

Some topics in L^p -theory for second-order elliptic operators with unbounded coefficients

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1. Introduction

In this paper we deal with the second-order elliptic operators of the form

$$(1.1) \quad Au(x) := -\operatorname{div}(a(x)\nabla u(x)) + F(x) \cdot \nabla u(x) + V(x)u(x), \quad x \in \mathbb{R}^N,$$

where $N \in \mathbb{N}$ and the coefficients (a, F, V) satisfy the following condition:

(A1) ${}^t a = a \in C^1(\mathbb{R}^N; \mathbb{R}^{N \times N})$, $F \in C^1(\mathbb{R}^N; \mathbb{R}^N)$, $V \in L^\infty_{\text{loc}}(\mathbb{R}^N; \mathbb{R})$ and $a(x)$ is positive-definite for every $x \in \mathbb{R}^N$, that is, $\langle a(x)\xi, \xi \rangle > 0$ for every $x \in \mathbb{R}^N$, $\xi \in \mathbb{C}^N \setminus \{0\}$.

Here $\langle \cdot, \cdot \rangle$ is the usual Hermitian product. Under condition (A1) we define the minimal and maximal realization of A in $L^p = L^p(\mathbb{R}^N)$ ($1 < p < \infty$) respectively as

$$\begin{cases} A_{p,\min} u := Au, \\ D(A_{p,\min}) := C_0^\infty(\mathbb{R}^N), \\ A_{p,\max} u := Au, \\ D(A_{p,\max}) := \{u \in L^p \cap W_{\text{loc}}^{2,p}(\mathbb{R}^N); Au \in L^p\}. \end{cases}$$

Our interest is the following properties of $A_{p,\min}$ and $A_{p,\max}$:

- essential m -accretivity of $A_{p,\min}$
- m -accretivity of $A_{p,\max}$
- m -sectoriality of $A_{p,\max}$

(see e.g., Goldstein [7]). These properties are strongly related to the evolution equation

$$(1.2) \quad \frac{du}{dt} + A_{p,\max} u = 0, \quad t \in (0, \infty), \quad u(0) = u_0.$$

The m -accretivity of $A_{p,\max}$ gives solvability of (1.2) and the m -sectoriality of $A_{p,\max}$ implies the smoothing effect of solutions to (1.2), which may be expected for equations of parabolic type. Here we only discuss the essential m -accretivity of $A_{p,\min}$.

As is well-known, second-order elliptic operators appear in the theories of non-relativistic quantum mechanics and stochastic analysis. In particular, some important

models of them are written by using the operator A in (1.1) with unbounded coefficients. For instance, in non-relativistic quantum mechanics the Schrödinger operators

$$Su(x) = -\Delta u(x) + V(x)u(x), \quad x \in \mathbb{R}^N$$

($a = (\delta_{jk})_{jk}$ and $F \equiv 0$, where δ_{jk} is the Kronecker delta) describe the motion of a quantum mechanical particle under the potential V . On the other hand, in stochastic analysis the Ornstein-Uhlenbeck operators

$$A_{OU}u(x) = -\Delta u(x) + Bx \cdot \nabla u(x), \quad x \in \mathbb{R}^N$$

($a = (\delta_{jk})_{jk}$, $F = (\sum_{j=1}^N B_{jk}x_j)_k$ and $V \equiv 0$) describe the process of random variables, where $B = (B_{jk})_{jk}$ is an $N \times N$ -matrix. Our interest is the m -accretivity of operators which have the differential expression A .

There exist many investigations dealing with these problems for uniformly elliptic operators with bounded coefficients (see e.g., Kato [9, Example V.3.34], Fattorini [5, Chapter 3], Lunardi [12, Chapter 3] and their references).

For unbounded coefficients, the Schrödinger operators $-\Delta + V$ have been considered in many previous works (see e.g., Kato [8, 11], Simon [18], Semenov [17], Okazawa [15, 16] and others). The operators A with unbounded diffusion and drift are also dealt with (see e.g., Cupini-Fornaro [3], Metafuno-Pallara-Prüss-Schnaubelt [13] and Fornaro-Lorenzi [6]).

Here we describe recent results for the (essential) m -accretivity of $A_{p,\min}$ and $A_{p,\max}$ with unbounded diffusions. In Eberle [4], it is shown that under

$$\frac{\langle a(x)x, x \rangle}{|x|^2} \leq \alpha(|x| \log(e + |x|))^2, \quad x \in \mathbb{R}^N \setminus \{0\},$$

the operator $A_{p,\min}$ with $F \equiv 0$ and $V \equiv 0$ is essentially m -accretive in L^p . In Metafuno-Pallara-Rabier-Schnaubelt [14], they proved the essential m -accretivity of $A_{p,\min}$ under the following conditions: there exist $\rho \in C^N(\mathbb{R}^N)$ satisfying $\lim_{|x| \rightarrow \infty} \rho(x) = \infty$ and $|\nabla \rho| \neq 0$ a.e. on $\mathbb{R}^N$ and constants $s, s' > 0$ satisfying $0 < s' < s$ and

$$(1.3) \quad V - \frac{\operatorname{div} F}{p} \geq 0;$$

$$(1.4) \quad V - \frac{\operatorname{div} F}{p} \geq -s \left(\langle F, \nabla \rho \rangle - \left(1 - \frac{2}{p}\right) \operatorname{div}(a \nabla \rho) \right) + s^2 \langle a \nabla \rho, \nabla \rho \rangle;$$

$$(1.5) \quad e^{-p's'\rho} \langle F, \nabla \rho \rangle \in L^\infty(\mathbb{R}^N);$$

$$(1.6) \quad e^{-p's'\rho} \langle a \nabla \rho, \nabla \rho \rangle \in L^\infty(\mathbb{R}^N).$$

They also deal with the m -accretivity of $\tilde{A}_{p,\min}$ under a similar restriction as (1.3)–(1.6) mentioned above. Although their result enables us to deal with general coefficients if $p = 2$, a certain restriction on the derivative of the diffusion a is required when $p \neq 2$ (see condition (1.4)). Because of this gap between $p = 2$ and $p \neq 2$, it seems to be unnatural from the view point of L^p -generalization.

