Tropical geometry and algebraic cycles

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博　士　論　文

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（トロピカル幾何学と代数的サイクル）

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To my family.

Abstract

In this thesis, we study tropical geometry and its relation with algebraic cycles. Tropical geometry, a relatively new area of mathematics, is a combinatorial shadow of algebraic geometry. Several kinds of problems of algebraic cycles are proved by reducing them to tropical analogs, e.g., calculations of Gromov-Witten invariants.

In the first part of this thesis, we give a tropical characterization of closed algebraic subvarieties of toric varieties over non-archimedean fields among all Zariski closed analytic subvarieties. This is a converse of the celebrated result of Bieri-Groves.

The second and third parts are devoted to a very challenging program. We propose a tropical approach to some of the most important problems on algebraic cycles, i.e., some problems on algebraic classes of cohomology groups, such as the Hodge conjecture. As a part of this approach, we prove a tropical analog of the Hodge conjecture for smooth algebraic varieties over trivially valued fields. To prove this, in the second part of this paper, inspired by Liu's construction of tropical cycle class maps by Milnor K-groups, we introduce tropical analogs of Milnor K-groups and show the exactness of the Gersten complexes for the Zariski sheaf cohomology of the sheaves of them on smooth algebraic varieties over trivially valued fields. Then the tropical analog of the Hodge conjecture follows from that the tropical cohomology groups for smooth algebraic varieties over trivially valued fields are isomorphic to the Zariski sheaf cohomology groups of the sheaves of tropical analogs of Milnor K-groups. In the third part of this paper, we prove these isomorphisms of cohomology groups by a theorem for general "cohomology theories", developed by many mathematicians, e.g., Quillen, and explicit calculations of tropical cohomology of the trivial line bundle by non-archimedean geometry. Usually, non-archimedean geometry of height 1 valuations is used to study tropical geometry. To study tropical analogs of Milnor K-groups and tropical cohomology, we introduce tropicalizations of valuations of higher heights.

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Contents

CHAPTER 1

Introduction

1.1. Introduction

In this thesis, we study tropical geometry and its relation with algebraic cycles. Tropical geometry is a combinatorial shadow of algebraic geometry and deals with certain weighted balanced polyhedral complexes. Tropical geometry is used to study many areas of mathematics, for examples, birational geometry, linear series of algebraic curves, non-archimedean arakelov geometry, and the Gross-Siebert program in mirror symmetry. Among these many kinds of applications, one of the most important applications is those to algebraic cycles. Tropical geometry enables us to reduce some problems of algebraic cycles to combinatorial problems, which are easier to deal with, e.g., Mikhalkin's combinatorial descriptions of the Gromov-Witten invariants of toric surfaces. In this thesis, we study tropical geometry and its relation with algebraic cycles further. This thesis consists of three parts based on three papers [**M18**], [**M20-1**], and [**M20-2**].

In the first part of this thesis, we give a tropical characterization of closed algebraic subvarieties of toric varieties over complete non-archimedean valuation fields among Zariski closed analytic subvarieties. In 1984, Bieri and Groves showed that tropicalizations of closed algebraic subvarieties of tori over non-archimedean fields are finite unions of polyhedra. This is a fundamental result of tropical geometry. In 2007, Gubler showed that tropicalizations of Zariski closed analytic subvarieties of tori over complete non-archimedean fields are locally finite unions of polyhedra. In the first part of this thesis, we show that if the tropicalizations of Zariski closed analytic subvarieties of toric varieties over complete non-archimedean fields are finite unions of polyhedra, they are algebraic subvarieties. This is a converse of Bieri-Groves' theorem and a generalization of Chow's lemma over complete non-archimedean fields (Remark 5.1.3).

Theorem 1.1.1 (Theorem 5.1.1). *Let $X \subset T^{\mathrm{Ber}}$ be a Zariski closed analytic subvariety of the Berkovich analytification T^{Ber} of a toric variety T over a complete non-archimedean field of height 1. Assume that the tropicalization $\mathrm{Trop}(X)$ is a finite union of polyhedra. Then X is the Berkovich analytification of a Zariski closed algebraic subvariety of T.*

The second and third parts are devoted to a very challenging program. We propose a tropical approach to some of the most important problems of algebraic cycles, i.e., some problems of algebraic classes of cohomology groups such as the Hodge conjecture, the Tate conjecture, and Grothendieck's standard conjectures. We remind two conjectures which are especially related to this thesis.

Conjecture 1.1.2 (the Hodge conjecture). *Let X be a smooth projective variety over $\mathbb{C}$. Then, for any p, the image of the cycle class map*

$$\mathrm{CH}^p(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to H^{2p}_{\mathrm{sing}}(X, \mathbb{Q})$$

is the group of Hodge classes $H^{2p}_{\mathrm{sing}}(X,\mathbb{Q})\cap H^{p,p}(X)$*. Here,* $\mathrm{CH}^p(X)$ *is the Chow group of algebraic subvarieties of codimension* p*.*

By Hodge theory, the Hodge conjecture means the following corollary.

CONJECTURE 1.1.3 (Grothendieck's standard conjecture D). *Let* X *be a smooth projective variety over* $\mathbb{C}$*. Then, for any* p*, the kernel of the cycle class map*

$$\mathrm{CH}_p(X)\otimes_{\mathbb{Z}}\mathbb{Q}\to H^{\mathrm{sing}}_{2p}(X,\mathbb{Q})$$

is the $\mathbb{Q}$*-vector subspace of algebraic cycles* V *of numerically equivalent to* 0*, i.e., for any algebraic subvariety* $W\subset X$ *of codimension* p*, the intersection number* (V,W) *is* 0*.*

The goal of the second and third parts of this thesis is to prove a tropical analog of the Hodge conjecture for smooth algebraic varieties over trivially valued fields.

THEOREM 1.1.4 (Theorem 7.1.1). *Let* X *be a smooth algebraic variety over a trivially valued field. Then, for any* $p\geq 0$*, Liu's tropical cycle class map*

$$\mathrm{CH}^p(X)\otimes_{\mathbb{Z}}\mathbb{Q}\to H^{p,p}_{\mathrm{Trop}}(X)$$

is an isomorphism, here $H^{p,p}_{\mathrm{Trop}}(X)$ *is the* $\mathbb{Q}$*-coefficient tropical cohomology group.*

As a corollary, we also prove a tropical analog of Grothendieck's standard conjecture D for smooth proper varieties embeddable into toric varieties over trivially valued fields.

COROLLARY 1.1.5 (Corollary 7.1.5). *Let* X *be a smooth proper algebraic variety over a trivially valued field. We assume that there is a closed immersion of* X *to a normal toric variety. Then, for any* $p\geq 0$*, the kernel of the tropical cycle class map*

$$\mathrm{Z}_p(X)\otimes\mathbb{Q}\to H^{\mathrm{Trop}}_{p,p}(X)$$

from the $\mathbb{Q}$*-vector space* $\mathrm{Z}_p(X)\otimes\mathbb{Q}$ *of algebraic cycles of dimension* p *to the tropical homology group* $H^{\mathrm{Trop}}_{p,p}(X)$ *is the* $\mathbb{Q}$*-vector subspace generated by algebraic cycles numerically equivalent to* 0*.*

In the second part, inspired by Liu's construction of tropical cycle class maps by Milnor K-groups, we introduce tropical analogs K^*_T of Milnor K-groups, and show the exactness of the Gersten complexes for the Zariski sheaf cohomology of the sheaves of them on smooth algebraic varieties over trivially valued fields.

THEOREM 1.1.6 (Theorem 6.1.1). *Tropical* K*-groups* K^*_T *form a cycle module in the sense of Rost* [**Ros96**, Definition 2.1]*. In particular, for any* $p\geq 0$*, their Zariski sheafification* $\mathscr{K}^p_T$ *on a smooth algebraic variety* X *over a trivially valued field has the Gersten resolution, i.e., a flasque resolution*

$$0\to\mathscr{K}^p_T\xrightarrow{d}\bigoplus_{x\in X^{(0)}}i_{x*}K^p_T(k(x))\xrightarrow{d}\bigoplus_{x\in X^{(1)}}i_{x*}K^{p-1}_T(k(x))\xrightarrow{d}\bigoplus_{x\in X^{(2)}}i_{x*}K^{p-2}_T(k(x))\xrightarrow{d}\dots$$
$$\xrightarrow{d}\bigoplus_{x\in X^{(p)}}i_{x*}K^0_T(k(x))\xrightarrow{d}0,$$

where $X^{(i)}$ *is the set of points of codimension* i*, the residue field at* $x\in X$ *is denoted by* $k(x)$*, the maps* $i_x\colon\mathrm{Spec}(k(x))\to X$ *are the natural morphisms, and we identify the groups* $K^*_T(k(x))$ *and the Zariski sheaf defined by them on* $\mathrm{Spec}\,k(x)$*.*

In particular, we have the following corollary.

