から X^* への linear, bounded な関数の全体を表し、 $T_K(x)$ は x における K の tangent cone のことで、 $T_K(x) := \text{cl}(\cup_{h>0}(K-x)/h)$ であるとする。

Definition 2.1 $\partial f(x)$ is said to satisfy tangential condition with respect to $A \in \mathfrak{L}(X, X^*)$ and a subset $K \subset X$ if

$$\forall x \in K, \quad \partial f(x) \cap \text{cl}(AT_K(x)) \neq \phi. \quad (2.1)$$

ただし、

$$\partial f(x) := \{\xi \in X^* | f(x) - f(y) \leq \langle x - y, \xi \rangle, \text{ for all } y \in X\}.$$

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Theorem 2.2 K を Banach 空間 X の弱コンパクト凸集合とし, 実関数 $f : X \rightarrow \mathbb{R}$ を X 上で連続かつ凸な写像とする. さらに, $A \in \mathcal{L}(X, X^*)$ が次を満たすとする.

$$\forall x \in K, \quad \partial f(x) \cap \text{cl}(AT_K(x)) \neq \phi.$$

このとき,

$$\exists \bar{x} \in K, \quad \text{s.t.} \quad \theta_{X^*}^* \in \partial f(\bar{x}), \quad (2.2)$$

が成立する.

Proof. We proceed by contradiction, assuming that the conclusion is false. Hence, for any $x \in K$, $\theta_{X^*}^*$ does not belong to $\partial f(x)$. Since the sets $\partial f(x)$ are w^* -closed and convex, the Hahn-Banach Separation Theorem implies

$$\exists p_x \in X \setminus \{\theta_X\} \quad \text{such that} \quad \sigma(\partial f(x), p_x) < 0,$$

where θ_X is the null vector of X . We set $\Gamma_p := \{x \in X \mid \sigma(\partial f(x), p) < 0\}$. Then K is covered by the subsets Γ_p when p ranges over X . These subsets are weak open. So, K can be covered by n such weak open subsets Γ_{p_i} .

Let us consider a continuous partition of unity $\{\alpha_i\}_{i=1, \dots, n}$ associated with $\{\Gamma_{p_i}\}_{i=1, 2, \dots, n}$ and introduce the function $\varphi : K \times K \rightarrow \mathbb{R}$ defined by

$$\varphi(x, y) := \sum_{i=1}^n \alpha_i(x) \langle A^* p_i, x - y \rangle.$$

Being continuous with respect to x and affine with respect to y , the assumptions of Theorem 2.1 are satisfied. Hence there exists $\bar{x} \in K$ such that

$$\forall y \in K, \quad \varphi(\bar{x}, y) = \langle A^* p^*, \bar{x} - y \rangle \leq 0,$$

where we have set $p^* := \sum_{i=1}^n \alpha_i(\bar{x}) p_i$. In other words, $-A^* p^*$ belongs to the normal cone $N_K(\bar{x})$. The dual tangential condition implies that

$$\sigma(\partial f(\bar{x}), p^*) \geq 0.$$

But this inequality is false. To see that, we let I be the subset of the indices i such that $\alpha_i(\bar{x}) > 0$. I is non-empty since $\sum_{i=1}^n \alpha_i(\bar{x}) = 1$. If i belongs to I , then $\bar{x}$ belongs to Γ_{p_i} and consequently

$$\begin{aligned} \sigma(\partial f(\bar{x}), \bar{p}) &= \sigma(\partial f(\bar{x}), \sum_{i=1}^n \alpha_i(\bar{x}) p_i) \\ &\leq \sum_{i=1}^n \alpha_i(\bar{x}) \sigma(\partial f(\bar{x}), p_i) \\ &< 0. \end{aligned}$$

Thus, we have obtained a contradiction and prove our theorem. $\square$

Definition 2.2 X を Banach 空間とする. このとき, $x \in X$ に対して, X^* の部分集合

$$J(x) := \{x^* \in X^* \mid \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\} \quad (2.3)$$

を対応させる写像 J のことを *duality mapping* と呼ぶ.

Lemma 2.1 X を reflexive な Banach 空間とする. このとき, $\|x^*\| = 1$ である $x^* \in B^*$ に対して次が成立する.

$$T_{B^*}(x^*) = \bigcap_{y \in J^{-1}(x^*)} \{p \in X^* \mid \langle p, y \rangle \leq 0\}. \quad (2.4)$$

ただし, B^* は X^* の unit ball であり, J^{-1} は duality mapping J の逆写像である.