In [19], the m -accretivity of $A_{p,\max}$ and coincidence of $A_{p,\max}$ and the closure of $A_{p,\min}$ are proved if the coefficients a , F and V satisfy that there exists a nonnegative auxiliary function $\Psi_p \in L_{\text{loc}}^\infty(\mathbb{R}^N; [0, \infty))$ such that

$$(1.7) \quad \frac{\langle a(x)x, x \rangle}{|x|^2} \leq (1 + \Psi_p(x))^{1-\frac{2}{r}} f(|x|)^2, \quad x \in \mathbb{R}^N \setminus B_R,$$

$$(1.8) \quad \frac{|\langle F(x), x \rangle|}{|x|} \leq (1 + \Psi_p(x))^{1-\frac{1}{r}} f(|x|), \quad x \in \mathbb{R}^N \setminus B_R,$$

$$(1.9) \quad V(x) - \frac{\text{div} F(x)}{p} \geq \Psi_p(x), \quad x \in \mathbb{R}^N$$

with $f(r) = r \log r$ for constants $R > 1$ and $r \in [2, \infty)$, where B_R is the N -dimensional ball with center at the origin and radius R . More generally, the case where

$$f \in \mathcal{F}_R := \left\{ f \in C([R, \infty); (0, \infty)) ; \int_R^\infty f(s)^{-1} ds = \infty \right\}$$

is dealt with in [20]; note that the case where $V \equiv 0$ and $\Psi_p \equiv 0$ is proved in [4, Theorem 2.3] and $\mathcal{F}_R$ essentially appears in its proof. The optimality of $\mathcal{F}_R$ is shown in [2, Example 3.5]. This result may be regarded as a natural generalization from $p = 2$ to $p \neq 2$. In [19], the following view point is crucial.

Proposition 1.1 (see [19, Section 1]). *Assume that (A1) is satisfied. Then for every $1 \leq q \leq \infty$, $w \in W_{\text{loc}}^{1,1}(\mathbb{R}^N)$ and $\psi \in C_0^\infty(\mathbb{R}^N)$,*

$$(1.10) \quad \begin{aligned} \int_{\mathbb{R}^N} (A\psi)\bar{w} dx &= \int_{\mathbb{R}^N} \left[\langle a\nabla\psi, \nabla w \rangle + \left(V - \frac{\text{div} F}{q} \right) \psi\bar{w} \right] dx \\ &\quad + \int_{\mathbb{R}^N} \left[\frac{1}{q'} \langle \bar{w}\nabla\psi, F \rangle - \frac{1}{q} \langle \psi\nabla\bar{w}, F \rangle \right] dx, \end{aligned}$$

where $q' = \frac{q}{q-1}$ (the Hölder conjugate of q).

The equality (1.10) may be regarded as a generalization in L^p of decomposition formula for sesquilinear form in L^2 into symmetric and anti-symmetric parts.

Recently, in [21], the endpoint case $r = \infty$ of [19] is discussed under an additional condition similar to the oscillation condition $|\nabla\Psi_p|^2 \leq \gamma\Psi_p^3$ (see (1.15) below).

On the other hand, in Kato [10], the essential selfadjointness of the Schrödinger operators $(-\text{div}(a\nabla\cdot) + V)_{2,\min}$ with the following coefficients is posed:

$$(K) \quad \begin{cases} \frac{\langle a(x)x, x \rangle}{|x|^2} \leq k(1 + |x|)^{\ell+2}, & x \in \mathbb{R}^N \setminus B_R, \\ V(x) \geq c|x|^\ell, & x \in \mathbb{R}^N, \end{cases}$$

with $k, c, \ell > 0$. This problem is partially solved in [14] under the additional condition $c > \ell^2/4$. In the case $c < \ell^2/4$, the negative answer (counterexample) is given in [22] which is written in L^p -framework.

The first purpose of this paper is to prove the assertions of the essential m -accretivity of $A_{p,\min}$ in [19] and [21] via a unified approach. The second is to give a summary of answer to Kato's selfadjointness problem and its proof in L^2 -framework (which is simpler than [22]).

Here we introduce the main assumption of this paper (almost the same setting as (1.7)–(1.9)).

(A2) There exist constants $\alpha, \beta > 0, r \in [2, \infty], R > 0$ and a nonnegative auxiliary function $\Psi_p \in L^\infty_{\text{loc}}(\mathbb{R}^N; \mathbb{R})$ such that

$$(1.11) \quad \frac{\langle a(x)x, x \rangle}{|x|^2} \leq \alpha(1 + \Psi_p(x))^{1-\frac{2}{r}} (|x| \log |x|)^2 \quad \text{a.a. } x \in \mathbb{R}^N \setminus B_R;$$

$$(1.12) \quad \frac{\langle F(x), x \rangle}{|x|} \leq \beta(1 + \Psi_p(x))^{1-\frac{1}{r}} (|x| \log |x|) \quad \text{a.a. } x \in \mathbb{R}^N \setminus B_R;$$

$$(1.13) \quad V - \frac{\operatorname{div} F}{p} \geq \Psi_p \quad \text{a.e. on } \mathbb{R}^N.$$

Now we are in a position to state our main result. The first theorem is the assertion for essential m -accretivity in the case where $r \in [2, \infty)$.

Theorem 1.1 ([19, Theorem 1.1]). *Let $1 < p < \infty$. Assume that (A1) and (A2) are satisfied with $r \in [2, \infty)$. Then $A_{p,\min}$ is essentially m -accretive in L^p , that is,*

$$(1.14) \quad \operatorname{Re} \int_{\mathbb{R}^N} (A_{p,\min} u) \bar{u} |u|^{p-2} dx \geq 0 \quad \forall u \in D(A_{p,\min}), \quad \overline{R(1 + A_{p,\min})} = L^p,$$

where $R(1 + A_{p,\min})$ is the range of $1 + A_{p,\min}$.

The second is the assertion for essential m -accretivity in the endpoint case $r = \infty$ of Theorem 1.1.