Corollary 1.1.7 (Corollary 6.1.2).

$$H^p_{Zar}(X, \mathscr{K}^p_T) \cong CH^p(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

In the third part, we finish proving the tropical analog of the Hodge conjecture. It follows from that the tropical cohomology groups for smooth algebraic varieties over trivially valued fields are isomorphic to the Zariski sheaf cohomology groups of the sheaves of tropical analogs of Milnor K-groups.

Theorem 1.1.8 (Theorem 7.1.2). *Let X be a smooth algebraic variety over a trivially valued field. For any p, q, we have*

$$H^{p,q}_{\mathrm{Trop}}(X) \cong H^q(X_{\mathrm{Zar}}, \mathscr{K}^p_T).$$

We prove Theorem 1.1.8 by a theorem for general "cohomology theories", which is first proved by Quillen [**Qui73**] for algebraic K-theory and developed by many mathematicians including Bloch-Ogus [**BO74**], Gabber [**Gab94**], Rost [**Ros96**], and Collit-Thélène-Hoobler-Kahn [**CTHK97**], and explicit calculations of tropical cohomology of the trivial line bundle by non-archimedean geometry. The latter is the most technical part of this thesis.

Usually, non-archimedean geometry of height 1 valuations (Berkovich analytic spaces) are used to study tropical geometry. To study tropical analogs of Milnor K-groups and tropical cohomology, we introduce tropicalizations of noon-archimedean spaces of higher heights valuations (Huber's adic spaces and Zariski-Riemann spaces).

The organization of this paper is as follows. Chapter 2-4 are devoted to basics on valuations, non-archimedean analytic spaces, thier tropicalizations, tropical cohomology. In Chapter 5, we prove Theorem 1.1.1. In Chapter 6, we introduce and study tropical K-groups. In Chapter 7, we prove a tropical analog of the Hodge conjecture over trivially valued fields. A key proposition, $\mathbb{A}^1$-homotopy invariance of tropical cohomology over trivially valued fields, is proved in Chapter 9. In Chapter 8, we give explicit descriptions of analytifications (in the sense of both Huber and Berkovich) and tropicalizations of the projective line $\mathbb{P}^1$ over non-archimedean valuation fields of height 1.

1.2. Notations and terminologies

For a $\mathbb{Z}$-module G and a commutative ring R, we put $G_R := G \otimes_{\mathbb{Z}} R$. For a subgroup $\Gamma \subset \mathbb{R}$, we put $\sqrt{\Gamma}$ the minimal devisible subgroup of $\mathbb{R}$ containing Γ. For a field L, we denote its algebraic closure by L^{alg}. Integral separated schemes of finite type over fields are called algebraic varieties. Cones mean strongly convex rational polyhedral cones. Toric varieties are assumed to be normal. For a valuation v of a field L, we put L_v the completion of L. When there is no confusion, we also denote it by $\hat{L}$. We denote the residue field of a valuation v by $\kappa(v)$. We denote the residue field of the structure sheaf at a point x of a scheme X by $k(x)$. For an extension of fields L/K, we denote its transcendental degree by $\mathrm{tr.deg}(L/K)$.

CHAPTER 2

Valuations and non-archimedean analytic spaces

In this chapter, we give a quick review on (non-archimedean) *valuations* (Section 2.1) and non-archimedean analytic spaces: *Berkovich analytic spaces* (Subsection 2.2.1), *Zariski-Riemann spaces* (Subsection 2.2.2), and *Huber's adic spaces* (Subsection 2.2.3). (See Chapter 8 for Berkovich's and Huber's analytification of the projective line, which is used in Chapter 9.) We refer to [**HK94**] and [**Bou72**, Chapter 6] for valuations [**Hub93**] and [**Hub94**] for valuations and Huber's adic spaces, [**Ber90**], [**Ber93**], and [**Tem15**] for Berkovich analytic spaces, and [**Tem11**] for Zariski-Riemann spaces.

2.1. Valuations

In this section, rings are assumed to be commutative with a unit element.

Definition 2.1.1. *We define a* valuation v *of a ring* R *as a map* $v\colon R \to \Gamma'_v \cup \{\infty\}$ *satisfying the following properties:*

- Γ'_v *is a totally ordered abelian group,*
- $v(ab) = v(a) + v(b)$ *for any* $a, b \in R$*, where we extend the group law of* Γ'_v *to* $\Gamma'_v \cup \{\infty\}$ *by* $\gamma + \infty = \infty + \gamma = \infty$ *for any* $\gamma \in \Gamma'_v$*,*
- $v(0) = \infty$ *and* $v(1) = 0$*,*
- $v(a + b) \geq \min\{v(a), v(b)\}$*, where we extend the order of* Γ'_v *to* $\Gamma'_v \cup \{\infty\}$ *by* $\infty \geq \gamma$ *for any* $\gamma \in \Gamma'_v$*.*

The subgroup of Γ'_v generated by $v(R) \setminus \{\infty\}$ is called the *value group* of v. We denote it by Γ_v. We put $\mathrm{supp}(v) := v^{-1}(\infty)$. We call it the *support* of v. We put

$$\mathcal{O}_v := \{a \in \mathrm{Frac}(R/\mathrm{supp}(v)) \mid v(a) \geq 0\},$$

which is called the *valuation ring* of v. The valuation v extends to a valuation on $\mathcal{O}_v$, which is also denoted by v. We put

$$\kappa(v) := \mathrm{Frac}(\mathcal{O}_v/\{a \in \mathcal{O}_v \mid v(a) > 0\}),$$

which is called the *residue field* of v. If $R/\mathrm{supp}(v) \subset \mathcal{O}_v$, we call the image of the maximal ideal under the canonical morphism $\mathrm{Spec}\,\mathcal{O}_v \to \mathrm{Spec}\,R$ the *center* of v.

Definition 2.1.2. *We call two valuations* v *and* w *of a ring* R *are* equivalent *if there exists an isomorphism* $\varphi\colon \Gamma_v \xrightarrow{\sim} \Gamma_w$ *of totally ordered abelian groups satisfying* $\varphi' \circ v = w$*, where* $\varphi'\colon \Gamma_v \cup \{\infty\} \to \Gamma_w \cup \{\infty\}$ *is the extension of* φ *defined by* $\varphi'(\infty) = \infty$*.*

We call the rank of a totally ordered abelian group Γ as an abelian group the *rational rank* of Γ.

Definition 2.1.3. *Let* Γ *be a totally ordered abelian group. A subgroup* H *of* Γ *is called* convex *if every element* $\gamma \in \Gamma$ *satisfying* $h < \gamma < h'$ *for some* $h, h' \in H$ *is contained in* H*.*

Definition 2.1.4. *We call the number of proper convex subgroups of a totally ordered abelian group* Γ *the* height *of* Γ*. We denote it by* $\mathrm{ht}\,\Gamma$*.*

The following well-known theorem is called the Harn embedding theorem.

THEOREM 2.1.5 (Clifford [**Cli54**], Hausner-Wendel [**HW52**]). *Every totally ordered abelian group of finite height n has an embedding into the additive group $\mathbb{R}^n$ with the lexicographic order.*

We call the rational rank (resp. height) of the value group of a valuation v the rational rank (resp. height) of v. Equivalent valuations have the same rational ranks and heights. Hence rational ranks and heights of equivalence classes of valuations are well-defined.

