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QUANTIZATION OF INFORMATION THEORY

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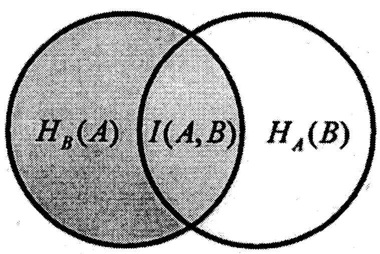
Abstract: In scope of CP-convexity theory for C*-algebras (quantization of convexity, measure, entropy for completely positive maps), we investigate the operational structure of quantum interactions of entangled systems, and propose new information quantities which naturally generalize the classical information theory.

1. Introduction

It is well known that in quantum information theory we do not have the natural generalization of classical information quantities, such as joint entropy, mutual entropy, conditional entropies. For example, consider an entangled pure state with marginal entropies $H(A)$ and $H(B)$, then the joint entropy $H(A,B)$ satisfies $H(A,B)=0<H(A),H(B)$ which is impossible in the classical information theory. Moreover, the mutual entropy $I(A,B)$ is customarily defined by the relation $I(A,B)=H(A)+H(B)-H(A,B)$, so in this case $I(A,B)=2H(A)>H(A)$, which does not happen in the classical theory. Also, note that the conditional entropy is defined by $H_B(A)=H(A)-I(A,B)$ in the classical case, but in this case $H_B(A)=-H(A)<0$ which would be unacceptable. These situations are illustrated as follows:

Classical case

$H(A)$ $H(B)$

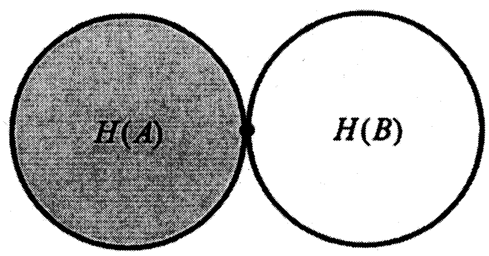


$H(A,B)$

$H(A,B)\geq H(A),H(B)$
$I(A,B)=H(A)+H(B)-H(A,B)\leq H(A),H(B)$
$H_B(A)=H(A)-I(A,B)\geq 0$

Quantum case

Entangled pure state



$H(A,B)=0$

$H(A,B)<H(A),H(B)$??
$I(A,B)=2H(A)>H(A)$??
$H_B(A)=-H(A)<0$??

The purpose of this note is to define a new informational joint entropy $H(A,B)$ so that it should include the information from the entanglement of the compound system, it is symmetric with respect to A and B , and it satisfies the inequality $H(A,B) \leq H(A)+H(B)$. Once $H(A,B)$ is constructed, then the natural generalization of other entropies would automatically follow.

Note that there exists one-to-one correspondence between a normal state ω on the compound system $B(K)\otimes B(H)$ and a normal completely positive map φ_ω from $B(K)$ to $T(H)$ such that $\omega(a\otimes b)=\mathrm{Tr}(\varphi_\omega(a)^t b)$ for $a\in B(K), b\in B(H)$. Therefore, our scheme can be reduced to find an appropriate definition of the *entropy* of the completely positive map φ_ω. Recall that the notion of entropy is closely related with those of *probability* or *measure*, i.e., the decomposition into extreme elements in convexity theory. Then the question is: "What is the set of *extreme* elements of the set of CP-maps? How can a CP-map be decomposed into those extreme elements? This problem has been solved by introducing *CP-convexity* and *CP-measure and integration theory* in [1-3].

2. Preliminaries

Recall that every completely positive map ψ from a C*-algebra A to $B(H)$ is represented as $\psi(a)=V^*\pi(a)V$ $(a\in A)$, $\pi\in\mathrm{Rep}(A)$, $V\in B(H,H_\pi)$. We denote the set of all CP-maps from a C*-algebra A to $B(H)$ by $CP(A,B(H))$, and the set of all contractive ones by $Q_H(A)$, called the *CP-state space* of A for H. We say that $\psi\in CP(A,B(H))$ is a *CP-convex combination* of $\psi_i\in CP(A,B(H))$ if

$$\psi=\sum_i S_i^*\psi_i S_i \quad \text{with } S_i\in B(H) \text{ such that } \sum_i S_i^* S_i = I_H,$$

which we shall abbreviate by $\psi=\mathrm{CP}\text{-}\sum_i S_i^*\psi_i S_i$.