Theorem 1.2 ([21, Theorem 1.1]). *Let $1 < p < \infty$. Assume that (A1) and (A2) are satisfied with $r = \infty$. Assume further that $\Psi_p \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^N)$ and there exists $\gamma_p > \frac{p-1}{4}$ such that*

$$(1.15) \quad \Psi_p \geq \frac{1}{p'} \frac{\langle F, \nabla \Psi_p \rangle}{1 + \Psi_p} + \gamma_p \frac{\langle a \nabla \Psi_p, \nabla \Psi_p \rangle}{(1 + \Psi_p)^2} \quad \text{a.e. on } \mathbb{R}^N.$$

Then $A_{p,\min}$ is essentially m -accretive in L^p .

The conditions (1.13) and (1.15) in Theorem 1.2 can be replaced with a weaker condition.

Theorem 1.3. *Let $1 < p < \infty$. Assume that (A1) and (1.11) and (1.12) are satisfied with $r = \infty$. Assume further that $\Phi_p \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^N)$ and there exists $\gamma_p > \frac{p-1}{4}$ such that*

$$(1.16) \quad V - \frac{\operatorname{div} F}{p} \geq \max \left\{ 0, \frac{1}{p'} \frac{\langle F, \nabla \Psi_p \rangle}{1 + \Psi_p} + \gamma_p \frac{\langle a \nabla \Psi_p, \nabla \Psi_p \rangle}{(1 + \Psi_p)^2} \right\} \quad \text{a.e. on } \mathbb{R}^N.$$

Then $A_{p,\min}$ is essentially m -accretive in L^p .

The last theorem is a summary of the answer to Kato's selfadjointness problem.

Theorem 1.4 ([21, Theorem 3.1] and [22, Theorem 1.1], $p = 2$). *The following assertions hold:*

- (i) *If (a, V) satisfies (A1) and (K) with $c > \frac{k\ell^2}{4}$, then $A_{2,\min}$ is essentially selfadjoint.*
- (ii) *If $N \geq 5$ and $c_0 < \frac{k_0\ell^2}{4}$, then there exists a pair (a, V) such that (a, V) satisfies (A1) and (K) with $k = k_0$ and $c = c_0$ and $A_{2,\min}$ is not essentially selfadjoint.*

The plan of this paper is as follows. Theorems 1.1, 1.2 are proved in Section 2 via a unified approach. In Section 3, We prove Theorem 1.4 (i) and (ii). The proof of (i) is based on Theorem 1.3 and the other is a simplified version of that in [22].

2. Proofs of Theorems 1.1 and 1.3

First we show that $A_{p,\min}$ is accretive in L^p (the first part of (1.14)).

Proof of Theorems 1.1 and 1.3 (accretivity). Let $u \in C_0^\infty(\mathbb{R}^N)$. If $2 \leq p < \infty$, then taking the real part of (1.10) with $q = p$, $w = |u|^{p-2}u$ and $\psi = u$, we see from (A1) and $V - \frac{\operatorname{div} F}{p} \geq 0$ that

$$\begin{aligned} \operatorname{Re} \int_{\mathbb{R}^N} (Au) \bar{u} |u|^{p-2} dx &= (p-1) \int_{\mathbb{R}^N} |u|^{p-4} \langle a \operatorname{Re}(\bar{u} \nabla u), \operatorname{Re}(\bar{u} \nabla u) \rangle dx \\ &\quad + \int_{\mathbb{R}^N} |u|^{p-4} \langle a \operatorname{Im}(\bar{u} \nabla u), \operatorname{Im}(\bar{u} \nabla u) \rangle dx \\ &\quad + \int_{\mathbb{R}^N} \left(V - \frac{\operatorname{div} F}{p} \right) |u|^p dx \geq 0. \end{aligned}$$

If $1 < p < 2$, then we use (1.10) with $q = p$, $w = (|u|^2 + \varepsilon)^{\frac{p-2}{2}}u$ ($\varepsilon > 0$) and $\psi = u$. Letting $\varepsilon \downarrow 0$, we obtain the accretivity of $A_{p,\min}$ for $1 < p < 2$. $\square$

Next we prove the (essential) maximality of $A_{p,\min}$, that is, $R(1 + A_{p,\min})$ (the range of $1 + A_{p,\min}$) is dense in L^p . We only prove the case $2 \leq p' < \infty$ ($1 < p \leq 2$) in order to avoid the complicated computation. The case $1 < p' < 2$ can be verified via a procedure similar to the other case with ε -regularization as in the proof of accretivity (see [19, Theorem 1.1], [20, Theorem 1.1] and [21, Theorem 1.1]).

Here we need the following lemma.

Lemma 1. *Let $v \in L^{p'}$ be real-valued. Assume that (A1) and*

$$(2.1) \quad \int_{\mathbb{R}^N} v(\varphi + A\varphi) dx = 0 \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N).$$

Then $v \in H_{\text{loc}}^1(\mathbb{R}^N) \cap C(\mathbb{R}^N)$. Moreover, if $\Phi \in W^{1,\infty}(\mathbb{R}^N)$ has a compact support in $\mathbb{R}^N$, then

$$\begin{aligned} \int_{\mathbb{R}^N} \left[(p'-1) \langle a \nabla v, \nabla v \rangle \Phi |v|^{p-2} + \langle a \nabla v, \nabla \Phi \rangle v |v|^{p-2} \right] dx \\ + \int_{\mathbb{R}^N} \left[\frac{1}{p'} \langle F, \nabla \Phi \rangle + \left(1 + V - \frac{\operatorname{div} F}{p} \right) \Phi \right] |v|^{p'} dx = 0. \end{aligned}$$

Proof. Using the elliptic regularity (see e.g., Agmon [1, Lemma 5.1]) iteratively, we see $v \in H_{\text{loc}}^1(\mathbb{R}^N) \cap C(\mathbb{R}^N)$. Then using (1.10) with $q = p$ and $w = v$ and $\psi = \varphi$, we deduce

$$\int_{\mathbb{R}^N} \left[\langle a \nabla v, \nabla \varphi \rangle + \frac{1}{p'} \langle F, v \nabla \varphi \rangle - \frac{1}{p} \langle F, \varphi \nabla v \rangle + \left(1 + V - \frac{\text{div} F}{p} \right) v \varphi \right] dx = 0.$$