DEFINITION 2.1.6. *A field L equipped with a valuation $v\colon L \to \Gamma_v \cup \{\infty\}$ is called a* valuation field. *A valuation field (L, v) is called trivially valued if its value group Γ_v is the trivial group $\{0\}$.*

Let R be a ring. We call the set of all equivalent classes of valuations of R the *valuation spectrum* of R. We denote it by $\operatorname{Spv}(R)$. We equip $\operatorname{Spv}(R)$ with the topology which is generated by the sets

$$\{v \in \operatorname{Spv}(R) \mid v(a) \geq v(b) \neq \infty\} \qquad (a, b \in R).$$

Let $v\colon R \to \Gamma_v \cup \{\infty\}$ be a valuation. For every convex subgroup $H \subset \Gamma_v$, we define two mappings

$$v/H\colon R \to (\Gamma_v/H) \cup \{\infty\}, \quad a \mapsto \begin{cases} v(a) \bmod H & \text{if } v(a) \neq \infty \\ \infty & \text{if } v(a) = \infty, \end{cases}$$

$$v|H\colon R \to H \cup \{\infty\}, \quad a \mapsto \begin{cases} v(a) & \text{if } v(a) \in H \\ \infty & \text{if } v(a) \notin H. \end{cases}$$

LEMMA 2.1.7 (Huber-Knebusch [**HK94**, Lemma 1.2.1]).
(1) *The map v/H is a valuation of R, and it is a generalization of v in $\operatorname{Spv}(R)$.*
(2) *The map $v|H$ is a valuation of R if and only if the image of R in $\operatorname{Frac}(R/\operatorname{supp}(v))$ is contained in the localization $\mathcal{O}_{v,p\mathcal{O}_v}$ of the valuation ring $\mathcal{O}_v$ at a prime ideal $p\mathcal{O}_v$, where we put $p := \{a \in R \mid v(a) > H\}$. Moreover, in this case, the valuation $v|H$ is a specialization of v in $\operatorname{Spv}(R)$.*

DEFINITION 2.1.8. *We call a generalization of v in $\operatorname{Spv}(R)$ of the form v/H for a convex subgroup $H \subset \Gamma_v$ a vertical (or primary) generalization, and a specialization of v in $\operatorname{Spv}(R)$ of the form $v|H$ for a convex subgroup $H \subset \Gamma_v$ a horizontal (or secondary) spciaization of v.*

LEMMA 2.1.9. *Let (L, v) be a valuation field, $f = \sum_i a_i T^i \in L[T]$ a polynomial, and $b \in L^{\mathrm{alg}}$ a root of it. Then $v(b)$ is contained in a finite set determined by $(v(a_i))_i$.*

PROOF. This follows from the folloing: for $c, d \in L$ such that $v(a) > v(b)$, we have $v(a + b) = v(b)$ [**BGR84**, Proposition 3 of Section 1.1.1]. □

REMARK 2.1.10 (Bourbaki [**Bou72**, Proposition 2 in Section 4 in Chapter 6]). *For a valuation v of a field K, there is a natural bijection between vertical specializations of v in $\operatorname{Spv}(K)$ and valuations on the residue field $\kappa(v)$ of v.*

2.2. Non-archimedean analytic spaces

2.2.1. Berkovich analytic spaces. In [**Ber90**, Section 3.5], Berkovich introduced the *Berkovich analytic space* associated to a scheme X locally of finite type over a complete valuation field $(L, v_L \colon L^\times \to \mathbb{R})$ of height ≤ 1. We denote it by X^{Ber}. Berkovich analytic spaces are, as sets, the sets of bounded multiplicative seminorms. There exists a bijection between multiplicative seminorms $|\ |$ and valuations v of height ≤ 1 defined by $|\ | \mapsto -\log|\ |$. By this bijection, we consider multiplicative seminorms as valuations.

The scheme X is reduced, pure d-dimensional, irreducible, or separated if and only if X^{Ber} has the same property; see [**Duc18**, Proposition 2.7.16] for irreducibility, [**Ber90**, Proposition 3.4.6] for separatedness, and [**Ber90**, Proposition 3.4.3] for the others.

For an algebraic variety X over K, a Zariski closed Berkovich analytic subvarity of X^{Ber} means a reduced Zariski closed Berkovich K-analytic subspace of X^{Ber}. We say that a Zariski closed Berkovich analytic subvarity of $W \subset X^{\mathrm{Ber}}$ is *algebraic* if W is the Berkovich analytification of a Zariski closed algebraic subvariety of X.

For a Berkovich analytic space Z and a coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_Z$, the Zariski closed Berkovich analytic subspace corresponding to $\mathcal{I}$ is denoted by $V(\mathcal{I}) \subset Z$; see [**Ber90**, Proposition 3.1.4].

In this paper, an *affinoid algebra* A over L means an L-affinoid algebra A in the sense of [**Ber90**, Definition 2.1.1]. We put $A^\circ \subset A$ the ring of power bounded elements, i.e., the subring consists of elements whose spectral radiuses are ≤ 1, see [**Ber90**, Section 1.3]. We denote the Berkovich analytic space associated with A by $\mathscr{M}(A)$ [**Ber90**, Section 1.2]. There exists a unique minimal subset $B(A)$ of $\mathscr{M}(A)$ on which every valuation of A has its minimum [**Ber90**, Corollary 2.4.5]. (Note that the minimum as valuations is the maximum as multiplicative seminorms.) It is called the *Shilov boundary* of $\mathscr{M}(A)$. It is a finite set [**Ber90**, Corollary 2.4.5].

By [**Ber90**, Proposition 2.4.4], we have the following.

LEMMA 2.2.1. *For a pure d-dimensional affinoid domain A, the Shilov boundary $B(A)$ does not intersect with any Zariski closed subset of dimension $\leq (d-1)$.*

LEMMA 2.2.2. *Let W be a Zariski closed Berkovich K-analytic subvarity of the analytification Z^{Ber} of an algebraic variety Z over K. Then for any Berkovich analytic space U, a morphism $\phi \colon W \to U$ of Berkovich K-analytic spaces is separated, and the relative boundary of ϕ is empty.*

PROOF. First, we show that ϕ is separated. Since Z is separated, the analytification Z^{Ber} is separated by [**Ber90**, Proposition 3.4.6]. The Zariski closed Berkovich analytic subvarity $W \subset Z^{\mathrm{Ber}}$ is separated by [**Ber90**, Proposition 3.1.5]. Hence, ϕ is separated by [**Ber90**, Proposition 3.1.5].

Second, we show that the relative boundary of ϕ is empty. The boundary of Z^{Ber} is empty by [**Ber90**, Theorem 3.4.1]. (We note that, in [**Ber90**], a Berkovich K-analytic space is said to be closed if its boundary is empty; see [**Ber90**, Section 3.1, p.49].) Since the closed immersion $W \hookrightarrow Z^{\mathrm{Ber}}$ has no boundary by [**Ber90**, Corollary 2.5.13 (i) and Proposition 3.1.4 (i)], the boundary of W is empty by [**Ber90**, Proposition 3.1.3 (ii)]. Hence the relative boundary of ϕ is empty by [**Ber90**, Proposition 3.1.3 (ii)]. □

2.2.2. Zariski-Riemann spaces. For a finitely generated extension L/K of fields, we put $\mathrm{ZR}(L/K)$ the set of equivalence classes of valuations of L which are trivial on K.

We call $\mathrm{ZR}(L/K)$ with the restriction of the topology of $\mathrm{Spv}(L)$ the *Zariski-Riemann space.*

There is another expression. For each $v \in \mathrm{ZR}(L/K)$ and proper algebraic variety X over K with function field L, by the valuative criterion of properness, there exists a unique canonical morphism $\operatorname{Spec} \mathcal{O}_v \to X$. This induces a map from $\mathrm{ZR}(L/K)$ to the inverse limit $\varprojlim X$ (as topological spaces) of birational morphisms of proper algebraic varieties X over K whose function fields are L.

The following Propositon is well-known.

Proposition 2.2.3. *The map* $\mathrm{ZR}(L/K) \to \varprojlim X$ *is a homeomorphism.*

Proof. See [**Tem11**, Corollary 3.4.7]. $\square$

Remark 2.2.4. *For any* $v \in \mathrm{ZR}(L/K)$*, we have* $\mathrm{rank}(v) \leq \mathrm{tr.deg}(L/K)$*. The equality holds for some* v*.*

Definition 2.2.5. *We put* $(\operatorname{Spec} L/K)^{\mathrm{Ber}}$ *the subspace of the analytification* X^{Ber} *of a model* X *of* L/K *(i.e., an algebraic variety over* K *whose function field is* L*,) consisting of points whose supports are the generic point of* X*. This definition is independent of the choice of a model* X*.*

The following slight modification of [**Duc18**, Proposition 7.1.3] is proved in the same way by using $K(s_1^{-1}S_1, \dots, s_l^{-1}S_l)$ $(s_i \in \mathbb{R}_{>0})$ instead of $K(S_1, \dots, S_l)$. (This notation is what used in [**Duc18**].)