Definition. (i) A CP-state $\psi\in Q_H(A)$ is define to be *CP-extreme* if $\psi=\mathrm{CP}\text{-}\sum_i S_i^*\psi_i S_i$ implies that ψ_i is unitarily equivalent to ψ. We denote the set of all CP-extreme states by $D_H(A)$.

(ii) A CP-state $\psi\in Q_H(A)$ is defined to be a *conditionally CP-extreme state* if $\psi=\mathrm{CP}\text{-}\sum_i S_i^*\psi_i S_i$ with $S_i\geq 0$ implies that $\psi_i=\psi$. We denote the set of all conditionally CP-extreme states by $D_H^c(A)$.

Theorem. (i) *If H is infinite dimensional, then* $D_H(A) = \mathrm{Irr}(A:H)$, *and if H is finite dimensional, then* $D_H(A) = \{\psi = u^*\pi u \in Q_H(A);\ \pi \in \mathrm{Irr}(A),\ u^*u = I_H \text{ or } uu^* = I_{H_\pi}\}$.

(ii) $D_H^c(A) = \{\psi = u^*\pi u \in Q_H(A);\ \pi \in \mathrm{Irr}(A),\ u^*u = p_\psi$ (the support projection of ψ)$\}$.

Note that $D_H(A) \subset D_H^c(A)$. If $A = B(K)$, then an operation (a normal contractive CP-map) $\psi \in Q_H(B(K))_n$ is CP-extreme iff ψ is a *unitary transform*, i.e., $\psi = U^* \cdot U$ with a unitary U, and ψ is conditionally CP-extreme iff ψ is a *conditional transform*, i.e., $\psi = u^* \cdot u$ with a partial isometry u.

Suppose that an operation $\psi \in Q_H(B(K))_n$ has a Kraus representation $\psi = \sum_i V_i^* \cdot V_i$ with $V_i \in B(H,K)$, and let $V_i = u_i|V_i|$ be the polar decomposition of V_i, then we have the extreme decomposition $\psi = \sum_i |V_i| u_i^* \cdot u_i |V_i|$ where $u_i^* \cdot u_i \in D_H^c(B(K))$ with positive coefficients $|V_i| \geq 0$. From the analogy of scalar convexity theory, we can consider that ψ is represented by an "operation valued measure" $\lambda_\psi = \{|V_i| \cdot |V_i|\}$ supported by $\{u_i^* \cdot u_i\}$. In fact, we developed *CP-measure* (the continuous version of the above atomic case) and integration theory to show that every CP-state is represented by a CP-measure supported by the CP-extreme states $D_H(A)$ (cf. [2]).

Now let $\varphi = \sum_i v_i^* \cdot v_i \in CP(B(K),T(H))$ with $\sum_i v_i^* v_i = \rho \in T(H)_1$ (density operators), and have a CP-extreme decomposition

$$\varphi = \sum_i |v_i| u_i^* \cdot u_i |v_i| = \sum_i \lambda_i |\tilde{v}_i| u_i^* \cdot u_i |\tilde{v}_i| \text{ with } \lambda_i = \mathrm{Tr}\, v_i^* v_i \text{ and } \tilde{v}_i = \lambda_i^{-1/2} v_i$$

where $\lambda_i > 0, \sum_i \lambda_i = 1$ and $\tilde{v}_i^* \tilde{v}_i = \rho_i \in T(H)_1$. We shall denote by λ_φ the CP-measure corresponding to the above CP-decomposition.

Definition. (i) Let $S^L(\lambda_\varphi) := -\sum_i \lambda_i \ln \lambda_i$, and define $S^L(\varphi) := \inf_{\lambda_\varphi} S^L(\lambda_\varphi)$ which we call the *Lindblad entropy* of φ (cf. [5]).