The above equality is verified even for $\varphi \in H^1(\mathbb{R}^N)$ with a compact support. Here we choose $\varphi = \Phi v |v|^{p'-2}$. Then noting that

$$\frac{1}{p'} v \nabla \varphi - \frac{1}{p} \varphi \nabla v = \frac{1}{p'} \nabla \Phi |v|^{p'},$$

we obtain the desired assertion. $\square$

Proof of Theorem 1.1 (maximality). Assume (2.1) for $v \in L^{p'}(\mathbb{R}^N)$. It suffices to prove that $v = 0$ a.e. on $\mathbb{R}^N$. We may assume without loss of generality that v is real-valued. We take the cut-off functions $\{\zeta_n\}_n \subset W^{1,\infty}(\mathbb{R}^N)$ as

$$\zeta_n(x) := \begin{cases} 1 & \text{if } |x| < \exp \exp n, \\ 0 & \text{if } |x| > \exp \exp(n+1), \\ n+1 - \log \log |x| & \text{otherwise} \end{cases}$$

for $n \in \mathbb{N}$ and $x \in \mathbb{R}^N$. Applying Lemma 1 with $\Phi = \zeta_n^r$, we deduce that

$$\begin{aligned} & (p' - 1) \int_{K_n} \zeta_n^r \langle a \nabla v, \nabla v \rangle |v|^{p'-2} dx + r \int_{K_n \setminus K_{n-1}} \zeta_n^{r-1} \langle a \nabla v, \nabla \zeta_n \rangle |v|^{p'-2} v dx \\ & + \frac{r}{p'} \int_{K_n \setminus K_{n-1}} \zeta_n^{r-1} \langle F, \nabla \zeta_n \rangle |v|^{p'} dx + \int_{K_n} \zeta_n^r \left(1 + V - \frac{\text{div} F}{p} \right) |v|^{p'} dx = 0, \end{aligned}$$

where $K_n := \text{supp } \zeta_n$. By the Cauchy-Schwarz and Young inequalities, we have

$$\begin{aligned} (2.2) \quad \int_{K_n} \zeta_n^r \left(1 + V - \frac{\text{div} F}{p} \right) |v|^{p'} dx & \leq \frac{r^2}{4(p' - 1)} \int_{K_n \setminus K_{n-1}} \zeta_n^{r-2} \langle a \nabla \zeta_n, \nabla \zeta_n \rangle |v|^{p'} dx \\ & - \frac{r}{p'} \int_{K_n \setminus K_{n-1}} \zeta_n^{r-1} \langle F, \nabla \zeta_n \rangle |v|^{p'} dx. \end{aligned}$$

On the other hand, note that

$$\nabla \zeta_n(x) = \begin{cases} \frac{-x}{|x| \log |x|} & \text{if } x \in K_n \setminus K_{n-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus it follows from (1.11), (1.12) of the condition (A2) and Young's inequality that there exist constants $C_1, C_2 > 0$ such that for every $n \geq \log \log R$,

$$\begin{aligned} \zeta_n^{r-2} \langle a \nabla \zeta_n, \nabla \zeta_n \rangle & = \frac{\zeta_n^{r-2} \langle a(x)x, x \rangle}{|x|^4 (\log |x|)^2} \leq \alpha \zeta_n^{r-2} (1 + \Psi_p)^{1-\frac{2}{r}} \leq \frac{2(p' - 1)}{r^2} (C_1 + \zeta_n^r (1 + \Psi_p)), \\ -\zeta_n^{r-1} \langle F, \nabla \zeta_n \rangle & = \frac{\zeta_n^{r-1} \langle F(x), x \rangle}{|x|^2 \log |x|} \leq \beta (1 + \Psi_p)^{1-\frac{1}{r}} \zeta_n^{r-1} \leq \frac{p'}{2r} (C_2 + \zeta_n^r (1 + \Psi_p)). \end{aligned}$$

Therefore, combining (2.2), (1.13) and the above estimates, we have

$$\int_{K_n} \zeta_n^r (1 + \Psi_p) |v|^{p'} dx \leq (C_1 + C_2) \int_{K_n \setminus K_{n-1}} |v|^{p'} dx + \int_{K_n \setminus K_{n-1}} \zeta_n^r (1 + \Psi_p) |v|^{p'} dx.$$

Consequently, we see that

$$\int_{K_{n-1}} |v|^{p'} dx \leq (C_1 + C_2) \int_{K_n \setminus K_{n-1}} |v|^{p'} dx \rightarrow 0$$

as $n \rightarrow \infty$. This implies that $v = 0$ a.e. on $\mathbb{R}^N$, that is, $R(1 + A_{p,\min})$ is dense in L^p . $\square$

Proof of Theorem 1.3 (maximality). Assume (2.1) for real-valued function $v \in L^{p'}$. As in the proof of Theorem 1.1, we prove that $v = 0$ a.e. on $\mathbb{R}^N$. Applying Lemma 1 with $\Phi = \Theta_p^{-1} \zeta_n^2$ ($\Theta_p := 1 + \Psi_p$), we deduce that

$$(2.3) \quad (p' - 1) \int_{K_n} \frac{\zeta_n^2 \langle a \nabla v, \nabla v \rangle |v|^{p'-2}}{\Theta_p} dx + 2 \int_{K_n \setminus K_{n-1}} \frac{\zeta_n \langle a \nabla v, \nabla \zeta_n \rangle |v|^{p'-2} v}{\Theta_p} dx \\ - \int_{K_n} \frac{\zeta_n^2 \langle a \nabla v, \nabla \Psi_p \rangle |v|^{p'-2} v}{\Theta_p^2} dx - \frac{1}{p'} \int_{K_n \setminus K_{n-1}} \frac{\zeta_n^2 \langle F, \nabla \Psi_p \rangle |v|^{p'}}{\Theta_p^2} dx \\ + \frac{2}{p'} \int_{K_n \setminus K_{n-1}} \frac{\zeta_n \langle F, \nabla \zeta_n \rangle |v|^{p'}}{\Theta_p} dx + \int_{K_n} \frac{\zeta_n^2}{\Theta_p} \left(1 + V - \frac{\operatorname{div} F}{p} \right) |v|^{p'} dx = 0.$$