Lemma 2.2.6. *Let* $\psi\colon \operatorname{Spec} L' \to \operatorname{Spec} L$ *be a finite extension,* $U \subset (\operatorname{Spec} L'/K)^{\mathrm{Ber}}$ *a subset which is the intersection of* $(\operatorname{Spec} L'/K)^{\mathrm{Ber}}$ *and an affinoid subdomain of a model of* L'/K*. Then* $\psi(U) \subset (\operatorname{Spec} L/K)^{\mathrm{Ber}}$ *is a finite union of intersections of* $(\operatorname{Spec} L/K)^{\mathrm{Ber}}$ *and affinoid subdomains of a model of* L/K*.*

2.2.3. Huber's adic spaces. For an algebraic variety X over a trivially valued field K, we define the adic space X^{ad} associated to X as follows. (See [**Hub93**] and [**Hub94**] for notations and theory of adic spaces.) For each affine open subvariety $U = \operatorname{Spec} R \subset X$, we put $U^{\mathrm{ad}} := \mathrm{Spa}(R, R \cap K^{\mathrm{alg}})$, which is the space of equivalence classes of valuations on R trivial on K (here we consider R a ring equipped with the discrete topology). We define X^{ad} by glueing U_α^{ad} for an affine open covering $\{U_\alpha\}$ of X.

Remark 2.2.7. *Taking supports of valuations induces a surjective map* $X^{\mathrm{ad}} \twoheadrightarrow X$ *whose fiber of* $x \in X$ *is homeomorphic to* $\mathrm{ZR}(k(x)/K)$*.*

Remark 2.2.8. *Taking equivalence classes induces a map* $X^{\mathrm{Ber}} \to X^{\mathrm{ad}}$*. This induces a bijection*

$$X^{\mathrm{Ber}}/(\text{equivalence relations}) \cong X^{\mathrm{ad},\mathrm{ht}\leq 1}$$

to the subset $X^{\mathrm{ad},\mathrm{ht}\leq 1}$ *of* X^{ad} *consisting of equivalence classes of valuations of height* ≤ 1*.*

For an algebraic variety Y over a complete non-archimedean valuation field L of height 1, we consider the adic space Y^{ad} associated to Y in the sense of [**Hub94**]. What we need to consider in this paper are subvarieties of the projective line, whose structure will be explained precisely in Chapter 8.

CHAPTER 3

Tropicalizations

In this chapter, we shall recall basic properties of fans and polyhedral complexes (Section 3.1), *tropicalizations* of non-archimedean analytic spaces (Berkovich analytic spaces (Section 3.2), Huber's adic spaces (Subsection 3.4.1 and 3.4.3), and Zariski-Riemann spaces (Subsection 3.4.2)), and tropical compactifications (Section 3.3). Tropicalizations of algebraic varieties are usually defined to be tropicalizations as Berkovich analytic spaces. They have been studied by many mathematicians, e.g., see [**Gub13**], [**GRW16**], [**GRW17**], and [**Pay09-1**]. Tropicalizations of Huber's adic spaces and Zariski-Riemann spaces are the inverse limits of certain polyhedral complex structures of tropicalizations of Berkovich analytic spaces. They are used to give explicit descriptions of tropical analogs of Milnor K-groups (Corollary 6.2.4) and to prove $\mathbb{A}^1$-homotopy invariance of tropical cohomology in Chapter 9. For details of tropicalizations of the two analytifications of the projective line, see Chapter 8. (Note that our tropicalizations of adic spaces are different from Foster-Payne's adic tropicalizations; see [**Fos16**].)

Let M be a free $\mathbb{Z}$-module of finite rank n. We put $N := \operatorname{Hom}(M, \mathbb{Z})$. Let Σ be a fan in $N_{\mathbb{R}}$, and T_Σ the normal toric variety over a field associated to Σ. (See [**CLS11**] for basic notions and results on toric varieties.) In this paper, cones mean strongly convex rational polyhedral cones. Remind that there is a natural bijection between the cones $\sigma \in \Sigma$ and the torus orbits $O(\sigma)$ in T_Σ. The torus orbit $O(\sigma)$ is isomorphic to the torus $\operatorname{Spec} K[M \cap \sigma^\perp]$. Its Zariski closure $\overline{O(\sigma)}$ in T_Σ is a normal toric variety over K containing it as the open dense torus orbit. We put $N_\sigma := \operatorname{Hom}(M \cap \sigma^\perp, \mathbb{Z})$. We fix an isomorphism $M \cong \mathbb{Z}^n$, and identify $\operatorname{Spec} K[M]$ and $\mathbb{G}_m^n$.

3.1. Fans and polyhedral complexes

In this section, we recall the partial compactification $\bigsqcup_{\sigma\in\Sigma} N_{\sigma,\mathbb{R}}$ of $\mathbb{R}^n$ and define generalizations of fans and polyhedral complexes in it.

We define a topology on the disjoint union $\bigsqcup_{\sigma\in\Sigma} N_{\sigma,\mathbb{R}}$ as follows. We extend the canonical topology on $\mathbb{R}$ to that on $\mathbb{R} \cup \{\infty\}$ so that $(a, \infty]$ for $a \in \mathbb{R}$ are a basis of neighborhoods of ∞. We also extend the addition on $\mathbb{R}$ to that on $\mathbb{R} \cup \{\infty\}$ by $a + \infty = \infty$ for $a \in \mathbb{R} \cup \{\infty\}$. For each $\sigma \in \Sigma$, we put

$$\sigma^\vee := \{m \in M \otimes_{\mathbb{Z}} \mathbb{R} \mid n(m) \geq 0 \text{ for all } n \in \sigma\}.$$

We consider the set of semigroup homomorphisms $\operatorname{Hom}(M \cap \sigma^\vee, \mathbb{R} \cup \{\infty\})$ as a topological subspace of $(\mathbb{R} \cup \{\infty\})^{M\cap\sigma^\vee}$. We define a topology on $\bigsqcup_{\substack{\tau\in\Sigma \\ \tau\preceq\sigma}} N_{\tau,\mathbb{R}}$ by the canonical bijection

$$\operatorname{Hom}(M \cap \sigma^\vee, \mathbb{R} \cup \{\infty\}) \cong \bigsqcup_{\substack{\tau\in\Sigma \\ \tau\preceq\sigma}} N_{\tau,\mathbb{R}}.$$

Then we define a topology on $\bigsqcup_{\sigma\in\Sigma} N_{\sigma,\mathbb{R}}$ by glueing the topological spaces $\bigsqcup_{\substack{\tau\in\Sigma \\ \tau\preceq\sigma}} N_{\tau,\mathbb{R}}$ together.

We shall define fans and polyhedral complex in $\bigsqcup_{\sigma\in\Sigma} N_{\sigma,\mathbb{R}}$.

DEFINITION 3.1.1. *Let $\Gamma \subset \mathbb{R}$ be a subgroup. Then a subset of $\mathbb{R}^n$ is called a* Γ-rational polyhedron *if and only if it is the intersection of sets of the form*

$$\{x \in \mathbb{R}^n \mid \langle x, a\rangle \leq b\} \ (a \in \mathbb{Z}^n, b \in \Gamma),$$

here $\langle x, a\rangle$ is the usual inner product of $\mathbb{R}^n$.