(ii) Let $E(\lambda_\varphi) := -\sum_i \lambda_i S(\rho_i)$ be the *entanglement* of λ_φ, and $E(\varphi) := \inf_{\lambda_\varphi} E(\lambda_\varphi)$ is called the *entanglement of formation* of φ.

(iii) Let $S^{op}(\lambda_\varphi) := -\sum_i \mathrm{Tr}\, v_i^* v_i \ln v_i^* v_i = -\sum_i \lambda_i \ln \lambda_i + \sum_i \lambda_i S(\tilde{v}_i^* \tilde{v}_i) = S^L(\lambda_\varphi) + E(\lambda_\varphi)$, and then we define $S^{op}(\varphi) := \inf_{\lambda_\varphi} S^{op}(\lambda_\varphi)$ to be the *operator entropy* of φ.

Note that $S^L(\varphi)=S(\omega)$ where ω is the entangled state corresponding to φ, and $E(\varphi)=0$ iff ω is separable, and also that $S^{op}(\varphi)\geq S^L(\varphi)+E(\varphi)$. We remark that $S^L(\varphi)$ is concave, $E(\varphi)$ is convex, and $S^{op}(\varphi)$ is neither concave nor convex with respect to φ in general.

To consider the analogy in the classical theory, let $\rho=\sum_k \mu_k P_k$ be the spectral decomposition of ρ, and define $D(\varphi^*):=\inf\sum_k \mu_k S(\varphi^*(\rho^{-1/2}P_k\rho^{-1/2}))$ to be the *dissemination* of φ^*, where inf is taken over all decompositions of ρ. (Similarly, we can define the dissemination of φ in the inverse direction through the spectral decomposition of $\hat{\rho}=\varphi^*(1)=\sum_i v_i v_i^*$.)

3. Construction of new informational quantities

We shall define the informational joint entropy $H(A,B)$ such that

1. It includes the information from the entanglement of the entangled system.
2. It is symmetric with respect to A and B.
3. It satisfies the inequality $H(A,B)\leq H(A)+H(B)$.

We first note that some candidates in the classical theory do not work here in the quantum case. That is, $S^L(\varphi)$, $S(\rho)+D(\varphi^*)$, $S^{op}(\varphi)$ all represent $H(A,B)$ in the classical theory, but $S^L(\varphi)$ does not satisfy 1, $S(\rho)+D(\varphi^*)$ does not satisfy 2, and $S^{op}(\varphi)$ may not satisfy 3. (Actually, we cannot show a counterexample for the last case, but we can easily find some cases where there exists a CP-decomposition such that $S^{op}(\lambda_\varphi)>S(\rho)+S(\hat{\rho})$)).

Let us consider again the decomposition of φ,

$\varphi=\sum_i v_i^*\cdot v_i=\sum_i |v_i|u_i^*\cdot u_i|v_i|=\sum_i \lambda_i|\tilde{v}_i|u_i^*\cdot u_i|\tilde{v}_i|$ with $\lambda_i=\mathrm{Tr}\,v_i^*v_i$ and $\tilde{v}_i=\lambda_i^{-1/2}v_i$

and let $\rho_i:=\tilde{v}_i^*\tilde{v}_i=\sum_j \alpha_{ij}P_{ij}$, $\hat{\rho}_i:=\tilde{v}_i\tilde{v}_i^*=\sum_j \alpha_{ij}\hat{P}_{ij}$ be the spectral decomposition of ρ_i, and $\hat{\rho}_i$ respectively, and set $u_{ij}:=u_iP_{ij}$.

Definition. Let $\varphi_{\lambda_\varphi}^{at}:=\sum_{ij}\lambda_i\alpha_{ij}u_{ij}^*\cdot u_{ij}$ (i.e., the quantum pure operation $u_i^*\cdot u_i$ being replaced by atomic ones), and define $S^{at}(\varphi):=\inf_{\lambda_\varphi} S^L(\varphi_{\lambda_\varphi}^{at})$ which will be called the *atomic entropy* of φ.