By the Cauchy-Schwarz and Young inequalities, we have

$$\int_{K_n} \frac{\zeta_n^2}{\Theta_p} \left(1 + V - \frac{\operatorname{div} F}{p} - \frac{1}{p'} \frac{\langle F, \nabla \Psi_p \rangle}{\Theta_p} - \gamma_p \frac{\langle a \nabla \Psi_p, \nabla \Psi_p \rangle}{\Theta_p^2} \right) |v|^{p'} dx \\ = -(p' - 1) \int_{K_n} \frac{\zeta_n^2 \langle a \nabla v, \nabla v \rangle |v|^{p'-2}}{\Theta_p} dx - \gamma_p \int_{K_n} \frac{\zeta_n^2 \langle a \nabla \Psi_p, \nabla \Psi_p \rangle |v|^{p'}}{\Theta_p^3} dx \\ - \int_{K_n} \frac{\zeta_n^2 \langle a \nabla v, \nabla \Psi_p \rangle |v|^{p'-2} v}{\Theta_p^2} dx - 2 \int_{K_n \setminus K_{n-1}} \frac{\zeta_n \langle a \nabla v, \nabla \zeta_n \rangle |v|^{p'-2} v}{\Theta_p} dx \\ - \frac{2}{p'} \int_{K_n \setminus K_{n-1}} \frac{\zeta_n^2 \langle F, \nabla \zeta_n \rangle |v|^{p'}}{\Theta_p} dx \\ \leq - \left(p' - 1 - \frac{1}{4\gamma_p} \right) \int_{K_n} \frac{\zeta_n^2 \langle a \nabla v, \nabla v \rangle |v|^{p'-2}}{\Theta_p} dx \\ - 2 \int_{K_n \setminus K_{n-1}} \frac{\zeta_n \langle a \nabla v, \nabla \zeta_n \rangle |v|^{p'-2} v}{\Theta_p} dx - \frac{2}{p'} \int_{K_n \setminus K_{n-1}} \frac{\zeta_n \langle F, \nabla \zeta_n \rangle |v|^{p'}}{\Theta_p} dx \\ \leq \left(p' - 1 - \frac{1}{4\gamma_p} \right)^{-1} \int_{K_n \setminus K_{n-1}} \frac{\langle a \nabla \zeta_n, \nabla \zeta_n \rangle |v|^{p'}}{\Theta_p} dx - \frac{2}{p'} \int_{K_n \setminus K_{n-1}} \frac{\zeta_n \langle F, \nabla \zeta_n \rangle |v|^{p'}}{\Theta_p} dx.$$

Therefore we see from (1.11), (1.12) and (1.16) that

$$\int_{K_n} \frac{\zeta_n^2}{\Theta_p} |v|^{p'} dx \leq \left[\alpha \left(p' - 1 - \frac{1}{4\gamma_p} \right)^{-1} + \frac{2\beta}{p'} \right] \int_{K_n \setminus K_{n-1}} |v|^{p'} dx \rightarrow 0$$

as $n \rightarrow \infty$. This implies that $v = 0$ a.e. on $\mathbb{R}^N$. This completes the proof. $\square$

Remark 2.1. The difference between the proof of Theorem 1.1 and one of Theorem 1.1 is only the choice of sequence of Φ . If $r \in [2, \infty)$, then we do not need to assume the differentiability of Ψ_p . On the other hand, if $r = \infty$, then the differentiability of Ψ_p is required.

3. Proof of Theorem 1.4

Proof of Theorem 1.4 (i). To apply Theorem 1.2 with $p = 2$, we put

$$\Psi_2(x) := \max\{1, c|x|^\ell\} - 1.$$

Then we see from (K) that for every $x \in \mathbb{R}^N$ satisfying $|x| > c^{-1/\ell}$,

$$\begin{aligned} \frac{\langle a(x)\nabla\Psi_2(x), \nabla\Psi_2(x) \rangle}{(1 + \Psi_2(x))^2} &= \ell^2 \frac{\langle a(x)x, x \rangle |x|^{2\ell-4}}{c^2|x|^{2\ell}} \\ &\leq k\ell^2|x|^\ell \\ &\leq \frac{k\ell^2}{c}\Psi_p(x). \end{aligned}$$

Therefore if $c > k\ell^2/4$, then Theorem 1.2 is applicable to (a, V) , that is, $A_{2,\min}$ is essentially m -accretive in L^2 . Since $A_{2,\min}$ is symmetric, we obtain the essential selfadjointness of $A_{2,\min}$. $\square$

To give a clear proof, we show Theorem 1.4 (ii) for $\ell \in (0, 2]$ (For the other case, see [22]). Before starting, we give a strategy of the proof. The proof is divided into the following three parts.

Step 1. We consider the Schrödinger operators

$$(3.1) \quad B = -\Delta_y + \frac{\lambda}{y_N^2} + W(y) \quad \text{in } \mathbb{R}_+^N := \mathbb{R}^{N-1} \times (0, \infty)$$

with $0 \leq W \in C(\overline{\mathbb{R}_+^N})$ and prove that $B_{2,\min}$ (B defined on $C_0^\infty(\mathbb{R}_+^N)$) is nonnegative but not essentially selfadjoint in $L^2(\mathbb{R}_+^N)$ when $\lambda \in (-\frac{1}{4}, \frac{3}{4})$.

Step 2. Using diffeomorphism $\Phi : \mathbb{R}^N \rightarrow \mathbb{R}^N$, we translate the operator B in $\mathbb{R}_+^N$ into an operator A in $\mathbb{R}^N$; remark that A is not essentially selfadjoint in $L^2(\mathbb{R}^N)$.

Step 3. We construct (a, V) such that (K) is satisfied with $k = k_0$ and $c = c_0$ and corresponding operator $A_{2,\min}$ is not essentially selfadjoint in $L^2(\mathbb{R}^N)$.

3.1. Step 1

Lemma 2. Assume $-\frac{1}{4} < \lambda < \frac{3}{4}$. Then $B_{2,\min}$ is nonnegative but not essentially selfadjoint in $L^2(\mathbb{R}_+^N)$.