DEFINITION 3.1.2. *Let $\Gamma \subset \mathbb{R}$ be a subgroup. For a cone $\sigma \in \Sigma$ and a (Γ-rational) polyhedron (resp. a cone) $C \subset N_{\sigma,\mathbb{R}}$, we call its closure $P := \overline{C}$ in $\bigsqcup_{\sigma\in\Sigma} N_{\sigma,\mathbb{R}}$ a (*Γ-rational*)* polyhedron *(resp. a* cone*) in $\bigsqcup_{\sigma\in\Sigma} N_{\sigma,\mathbb{R}}$. In this case, we put* $\mathrm{rel.int}(P) := \mathrm{rel.int}(C)$*, and call it the* relative interior *of P. We put* $\dim(P) := \dim(C)$*.*

Let $\sigma_P \in \Sigma$ be the unique cone such that $\mathrm{rel.int}(P) \subset N_{\sigma_P,\mathbb{R}}$. A subset Q of a polyhedron (resp. a cone) P in $\bigsqcup_{\sigma\in\Sigma} N_{\sigma,\mathbb{R}}$ is called a *face* of P if it is the closure of the intersection $P^a \cap N_{\tau,\mathbb{R}}$ in $\bigsqcup_{\sigma\in\Sigma} N_{\sigma,\mathbb{R}}$ for some $a \in \sigma_P \cap M$ and some cone $\tau \in \Sigma$, where P^a is the closure of

$$\{x \in P \cap N_{\sigma_P,\mathbb{R}} \mid x(a) \leq y(a) \text{ for any } y \in P \cap N_{\sigma_P,\mathbb{R}}\}$$

in $\bigsqcup_{\sigma\in\Sigma} N_{\sigma,\mathbb{R}}$. A finite collection Λ of (Γ-rational) polyhedra (resp. cones) in $\bigsqcup_{\sigma\in\Sigma} N_{\sigma,\mathbb{R}}$ is called a (Γ-*rational*) *polyhedral complex* (resp. a *fan*) if it satisfies the following two conditions:

- For all $P \in \Lambda$, each face of P is also in Λ.
- For all $P, Q \in \Lambda$, the intersection $P \cap Q$ is a face of P and Q.

We call the union

$$\bigcup_{P\in\Lambda} P \subset \bigsqcup_{\sigma\in\Sigma} N_{\sigma,\mathbb{R}}$$

the *support* of Λ. We denote it by $|\Lambda|$. We also say that Λ is a (Γ-*rational*) *polyhedral complex* (resp. a *fan*) *structure* of $|\Lambda|$.

3.2. Tropicalizations of Berkovich analytic spaces

We recall basics on *tropicalizations* of Berkovich analytic spaces. Let $(L, v_L\colon L^\times \to \mathbb{R})$ be a complete valuation field of height ≤ 1. In this section, every algebraic variety is defined over L.

The *tropicalization map*

$$\mathrm{Trop}\colon O(\sigma)^{\mathrm{Ber}} \to N_{\sigma,\mathbb{R}} = \mathrm{Hom}(M \cap \sigma^\perp, \mathbb{R})$$

is the proper surjective continuous map given by the restriction

$$\mathrm{Trop}(v_x) := v_x|_{M\cap\sigma^\perp}\colon M \cap \sigma^\perp \to \mathbb{R}$$

for $v_x \in O(\sigma)^{\mathrm{Ber}}$; see [**Pay09-1**, Section 2]. (Here, as explained in Subsection 2.2.1, we consider elements of Berkovich analytic spaces as valuations.) We define the tropicalization map

$$\mathrm{Trop}\colon T_\Sigma^{\mathrm{Ber}} = \bigsqcup_{\sigma\in\Sigma} O(\sigma)^{\mathrm{Ber}} \to \bigsqcup_{\sigma\in\Sigma} N_{\sigma,\mathbb{R}}$$

by glueing the tropicalization maps $\mathrm{Trop}\colon O(\sigma)^{\mathrm{Ber}} \to N_{\sigma,\mathbb{R}}$ together. We note that the tropicalization map

$$\mathrm{Trop}\colon T_\Sigma^{\mathrm{Ber}} \to \bigsqcup_{\sigma\in\Sigma} N_{\sigma,\mathbb{R}}$$

is proper, surjective, and continuous; see [**Pay09-1**, Section 3].

For a morphism $\varphi\colon X \to T_\Sigma$ from a scheme X of finite type over L, the image $\mathrm{Trop}(\varphi(X^{\mathrm{Ber}}))$ of X^{Ber} under the composition $\mathrm{Trop}\circ\varphi$ is called a *tropicalization* of X^{Ber} (or X). For simplicity, we often write $\mathrm{Trop}(\varphi(X))$ instead of $\mathrm{Trop}(\varphi(X))$. For a closed immersion $\varphi\colon X \to T_\Sigma$, the tropicalization $\mathrm{Trop}(\varphi(X^{\mathrm{Ber}}))$ is a finite union of $\dim(X)$-dimensional $\sqrt{v(L^\times)}$-rational polyhedra by [**BG84**, Theorem A] and [**MS15**, Theorem 6.2.18].

More generally, for a closed immersion $\varphi'\colon \mathscr{X} \to T_\Sigma^{\mathrm{Ber}}$ from an irreducible Berkovich analytic space $\mathscr{X}$ over L, Gubler showed that the tropicalization $\mathrm{Trop}(\varphi'(\mathscr{X}))$ is a union of $\dim(\mathscr{X})$-dimensional $\sqrt{v(L^\times)}$-rational polyhedra, and for any cone $\sigma \in \Sigma$, the intersection $\mathrm{Trop}(\varphi'(\mathscr{X}))\cap N_{\sigma,\mathbb{R}}$ is a locally finite union of $\sqrt{v(L^\times)}$-rational polyhedra [**Gub07**, Theorem 1.1]. Here, a locally finite union U of polyhedra $\{P_i\}_i$ means that U is the union of $\{P_i\}_i$ and, for each compact subset $C \subset U$, only finitely many polydedra in $\{P_i\}_i$ intersect with C.

For a toric morphism $\psi\colon T_{\Sigma'} \to T_\Sigma$, there exists a morphism $\mathrm{Trop}(T_{\Sigma'}) \to \mathrm{Trop}(T_\Sigma)$ inducing a commutative diagram

$$\begin{tikzcd} T_{\Sigma'}^{\mathrm{Ber}} \arrow[r, "\psi"] \arrow[d, "\mathrm{Trop}"] & T_\Sigma^{\mathrm{Ber}} \arrow[d, "\mathrm{Trop}"] \\ \mathrm{Trop}(T_{\Sigma'}) \arrow[r] & \mathrm{Trop}(T_\Sigma). \end{tikzcd}$$

We also denote it by $\psi\colon \mathrm{Trop}(T_{\Sigma'}) \to \mathrm{Trop}(T_\Sigma)$. The restriction $\psi|_{N_{\sigma'}}\colon N_{\sigma'} \to N_\sigma$ of it to each orbits is a linear map.

REMARK 3.2.1. *Tropicalizations do not change under base extensions, i.e., for an extension L'/L of complete valuation fields of height ≤ 1, we have a commutative diagram*

$$\begin{tikzcd} X_{L'} \arrow[r, "\varphi_{L'}"] \arrow[d] & T_{\Sigma,L'} \arrow[d] \arrow[dr, "\mathrm{Trop}"] & \\ X \arrow[r, "\varphi"] & T_\Sigma \arrow[r, two heads, "\mathrm{Trop}"] & \mathrm{Trop}(T_\Sigma), \end{tikzcd}$$

here, $(-)_{L'}$ means the base change to L'. In particular, we have

$$\mathrm{Trop}(\varphi_{L'}(X_{L'})) = \mathrm{Trop}(\varphi(X)) \subset \mathrm{Trop}(T_\Sigma).$$

We recall tropical skeletons. Let $\varphi\colon X \to T_\Sigma$ be a closed immersion of an algebraic variety X. For $\sigma \in \Sigma$ and $a \in \mathrm{Trop}(O(\sigma))$, the fiber $(\mathbb{G}_m^n)_a := (\mathrm{Trop})^{-1}(a)$ is an affinoid subdomain of $O(\sigma)^{\mathrm{Ber}}$. The fiber $X_a := (\mathrm{Trop}\circ\varphi)^{-1}(a)$ is a Zariski-closed subspace of it. We put $I \subset \Gamma((\mathbb{G}_m^n)_a, \mathcal{O}_{(\mathbb{G}_m^n)_a})$ the defining ideal of $X_a \subset (\mathbb{G}_m^n)_a$. When the base field (L, v) is of height 1, for $a \in N_{\mathbb{Q}\cdot\Gamma_v} \cap \mathrm{Trop}(\varphi(X))$, the formal scheme

$$\mathrm{Spf}(\Gamma((\mathbb{G}_m^n)_a, \mathcal{O}_{(\mathbb{G}_m^n)_a})^\circ / I \cap \Gamma((\mathbb{G}_m^n)_a, \mathcal{O}_{(\mathbb{G}_m^n)_a})^\circ)$$

is an admissible formal model of a strictly L-affinoid domain X_a. We call its special fiber $\mathrm{in}_a(X)$ the *initial degeneration* of X at a. There is a reduction map

$$X_a \to \mathrm{in}_a(X),$$

which is surjective [**GRW17**, Proposition 2.17]. (See also [**Ber90**, Section 2.4] and [**GRW17**, Section 2.13].) By [**GRW17**, Prosposition 2.12 and 2.16], we have the following.