$S^{al}(\varphi)$ satisfies the requirements above, i.e., it includes the information both of the Lindblad entropy and the entanglement of formation of the bipartite system, symmetric with respect to A and B, and satisfies the inequality $H(A,B) \leq H(A)+H(B)$. We now propose to set $H(A,B) := S^{al}(\varphi)$ and $I(A,B) = H(A)+H(B)-H(A,B)$. We can then recover the desired inequalities such as

Theorem. $H(A),\ H(B) \leq H(A,B) \leq H(A)+H(B)$ and $I(A,B) \leq H(A),\ H(B)$.

The proof depends on some properties of separable states due to [4], [6].

4. Properties of the atomic entropy $S^{al}(\varphi)$

It would be desirable to discuss our arguments in the framework of quantum interactions which generate the entanglement, but to maintain this note in a reasonable size we take other way to use the technique of purification.

Thus, for the entangled state ω on $H_A \otimes H_B$ (where we set $H_A = H,\ H_B = K$), there exists a Hilbert space H_C and a pure state P_ς in $H_A \otimes H_B \otimes H_C$, such that $\mathrm{Tr}_{H_C} P_\varsigma = \omega$ and $\mathrm{Tr}_{H_A \otimes H_B} P_\varsigma = \rho^L$, where note that $\rho^L \cong \omega$. Let $\rho^L = \sum_i \lambda_i Q_i$ be the decomposition corresponding to the decomposition of $\varphi_\omega = \varphi$ in the previous section. Let z_i, x_{ij} be the supporting unit vectors of the one dimensional projections of Q_i and P_{ij} respectively, and set $y_{ij} := u_{ij} x_{ij}$, and define

$$\varsigma := \sum_{ij} \lambda_i \alpha_{ij}\, x_{ij} \otimes y_{ij} \otimes z_i \ \text{(generating vector for } \varphi)$$
$$\Delta_{\lambda_\varphi} := \sum_{ij} \lambda_i \alpha_{ij}\, P_{x_{ij} \otimes y_{ij} \otimes z_i} \in T(H_A \otimes H_B \otimes H_C)_1$$

The following results show the relation between the operator entropy $S^{op}(\varphi)$ and the atomic entropy $S^{al}(\varphi)$.

Theorem. (i) $\varphi^{al}_{\lambda_\varphi} = \mathrm{Tr}_{H_C} \Delta_{\lambda_\varphi}$

(ii) $S^{op}(\lambda_\varphi) = S(\Delta_{\lambda_\varphi}) \geq S(\varphi^{al}_{\lambda_\varphi})$, so that $S^{op}(\varphi) \geq S^{al}(\varphi)$

We finally give an interpretation of $S^{al}(\varphi)$ in statistical approach. Assume now that ω, hence φ is pure with $\mathrm{Tr}_{H_B}\, \omega = \rho$ and $\rho = \sum_j \alpha_j P_j$ be a spectral decomposition of ρ. In the sense of correlation, it would be reasonable to consider an *atomic approximation* of ω, i.e., $\omega \sim \sum_j \alpha_j P_{x_j \otimes y_j}$ (atomic approximation). Next, let $\omega = \sum_i \lambda_i \omega_i$ as before, for each pure ω_i, we have atomic approximations $\omega_i \sim \sum_j \alpha_{ij} P_{x_{ij} \otimes y_{ij}}$. Then in the sense of correlation, ω has an atomic approximation $\omega \sim \sum_{ij} \lambda_i \alpha_{ij} P_{x_{ij} \otimes y_{ij}}$, which can be considered as a joint distribution with marginal distributions $\rho = \sum_{ij} \lambda_i \alpha_{ij} P_{x_{ij}}$ and $\hat{\rho} = \sum_{ij} \lambda_i \alpha_{ij} P_{y_{ij}}$. Thus our definition of joint entropy $S^{al}(\omega) = \inf_{\lambda_i} S(\sum_{ij} \lambda_i \alpha_{ij} P_{x_{ij} \otimes y_{ij}})$ can be considered as the best atomic approximation in view of statistical correlation in quantum interactions.

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