Proof. (nonnegativity) Let $v \in C_0^\infty(\mathbb{R}_+^N)$. Then by integration by parts and the one-dimensional Hardy inequality (with respect to y_N), we have

$$\begin{aligned} \int_{\mathbb{R}_+^N} (B_{2,\min} v) \bar{v} \, dy &= \int_{\mathbb{R}_+^N} |\nabla v|^2 \, dy + \lambda \int_{\mathbb{R}_+^N} \frac{|v|^2}{y_N^2} \, dy \\ &\geq \int_{\mathbb{R}_+^N} \left| \frac{\partial v}{\partial y_N} \right|^2 \, dy + \lambda \int_{\mathbb{R}_+^N} \frac{|v|^2}{y_N^2} \, dy + \int_{\mathbb{R}_+^N} W |v|^2 \, dy \\ &\geq \left(\lambda + \frac{1}{4} \right) \int_{\mathbb{R}_+^N} \frac{|v|^2}{y_N^2} \, dy + \int_{\mathbb{R}_+^N} W |v|^2 \, dy \\ &\geq 0. \end{aligned}$$

Therefore we can find the Friedrichs extension B_F of $B_{2,\min}$. We remark that $D(B_F) \subset D(1/y_N)$. More precisely,

$$\left(\lambda + \frac{1}{4} \right) \int_{\mathbb{R}_+^N} \frac{|v|^2}{y_N^2} \, dy \leq \|v\|_{L^2(\mathbb{R}_+^N)} \|B_F v\|_{L^2(\mathbb{R}_+^N)} \quad \forall v \in D(B_F).$$

(Non-selfadjointness) First we prove Fix $\eta \in C_0^\infty(\mathbb{R}; [0, 1])$ such that $\eta(s) = 1$ for $|s| \leq 1$ and $\eta(s) = 0$ for $|s| \geq 2$ and set

$$\psi(y) = (y_N)^{\frac{1}{2} - \sqrt{\lambda + \frac{1}{4}}} \prod_{j=1}^N \eta(y_j).$$

Then noting that $\lambda \in (-\frac{1}{4}, \frac{3}{4})$, we have $\psi \in L^2(\mathbb{R}_+^N)$ and $y_N^{-1} \psi \notin L^2(\mathbb{R}_+^N)$ and therefore $\psi \notin D(B_F)$. Define

$$f := \psi + B\psi.$$

Then noting that $(-\frac{d^2}{ds^2} + \frac{\lambda}{s^2}) s^{\frac{1}{2} - \sqrt{\lambda + \frac{1}{4}}} = 0$, we have

$$|B\psi(y)| \leq C \left(1 + (y_N)^{\frac{1}{2} - \sqrt{\lambda + \frac{1}{4}}} \right) \chi_{(-2,2)^{N-1} \times (0,1)}$$

and hence $f \in L^2(\mathbb{R}_+^N)$. Setting $\psi_W := \psi - (1 + B_F)^{-1} f \notin D(B_F)$, we have $\psi_W + B\psi_W = 0$. This yields

$$\int_{\mathbb{R}_+^N} (v + B_{2,\min} v) \psi_W \, dx = \int_{\mathbb{R}_+^N} v (\psi_W + B\psi_W) \, dx = 0.$$

Hence $B_{2,\min}$ is not essentially selfadjoint in $L^2(\mathbb{R}_+^N)$. □

3.2. Step 2

The following elementary lemma gives a transform of operators in $\mathbb{R}^N$ into one in $\mathbb{R}_+^N$.

Lemma 3. Let $\Phi \in C^\infty(\mathbb{R}_+^N; \mathbb{R}^N)$ be a diffeomorphism with $\det D\Phi(y) = y_N^{1/2}$. Set $J : C_0^\infty(\mathbb{R}^N) \rightarrow C_0^\infty(\mathbb{R}_+^N)$ as

$$(3.2) \quad Ju(y) := y_N^{1/4} u(\Phi(y)).$$

Then J can be extended to an isometry from $L^2(\mathbb{R}^N)$ to $L^2(\mathbb{R}_+^N)$ and for every $u \in C_0^\infty(\Omega_2)$,

$$(3.3) \quad -\operatorname{div}(a^\Phi \nabla u) = J^{-1} \left(-\Delta - \frac{1}{4y_N^2} \right) Ju,$$

where $a^\Phi \in C^\infty(\Omega_2, \mathbb{R}^{N^2})$ is defined as $a^\Phi(x) = (D\Phi D\Phi^*)(\Phi^{-1}(x))$.

3.3. Step 3

Now we consider the operator

$$A^\Phi u = -\operatorname{div}(a^\Phi \nabla u) + c|x|^\ell u,$$

where Φ is determined later. Then we have

Lemma 4. Condition **(K)** for a^Φ is equivalent to the following condition:

$$(3.4) \quad \frac{1}{4} |\nabla |\Phi(y)|^2|^2 \leq k(1 + |\Phi(y)|)^{\ell+2} |\Phi(y)|^2 \quad \text{if } |\Phi(y)| \geq R.$$

Now we introduce a suitable Φ . We define $\Phi \in C^\infty(\mathbb{R}_+^N; \mathbb{R}^N)$ as follows: for $y = (y', y_N)$ with $y' \in \mathbb{R}^{N-1}$ and $y_N > 0$,

$$(3.5) \quad \Phi_j(y) := \left(\frac{(y_N)^{1/2}}{F'(y_N)} \right)^{\frac{1}{N-1}} y', \quad \Phi_N(y) := F(y_N).$$

Here $F \in C^\infty(\mathbb{R}_+; \mathbb{R})$ satisfies $F'(t) > 0$ and

$$(3.6) \quad F(t) = \begin{cases} -\left(\frac{\ell}{2}t\right)^{-\frac{2}{\ell}} & \text{if } 0 < t < \frac{2}{\ell}, \\ t^{3/2} & \text{if } \frac{4}{\ell} < t < \infty. \end{cases}$$

Remark 3.1. In view of (3.4), the choice of $\Phi_N = F$ may be essential because of

$$\frac{1}{4} \left| \frac{d}{dt} |F(t)|^2 \right|^2 = |F(t)|^{\ell+4}, \quad t \in \left(0, \frac{2}{\rho-2} \right).$$

This property will be used in Lemma 5.