Proof. For any $v^* \in T_{B^*}(x^*)$, there exists a sequence of elements $v_n^* \in (\cup_{h>0}(B^* - x^*)/h)$ converging to v^* . Hence, for any n , there exists $h_n > 0$ and $b_n^* \in B^*$ such that $v_n^* = (b_n^* - x^*)/h_n$. Since $\langle v_n^*, y \rangle \leq 0$ for any $y \in J^{-1}(x^*)$, $\langle v^*, y \rangle \leq 0$ for any $y \in J^{-1}(x^*)$. Hence,

$$v^* \in \bigcap_{y \in J^{-1}(x^*)} \{p \in X^* \mid \langle p, y \rangle \leq 0\}.$$

Assume that there exists $w_0^* \notin T_{B^*}(x^*)$ such that $\langle w_0^*, y \rangle \leq 0$ for any $y \in J^{-1}(x^*)$. Since the sets $T_{B^*}(x^*)$ are closed and convex, the Hahn-Banach Separation Theorem implies

$$\exists z \in X \setminus \{\theta\}, \quad \exists a \in \mathbb{R}, \quad \text{s.t.} \quad \langle w_0^*, z \rangle > a > \langle v^*, z \rangle, \quad \forall v^* \in T_{B^*}(x^*).$$

So, we have z belongs to the normal cone $N_{B^*}(x^*)$ and $a > 0$. We set $z' := z/\|z\|$, then we have $z' \in J^{-1}(x^*)$. Hence,

$$\langle w_0^*, z' \rangle > \frac{a}{\|z\|} > 0.$$

However from assumption $\langle w_0^*, z' \rangle \leq 0$. Thus, we have obtained a contradiction and proved our theorem. $\square$

Theorem 2.3 K を reflexive Banach 空間 X の閉凸集合とし, 実関数 $f: X \rightarrow \mathbb{R}$ を X 上で連続かつ凸な写像とし, このとき次の 1~3 を満たす $A \in \mathfrak{L}(X, X^*)$ が存在したとする:

1. $\forall x \in K, \quad \partial f(x) \cap \text{cl}(AT_K(x)) \neq \emptyset$;
2. $A(B_X) = B^*$;
3. A^{-1} exists .

更に 次の仮定

$$\limsup_{\substack{\|x\| \rightarrow \infty \\ x \in K}} \sup_{y \in J^{-1}(Ax)} f'(x; y) < 0, \quad (2.5)$$

を満たしていれば, (2.2) が成立する. ここで, $f'(x; d)$ は f の x での d 方向の方向微分であり, $f'(x; d) := \lim_{h \rightarrow 0+} \frac{f(x+hd) - f(x)}{h}$ である.

Proof. Assumption (2.5) implies that there exists $\varepsilon > 0$ and $a > 0$ such that

$$\sup_{\|Ax\| \geq a} \sup_{y \in J^{-1}(Ax)} \sigma(\partial f(x), y) \leq -\varepsilon, \quad \text{and} \quad AK \cap \text{int}(A(aB)) \neq \emptyset. \quad (2.6)$$

By Lemma 2.1, we know that for any Ax belongs to $A(aB)$ with $\|Ax\| = a$ then,

$$T_{A(aB)}(Ax) = \bigcap_{y \in J^{-1}(Ax)} \{p \in X^* \mid \langle p, y \rangle \leq 0\}. \quad (2.7)$$

Hence, from (2.6) and (2.7), it follows that $\forall Ax \in AK \cap A(aB)$,

$$\partial f(x) \subset \text{cl}(AT_{aB}(x)) = T_{A(aB)}(Ax). \quad (2.8)$$

Next, since θ_{X^*} belong to $\text{int}(K + aB)$ from (2.6), we know that

$$\forall Ax \in AK \cap A(aB), \quad T_{AK \cap A(aB)}(Ax) = T_A K(Ax) \cap T_{A(aB)}(Ax).$$

Hence, by assumption 3

$$\forall x \in K \cap aB, \quad \partial f(x) \cap \text{cl}(AT_{K \cap aB}(x)) \neq \emptyset.$$

So, $K \cap aB$ satisfies the tangential condition (2.1) and obviously to prove that convex and w-compact set.

Hence, by Theorem 2.2 there exists a solution $\bar{x} \in K$ of the inclusion $\theta^* \in \partial f(\bar{x})$. $\square$

Corollary 2.1 X を Hilbert 空間, 実関数 $f: X \rightarrow \mathbb{R}$ は X 上で連続かつ凸である写像とし, $A \in \mathcal{L}(X, X^*)$ は次を満たしているとする:

$$\limsup_{\|x\| \rightarrow \infty} f'(x; Ax) < 0. \quad (2.9)$$

このとき (2.2) が成立する.

References

- [1] J.P. AUBIN, *Optima and Equilibria*, Springer-Verlag 1993.
- [2] J.-P. AUBIN AND I. EKKELAND, *Applied Nonlinear Analysis*, A Wiley-Interscience Publication 1984.
- [3] J.-P. AUBIN AND A. CELLINA, *Differential Inclusion*, Springer-Verlag, Grundlehren der math 1984.
- [4] J.P. AUBIN AND H. FRANKOWSKA, *Set-Valued Analysis*, Birkhäuser Boston 1990.
- [5] V. BARBU AND TH. PRECUPANU, *Convexity and Optimization in Banach Spaces*, Editura Academiei, Bucharest, Romania 1986.
- [6] W.W. HOGAN, Point-To-Set Maps In Mathematical Programming *SIAM* **15**(3) 1973, 591-603.
- [7] D.G. LUENBERGER, *Optimization by Vector Space Methods*, John Wiley & Sons, inc., 1969.
- [8] R.T. ROCKAFELLAR, *Convex Analysis*, Princeton University Press., 1970.