LEMMA 3.2.2. *The image of Shilov boundary $B(X_a)$ under the reduction map is the set of generic points of* $\mathrm{in}_a(X)$.

We put

$$\mathrm{Sk}_\varphi(X) := \bigsqcup_{a \in \mathrm{Trop}(\varphi(X))} B(X_a).$$

We call it a *tropical skeleton* of X. When there is no confusion, we simply denote it by $\mathrm{Sk}(X)$.

3.3. Tropicalizations and partial compactifications

In this section, we shall recall a relation between tropicalizations and partial compactifications. Let $X \subset \mathbb{G}_m^n = \operatorname{Spec} K[M]$ be a closed subvariety over a trivially valued field K.

DEFINITION 3.3.1. *The closure $\overline{X}$ in the toric variety T_Σ associated with a fan Σ in $N_\mathbb{R}$ is called a* tropical compactification *if the multiplication map*

$$\mathbb{G}_m^n \times \overline{X} \to T_\Sigma$$

is faithfully flat and $\overline{X}$ is proper.

THEOREM 3.3.2 (Tevelev [**Tev07**, Theorem 1.2]). *There exists a fan Σ such that $\overline{X} \subset T_\Sigma$ is a tropical compactification.*

REMARK 3.3.3. *Let $\overline{X} \subset T_\Sigma$ be a tropical compactification of $X \subset \mathbb{G}_m^n$.*

(1) The fan Σ is a fan structure of $\mathrm{Trop}(X) \subset N_\mathbb{R}$ [**Tev07**, Proposition 2.5].

(2) For any refinement Σ' of Σ, the closure of X in $T_{\Sigma'}$ is also a tropical compactification [**Tev07**, Proposition 2.5].

REMARK 3.3.4. *For a tropical compactification $\overline{X} \subset T_\Sigma$ and $v \in X^{\mathrm{Ber}}$, since $\overline{X}$ is proper, by the valuative criterion of properness, there exists a unique canonical morphism*

$$\operatorname{Spec} \mathcal{O}_v \to \overline{X}.$$

The orbit $O \subset T_\Sigma$ which contains the image of the maximal ideal under this morphism corresponds to the cone $\sigma \in \Sigma$ whose relative interior contains $\mathrm{Trop}(v)$.

PROPOSITION 3.3.5 (Tevelev [**Tev07**, Proposition 2.3]). *For a fan Σ in $N_\mathbb{R}$, the closure $\overline{X}$ of X in T_Σ is proper if and only if* $\mathrm{Trop}(X)$ *is contained in the support of Σ.*

The following, which is well-known for specialists, can be proved in a similar way to the proof of [**MS15**, Proposition 6.4.7].

LEMMA 3.3.6. *Let $\sigma \subset \mathrm{Trop}(X)$ be a cone. We put $\overline{X} \subset T_\sigma$ the closure of X in the affine toric variety T_σ associated with the cone σ. Then, for any face $\tau \subset \sigma$, the intersection $\overline{X} \cap O(\tau)$ is non-empty and of pure codimensional* $\dim(\tau)$ *in $\overline{X}$.*

LEMMA 3.3.7. *Let Λ and Λ' be fan structures of* $\mathrm{Trop}(X)$ *inducing tropical compactifications of X. We assume that Λ' is a refinement of Λ.*

Then for $\Lambda' \ni P' \mapsto P \in \Lambda$, the morphism

$$\overline{X} \cap O(P') \to \overline{X} \cap O(P)$$

is the trivial $(\dim(P') - \dim(P))$*-dimensional torus bundle.*

PROOF. For $x \in \text{rel.int}(P') \subset \text{rel.int}(P)$, by [**Gub13**, Remark 12.7], the initial degeneration $\text{in}_x(X)$ is the trivial torus bundle over $\overline{X} \cap O(P)$ and is that over $\overline{X} \cap O(P')$. Hence Lemma 3.3.7 holds. $\square$

REMARK 3.3.8. *When Λ is smooth and $T_{\Lambda'} \to T_\Lambda$ is a iteration of blow-ups at smooth centers, Lemma 3.3.7 follows from that $\overline{X}$ and orbits intersect properly, i.e., the intersection $\overline{X} \cap O(P)$ $(P \in \Lambda)$ is of pure codimension* $\dim P$ *in $\overline{X}$. For our purpose, refinements of this form are enough.*

DEFINITION 3.3.9. *We put X° the subset of X^{Ber} consisting of valuations v such that there exists a natural morphism* $\operatorname{Spec} O_v \to X$.

When X is proper, by the valuative criterion of properness, we have $X^\circ = X^{\mathrm{Ber}}$.
The following follows from Remark 3.3.4.

LEMMA 3.3.10. *Let $\varphi\colon X \to T_\Sigma$ be a closed immersion, Λ a fan structure of* $\text{Trop}(\varphi(X))$. *Then we have*

$$X^\circ = \bigcup_{\textit{compact cones } P \in \Lambda} (\text{Trop} \circ \varphi)^{-1}(P).$$

REMARK 3.3.11. *Since tropicalization maps are proper, the subset X° is compact.*

3.4. Tropicalizations of adic spaces and Zariski-Riemann spaces

3.4.1. Tropicalizations of adic spaces. In this subsection, we introduce tropicalizations and tropical skeletons of adic spaces associated with algebraic varieties over a trivially valued field K, and study their basic properties.

We define tropicalizations. Let X be a subvariety of a torus $\mathbb{G}_m^n$ over K, and Λ a fan structure of $\text{Trop}(X) \subset N_\mathbb{R}$. We consider the composition

$$X^{\mathrm{ad}} \to \overline{X} \to \Lambda,$$

where the first morphism is taking center and the second one is defined by $\overline{X} \ni x \to \lambda \in \Lambda$ such that $x \in O(\lambda)$. (The closure $\overline{X}$ is taken in the toric variety T_Λ associated with the fan Λ.) Note that by Proposition 3.3.5 and the valuative criterion of properness, the map $X^{\mathrm{ad}} \to \overline{X}$ is well-defined. We denote this composition by $\text{Trop}_\Lambda^{\mathrm{ad}}\colon X^{\mathrm{ad}} \to \Lambda$.

DEFINITION 3.4.1. *Taking all fan structures Λ of* $\text{Trop}(X)$, *we have a surjective map*

$$\text{Trop}^{\mathrm{ad}}\colon X^{\mathrm{ad}} \to \varprojlim \Lambda$$

called the tropicalization map *of X^{ad}.*

For a subvariety Y of a toric variety T_Σ over K and Λ a fan structure of $\text{Trop}(Y)$, we put

$$\text{Trop}_\Lambda^{\mathrm{ad}}\colon Y^{\mathrm{ad}} \to \Lambda$$

the disjoint union of $\text{Trop}^{\mathrm{ad}}_{\Lambda \cap \text{Trop}(O(\sigma))}$ $(\sigma \in \Sigma)$.

DEFINITION 3.4.2. *Taking all fan structures Λ of* $\text{Trop}(Y)$, *we have a surjective map*

$$\text{Trop}^{\mathrm{ad}}\colon Y^{\mathrm{ad}} \to \varprojlim \Lambda$$

called the tropicalization map *of Y^{ad}.*

Remark 3.4.3. *By Proposition 2.2.3, the natural map*

$$X^{\mathrm{ad}} \to \varprojlim \overline{X}$$

is surjective, where $\overline{X}$ runs through all tropical compactifications of $X \subset \mathbb{G}_m^n$, and the map $\mathrm{Trop}^{\mathrm{ad}}$ induces a bijection

$$X^{\mathrm{ad}}/J \cong \varprojlim \Lambda,$$

where J is the equivalence relation generated by $v \sim w$ for v and w such that the restrictions $v|_M$ and $w|_M$ are the equivalent, i.e., there exists an isomorphism $\varphi \colon v(M) \cong w(M)$ of totally ordered abelian groups satisfying $\varphi \circ v|_M = w|_M$.