Next we verify **(K)** for a^Φ with precise constant $k > 0$.

Lemma 5. If $N \geq 5$, then there exists $R_0 > 0$ such that a^Φ satisfies **(K)** with $k = 1$.

Proof. Observe that $N \geq 3 + 2\beta$ yields $4\tilde{\beta} \leq 1$ for every $\rho > 2$ and hence $k_0 = 1$. Therefore we only prove the general case.

By virtue of Lemma 4, it suffices to prove **(K)** that (3.4) holds with $k = 1$ and for some R_0 . By the definition of Φ , we see that

$$(3.7) \quad \nabla|\Phi(y)|^2 = 2 \begin{pmatrix} \left(\frac{(y_N)^{1/2}}{F'(y_N)} \right)^{\frac{1}{N-1}} \hat{\Phi}(y) \\ \frac{1}{N-1} \left(\frac{1}{2y_N} - \frac{F''(y_N)}{F'(y_N)} \right) |\hat{\Phi}(y)|^2 + F'(y_N) \Phi_N(y) \end{pmatrix},$$

where $\hat{\Phi}(y) = (\Phi_1(y), \dots, \Phi_{N-1}(y))$. Here we prove (3.4) by dividing its proof into three cases:

(The case $y_N \geq \frac{4}{\ell}$). In this case $F(y_N) = (y_N)^{\beta+1}$ and hence we have

$$(3.8) \quad \nabla|\Phi(y)|^2 = 2 \begin{pmatrix} \left(\frac{2}{3} \right)^{\frac{1}{N-1}} \hat{\Phi}(y) \\ \frac{3}{2} |\Phi_N(y)|^{\frac{4}{3}} \end{pmatrix},$$

Therefore there exists $R_1 > 0$ such that (3.4) holds with $k = 1$ if $|\Phi(y)| \geq R_1$.

(The case $\frac{2}{\ell} \leq y_N < \frac{4}{\ell}$). In this case, F' , $1/F'$ and F'' are uniformly bounded on $[\frac{2}{\rho-2}, \frac{4}{\rho-2}]$. Hence

$$\frac{1}{2} |\nabla|\Phi(y)|^2| \leq C_1 |\Phi(y)| + C_2 |\hat{\Phi}(y)|^2 \leq (C_1 + C_2 |\Phi(y)|) |\Phi(y)|,$$

where

$$C_1 = \max \left\{ \sup_{t \in [\frac{2}{\ell}, \frac{4}{\ell}]} \left(\frac{t^{a/2}}{F'(t)} \right)^{\frac{1}{N-1}}, \sup_{t \in [\frac{2}{\ell}, \frac{4}{\ell}]} F'(t) \right\},$$

$$C_2 = \frac{1}{N-1} \sup_{t \in [\frac{2}{\ell}, \frac{4}{\ell}]} \left| \frac{1}{2t} - \frac{F''(t)}{F'(t)} \right|.$$

Thus there exists $R_2 > 0$ such that (3.4) holds with $k = 1$ if $|\Phi(y)| \geq R_2$.

(The case $0 < y_N < \frac{2}{\ell}$). Observe that

$$F'(t) = |F(t)|^{\frac{\ell+2}{2}}, \quad F''(t) = -\frac{\rho}{2} |F(t)|^{\ell+1}.$$

Hence

$$\nabla|\Phi(y)|^2 = 2 \begin{pmatrix} \left(\frac{\ell}{2} \right)^{-\frac{1}{2(N-1)}} |\Phi_N(y)|^{-\beta} \hat{\Phi}(y) \\ \beta |\Phi_N(y)|^{\frac{\ell}{2}} |\hat{\Phi}(y)|^2 - 2 |\Phi_N(y)|^{2+\frac{\ell}{2}} \end{pmatrix},$$

where $\beta = \frac{3\ell+4}{N-1}$. Noting that $|\Phi_N(y)| \geq 1$, we have

$$\left(\frac{\ell}{2}\right)^{-\frac{1}{2(N-1)}} |\Phi_N(y)|^{-\beta} |\hat{\Phi}(y)| \leq \left(\frac{\ell}{2}\right)^{-\frac{1}{2(N-1)}} |\Phi(y)|.$$

On the other hand, putting $|\Phi_N(y)| = |\Phi(y)|\sqrt{s}$ and $|\Phi_j(y)'| = |\Phi(y)|\sqrt{1-s}$ ($s \in [0, 1]$), we have

$$\frac{\partial}{\partial x_N} |\Phi(y)|^2 = 2|\Phi(y)|^{\frac{\ell}{2}+1} \left(\beta s^{\frac{\ell-2}{4}} (1-s) - s^{\frac{2+\ell}{4}} \right).$$

By the standard computation, we have

$$\left| \beta s^{\frac{\ell-2}{4}} (1-s) - s^{\frac{2+\ell}{4}} \right| \leq 1.$$

Hence we can choose $R_3 > 0$ such that (3.4) holds with $k = 1$ if $|\phi(y)| \geq R_3$. Consequently, taking $R_0 = \max\{R_1, R_2, R_3\}$, we have (3.4) with $k = k_0$ and $R = R_0$. $\square$

Next we define

$$W_\ell(y) := |\Phi(y)|^\ell, \quad y \in \mathbb{R}_+^N.$$

Then we have

Lemma 6. *Let $\ell > 0$. Then there exists $M > 0$ depending only on ℓ such that if $y \in \mathbb{R}_+^N$ satisfies $y_N \leq \frac{2}{\ell}$, then*

$$(3.9) \quad \frac{4}{\ell^2} \frac{1}{(y_N)^2} \leq W_\ell(y) \leq \frac{4}{\ell^2} \frac{1}{(y_N)^2} + M(y_N)^{\frac{3\ell+4}{2(N-1)}} |y'|^2.$$

Proof. By the definition of Φ and F , we have

$$W_\ell(y) = \left(\left(\frac{\ell}{2}\right)^{\frac{2(\ell+2)}{\ell(N-1)}} (y_N)^{\frac{1}{N-1} + \frac{2(\ell+2)}{\ell(N-1)}} |y'|^2 + \left(\frac{\ell}{2} y_N\right)^{-\frac{4}{\ell}} \right)^{\frac{\ell}{2}}.$$

Noting that $\ell \leq 2$, the triangle inequality yields

$$\begin{aligned} \left(\frac{\ell}{2} y_N\right)^{-2} &\leq W_\ell(y) \leq \left(\frac{\ell}{2} y_N\right)^{-2} + \left(\frac{\ell}{2}\right)^{\frac{\ell+2}{N-1}} (y_N)^{\frac{3\ell+4}{2(N-1)}} |y'|^2 \\ &\leq \frac{4}{\ell^2} \frac{1}{(y_N)^2} + \left(\frac{\ell}{2}\right)^{\frac{\ell+2}{N-1}} (y_N)^{\frac{\ell(\beta+1)+2}{N-1}} |y'|^2. \end{aligned}$$

This completes the proof. $\square$

Proof of Theorem 1.4 (ii). Setting $\widetilde{W}_\ell \geq 0$ as

$$\widetilde{W}_\ell(y) := W_\ell(y) - \frac{4}{\ell^2} \frac{1}{y_N^2} \in C(\overline{\mathbb{R}_+^N}),$$

we deduce

$$A^\Phi u = J^{-1} B^\Phi J u$$

with

$$B^\Phi = -\Delta + \left(\frac{4c}{\ell^2} - \frac{1}{4} \right) \frac{1}{y_N^2} + c\widetilde{W}_\ell(y).$$

Applying Lemma 2 to $\lambda = \frac{4c}{\ell^2} - \frac{1}{4} \in (-\frac{1}{4}, \frac{3}{4})$ and $W = c\widetilde{W}_\ell$, we see that $B_{2,\min}^\Phi$ is not essentially selfadjoint in $L^2(\mathbb{R}_+^N)$. Since J is an isometry, $A_{2,\min}^\Phi$ is not essentially selfadjoint in $L^2(\mathbb{R}^N)$. This completes the proof. $\square$

References

- [1] S. Agmon, *The L_p approach to the Dirichlet problem, I: Regularity theorems*, Ann. Scuola Norm. Sup. Pisa III **13** (1959), 405–448.
- [2] E. B. Davies, *L^1 properties of second order elliptic operators*, Bull. London Math. Soc. **17** (1985), 417–436.
- [3] G. Cupini, S. Fornaro, *Maximal regularity in $L^p(\mathbb{R}^N)$ for a class of elliptic operators with unbounded coefficients*, Differential Integral Equations **17** (2004), 259–296.
- [4] A. Eberle, “Uniqueness and Non-Uniqueness of Semigroups Generated by Singular Diffusion Operators,” Lecture Notes in Mathematics **1718**, Springer-Verlag, Berlin, 1999.
- [5] H.O. Fattorini, *On the angle of dissipativity of ordinary and partial differential operators*, Functional Analysis, Holomorphy and Approximation Theory. II (Rio de Janeiro, 1981), Math. Studies, vol. **86**, North-Holland, Amsterdam, 1984.
- [6] S. Fornaro, L. Lorenzi, *Generation results for elliptic operators with unbounded diffusion coefficients in L^p - and C_b -spaces*, Discrete Contin. Dyn. Syst. **18** (2007), 747–772.
- [7] J. A. Goldstein, “Semigroups of Linear Operators and Applications,” Oxford Math. Monogr., The Clarendon Press, Oxford University Press, New York, 1985.
- [8] T. Kato, *Schrödinger operators with singular potentials*, Israel J. Math. **13** (1972), 135–148.
- [9] T. Kato, “Perturbation theory for linear operators,” Die Grundlehren der mathematischen Wissenschaften, Band **132** Springer-Verlag New York, Inc., New York 1966.
- [10] T. Kato, *Remarks on the selfadjointness and related problems for differential operators*, Spectral theory of differential operators (Birmingham, Ala., 1981), 253–266, Math. Stud., **55**, North-Holland, Amsterdam-New York, 1981.
- [11] T. Kato, *L^p -theory for Schrödinger operators with a singular potential*, Aspects of Positivity in Functional Analysis (Tübingen, 1985), 63–78, Math. Stud., **122**, North-Holland, Amsterdam, 1986.

- [12] A. Lunardi, "Analytic semigroups and optimal regularity in parabolic problems," Birkhauser/Springer Basel AG, Basel, 1995.
- [13] G. Metafune, D. Pallara, J. Prüss, R. Schnaubelt, *L^p -theory for elliptic operators on $\mathbb{R}^d$ with singular coefficients*, Z. Anal. Anwendungen **24** (2005), 497–521.
- [14] G. Metafune, D. Pallara, P. J. Rabier, R. Schnaubelt, *Uniqueness for elliptic operators on $L^p(\mathbb{R}^N)$ with unbounded coefficients*, Forum Math. **22** (2010), 583–601.
- [15] N. Okazawa, *An L^p theory for Schrödinger operators with nonnegative potentials*, J. Math. Soc. Japan **36** (1984), 675–688.
- [16] N. Okazawa, *L^p -theory of Schrödinger operators with strongly singular potentials*, Japan. J. Math. **22** (1996), 199–239.
- [17] Yu. A. Semenov, *Schrödinger operators with L^p_{loc} -potentials*, Comm. Math. Phys. **53** (1977), 277–284.
- [18] B. Simon, *Essential self-adjointness of Schrödinger operators with positive potentials*, Math. Ann. **201** (1973), 211–220.
- [19] M. Sobajima, *L^p -theory for second-order elliptic operators with unbounded coefficients*, J. Evol. Equ. **12** (2012), 957–971.
- [20] M. Sobajima, *L^p -theory for second-order elliptic operators with unbounded drift*, Sūrikaiseikikenkyūsho Kōkyūroku **1856** (2013), 75–88.
- [21] M. Sobajima, *L^p -theory for second-order elliptic operators with unbounded coefficients in an endpoint class*, J. Evol. Equ. **14** (2014), 461–475.
- [22] M. Sobajima, *On the threshold for Kato's selfadjointness problem and its L^p -generalization*, Evol. Equ. Control Theory **3** (2014), 699–711.