To give another description of $\mathrm{Trop}^{\mathrm{ad}}(X^{\mathrm{ad}})$, we define an equivalence relation on $(N_{\mathbb{R}})^r$. For $l_i \in N_{\mathbb{R}}$, we can write $l_i = \sum_k a_{i,k} l_{i,k}$ by $\mathbb{Q}$-linear independent elements $\{a_{i,k}\}_k \subset \mathbb{R}$ and $\mathbb{Q}$-linear independent elements $\{l_{i,k}\}_k \subset N_{\mathbb{Q}}$. In other words, the subspace $\bigoplus_k \mathbb{Q} \cdot l_{i,k} \subset N_{\mathbb{Q}}$ is the minimal subspace of $N_{\mathbb{Q}}$ such that $(\bigoplus_k \mathbb{Q} \cdot l_{i,k} \subset N_{\mathbb{Q}})_{\mathbb{R}}$ contains l_i. We put

$$J_r := \{(l_1, \dots, l_r) \in (N_{\mathbb{R}})^r \mid \overline{l_i} \in N_{\mathbb{R}} / \sum_{j=1}^{i-1} \sum_k \mathbb{R} \cdot l_{j,k} \text{ is non-zero } (1 \leq i \leq r)\}.$$

We put I_r the equivalence relation on $(N_{\mathbb{R}})^r$ generated by $(l_1, \dots, l_r) \sim (l'_1, \dots, l'_r)$ for $(l_1, \dots, l_r)$ and $(l'_1, \dots, l'_r)$ satisfying

$$\mathbb{R}_{>0} \cdot \overline{l_i} = \mathbb{R}_{>0} \cdot \overline{l'_i} \in N_{\mathbb{R}} / \sum_{j=1}^{i-1} \sum_k \mathbb{R} \cdot l_{j,k} \ (1 \leq i \leq r).$$

To simplify notations, we put $(N_{\mathbb{R}})^0 := \{(0, \dots, 0) \in N_{\mathbb{R}}\}$.

Lemma 3.4.4. *Let $A \subset N_{\mathbb{R}}$ be the support of a fan. There is a bijection*

$$\bigcup_{r \geq 0} \{(l_1, \dots, l_r) \in J_r \mid \sum_{i=1}^{r} (\prod_{j=1}^{i} \epsilon_j) l_i \in A \textit{ for small } \epsilon_j \in \mathbb{R}_{>0} \ (1 \leq j \leq r)\} / \bigcup_{r \geq 1} I_r$$
$$\cong \varprojlim \Lambda$$

given by $(l_1, \dots, l_r) \mapsto (P_\Lambda)_\Lambda$, where Λ runs through all fan structures of A and $(P_\Lambda)_\Lambda$ satisfies

$$\sum_{i=1}^{r} (\prod_{j=1}^{i} \epsilon_{\Lambda,j}) l_i \in \mathrm{rel.int}(P_\Lambda)$$

for any Λ and sufficiently small $\epsilon_{\Lambda,j}$ $(1 \leq j \leq r)$.

Proof. We prove Lemma 3.4.4 by induction on $\dim(A)$, i.e., the maximum of dimension of cones contained in A. When $\dim(A) = 0$, it is trivial. We assume $\dim(A) \geq 1$. We fix $l \in N_{\mathbb{R}} \setminus \{0\}$. We put $(P_{l,\Lambda})_\Lambda$ the image of l under the map in Lemma 3.4.4. Then for a fan structure Λ of A such that $\dim(P_{l,\Lambda})$ achieves its minimum, the set $\overline{A} \cap N_{P_{l,\Lambda},\mathbb{R}} \subset \mathrm{Trop}(T_\Lambda)$ does not depend on the choice of Λ under canonical identification of $N_{P_{l,\Lambda},\mathbb{R}}$ by bijective maps of tropicalizations induced from blow-ups of toric varieties. We fix such a fan structure, and denote it by Λ_0. Then, by applying the assumption of the induction to $\overline{A} \cap N_{P_{l,\Lambda_0},\mathbb{R}}$, the map in Lemma 3.4.4 for $\overline{A} \cap N_{P_{l,\Lambda_0},\mathbb{R}}$ induces a bijection between the subset of

$$\bigcup_{r \geq 0} \{(l_1, \dots, l_r) \in J_r \mid \sum_{i=1}^{r} (\prod_{j=1}^{i} \epsilon_j) l_i \in A \text{ for small } \epsilon_j \in \mathbb{R}_{>0} \ (1 \leq j \leq r)\} / \bigcup_{r \geq 1} I_r$$

consisting of the equivalent classes of $(l_1, \dots, l_r)$ such that $\mathbb{R}_{>0} \cdot l_1 = \mathbb{R}_{>0} \cdot l$ and the subset of $\varprojlim \Lambda$ consisting of $(P_\Lambda)_\Lambda$ such that $\mathbb{R}_{>0} \cdot l_{(P_\Lambda)} = \mathbb{R}_{>0} \cdot l$, where $l_{(P_\Lambda)_\Lambda}$ is a fixed point in the relative interior of the intersection $\bigcap_\Lambda P_\Lambda$. (Note that by compactness of S^{n-1} and the definition, the intersection $\bigcap_\Lambda P_\Lambda$ is a half-line.) Since $l \in N_\mathbb{R} \setminus \{0\}$ is arbitrary, Lemma 3.4.4 holds. □

REMARK 3.4.5. *Let $(l_1, \dots, l_r) \mapsto (P_\Lambda)_\Lambda$ as in Lemma 3.4.4. Then for a sufficiently fine fan structure Λ_1 of A, we have*

$$\mathrm{Span}_\mathbb{R}(P_{\Lambda_1}) = \sum_{i,k} \mathbb{R} \cdot l_{i,k}.$$

By definitions, we have the following.

LEMMA 3.4.6. *Let Λ be a fan structure of* $\mathrm{Trop}(X)$*, $v \in X^{\mathrm{ad}}$, and $w \in X^{\mathrm{ad}}$ a vertical specialization of v. We put $\overline{w} \in \overline{X} \cap O(\mathrm{Trop}_\Lambda^{\mathrm{ad}}(v))$ the unique vertical specialization v_0 of the trivial valuation on $k(\mathrm{center}(v))$ such that $\overline{w}$ is a horizontal specialization of w (*[**HK94**, Lemma 1.2.5(i)]*). Here, we consider v_0 as an element in $\overline{X}^{\mathrm{ad}}$.*

Then we have

$$\mathrm{Trop}^{\mathrm{ad}}_{\{\overline{P} \cap N_{\mathrm{Trop}_\Lambda^{\mathrm{ad}}(v)}\}_{P \in \Lambda}}(\overline{w}) = \overline{\mathrm{Trop}_\Lambda^{\mathrm{ad}}(w)} \cap N_{\mathrm{Trop}_\Lambda^{\mathrm{ad}}(v)}.$$

We consider $(l_1, \dots, l_r) \in (N_\mathbb{R})^r$ as a map

$$M \ni m \mapsto (l_1(m), \dots, l_r(m)) \in \mathbb{R}^r.$$

LEMMA 3.4.7. *Let $v \in X^{\mathrm{ad}}$ be a valuation, and $(l_1, \dots, l_r) \in (N_\mathbb{R})^r$ corresponds to* $\mathrm{Trop}^{\mathrm{ad}}(v)$ *via the bijection in Lemma 3.4.4 for* $\mathrm{Trop}(X)$*. Then there exists a ordered group isomorphism $\phi \colon v(M) \cong (l_1, \dots, l_r)(M)$ satisfying $\phi \cdot v|_M = (l_1, \dots, l_r)$. Here, we consider $\mathbb{R}^r$ equipped with the lexicographic order. (We denote a representative of the equivalence class v by v.)*

PROOF. When $r \leq 1$, the assertion is trivial. When $r \geq 2$, the assertion follows from induction on r in a similar way to Lemma 3.4.4 by Lemma 3.4.6. □

DEFINITION 3.4.8. *For $x \in \mathrm{Trop}^{\mathrm{ad}}(X^{\mathrm{ad}})$ whose inverse image under the bijection in Lemma 3.4.4 for* $\mathrm{Trop}(X)$ *is $(l_1, \dots, l_r)$, we put $\mathrm{ht}(x) := r$.*

We omit the proof of the following lemma because it immediately follows from Lemma 3.4.7.

LEMMA 3.4.9. *For $v \in X^{\mathrm{ad}}$, we have*

$$\mathrm{ht}(v(M)) = \mathrm{ht}(\mathrm{Trop}^{\mathrm{ad}}(v)).$$

REMARK 3.4.10. *The map in Lemma 3.4.4 induces a bijection*

$$\mathrm{Trop}(X^{\mathrm{Ber}})/I \cong \mathrm{Trop}^{\mathrm{ad}}(X^{\mathrm{ad}})^{\mathrm{ht} \leq 1}.$$

Here, the subset $\mathrm{Trop}(X^{\mathrm{ad}})^{\mathrm{ht}\leq 1}$ *of* $\mathrm{Trop}(X^{\mathrm{ad}})$ *consisting of elements of height* ≤ 1*, and* I *is the equivalence relation generated by* $a \sim a'$ *for* a *and* a' *such that* $\mathbb{R}_{>0}\cdot a = \mathbb{R}_{>0}\cdot a'$*. We have a commutative diagram*

$$\begin{array}{ccccc} X^{\mathrm{Ber}}/(\text{equivalence relations}) & \xrightarrow{\cong} & X^{\mathrm{ad},\mathrm{ht}\leq 1} & \subset & X^{\mathrm{ad}} \\ \downarrow{\scriptstyle \mathrm{Trop}} & & \downarrow{\scriptstyle \mathrm{Trop}^{\mathrm{ad}}} & & \downarrow{\scriptstyle \mathrm{Trop}^{\mathrm{ad}}} \\ \mathrm{Trop}(X^{\mathrm{Ber}})/I & \xrightarrow{\cong} & \mathrm{Trop}^{\mathrm{ad}}(X^{\mathrm{ad}})^{\mathrm{ht}\leq 1} & \subset & \mathrm{Trop}^{\mathrm{ad}}(X^{\mathrm{ad}}). \end{array}$$

The following is trivial.

Lemma 3.4.11. *Let* $v \in X^{\mathrm{ad}}$*. We write* $\mathrm{Trop}^{\mathrm{ad}}(v) = (P_\Lambda)_\Lambda$*. Then we have*

$$\operatorname{rank} v(M) = min_\Lambda \dim P_\Lambda.$$

Definition 3.4.12. *We define* tropical skeleton $\mathrm{Sk}(X^{\mathrm{ad}})$ *of* X^{ad} *to be the subset of* X^{ad} *consisting of generalizations of* $v \in X^{\mathrm{ad}}$ *such that* $\operatorname{rank}_{\mathbb{Z}} v(M) = \dim X$*. For a closed immersion* $\varphi\colon Y \to T_\Sigma$ *to a toric variety* T_Σ *over* K*, we put*

$$\mathrm{Sk}_\varphi(Y^{\mathrm{ad}}) := \bigsqcup_{\sigma\in\Sigma} \mathrm{Sk}(\varphi(Y^{\mathrm{ad}}) \cap O(\sigma)^{\mathrm{ad}}).$$

Proposition 3.4.13. *Let* $v \in X^{\mathrm{ad}}$*. The followings are equivalent.*

(1) $v \in \mathrm{Sk}(X^{\mathrm{ad}})$*.*
(2) There exists a specialization $w \in X^{\mathrm{ad}}$ *of* v *such that for any fan structure* Λ *of* $\mathrm{Trop}(X)$*, the tropicalization* $\mathrm{Trop}^{\mathrm{ad}}_\Lambda(w) \in \Lambda$ *is of dimension* $\dim(X)$*.*
(3) For any fan structure Λ *of* $\mathrm{Trop}(X)$ *such that* $\overline{X} \subset T_\Lambda$ *is a tropical compactification, the center* $\mathrm{center}(v) \in \overline{X}$ *is a generic point of an irreducible component of* $\overline{X} \cap O(\mathrm{Trop}^{\mathrm{ad}}_\Lambda(v))$*.*

Proof. (1) $\Leftrightarrow$ (2) follows from Lemma 3.4.11. (2) $\Leftrightarrow$ (3) follows from Remark 2.1.10 and Lemma 3.3.7. $\square$

Remark 3.4.14. *For any* $x = (P_\Lambda)_\Lambda \in \mathrm{Trop}^{\mathrm{ad}}(X^{\mathrm{ad}})$*, there exists* $v \in \mathrm{Sk}(X^{\mathrm{ad}}) \cap (\mathrm{Trop}^{\mathrm{ad}})^{-1}(x)$*. More precisely, for* $a = (a_\Lambda)_\Lambda \in \varprojlim \overline{X}$ *(where the limits is taken for all fan structures* Λ *of* $\mathrm{Trop}(X)$*) such that for any* Λ*, the point* a_Λ *is the generic point of an irreducible component of the intersection* $\overline{X} \cap O(P_\Lambda)$*, there exists* $v \in X^{\mathrm{ad}}$ *whose center is* a_Λ *for any* Λ*. Then, by Proposition 3.4.13, the valuation* v *is contained in* $\mathrm{Sk}(X^{\mathrm{ad}})$*.*

Proposition 3.4.15.

$$\mathrm{Sk}(X^{\mathrm{ad}}) \cap X^{\mathrm{ht}\leq 1} = \mathrm{Sk}\, X^{\mathrm{Ber}}/(\textit{the equivalence relations of valuations}).$$

Proof. This follows from Lemma 3.3.7, [**Ber90**, Proposition 2.4.4], and [**Gub13**, Remark 12.7], $\square$

Proposition 3.4.16. *For* $v \in \mathrm{Sk}(X^{\mathrm{ad}})$*, we have*

$$\mathrm{ht}(v) = \mathrm{ht}(\mathrm{Trop}^{\mathrm{ad}}(v)).$$

Proof. This follows from Lemma 3.4.9. $\square$

Remark 3.4.17. *We have*

$$\mathrm{Trop}^{\mathrm{ad}}(X^{\mathrm{ad}}) = \mathrm{Trop}^{\mathrm{ad}}(\mathrm{ZR}(K(X)/K)).$$

Here, we consider $\mathrm{ZR}(K(X)/K) \subset X^{\mathrm{ad}}$*.*

3.4.2. Tropicalizations of Zariski-Riemann spaces. In this subsection, we shall introduce tropicalizations of explicit subsets of Berkovich analytic spaces and *tropicalizations of Zariski-Riemann spaces*. They will be used to compute tropical analogs of Milnor K-groups. Let K be a trivially valued field, L/K a finitely generated extension of fields. For a morphism $\varphi\colon \operatorname{Spec} L \to \mathbb{G}_m^n$ over K, there is a canonical morphism

$$\varphi\colon (\operatorname{Spec} L/K)^{\mathrm{Ber}} \to \mathbb{G}_m^{n,\mathrm{Ber}}.$$

DEFINITION 3.4.18. *We call*

$$\operatorname{Trop}(\varphi(\operatorname{Spec} L/K)) := \operatorname{Trop}(\varphi((\operatorname{Spec} L/K)^{\mathrm{Ber}}))$$

a tropicalization *of* $(\operatorname{Spec} L/K)^{\mathrm{Ber}}$.

REMARK 3.4.19. *We have*

$$\operatorname{Trop}(\varphi(\operatorname{Spec} L/K)) = \operatorname{Trop}(\overline{\varphi(\operatorname{Spec} L)}),$$

where $\overline{\varphi(\operatorname{Spec} L)}$ *is the closure in* $\mathbb{G}_m^n$. *(This is because any affinoid subdomain of* $\overline{\varphi(\operatorname{Spec} L)}^{\mathrm{Ber}}$ *has a point whose support is the generic point of* $\overline{\varphi(\operatorname{Spec} L)}$.*) In particular,* $\operatorname{Trop}(\varphi(\operatorname{Spec} L/K))$ *is a finite union of cones.*

DEFINITION 3.4.20. *For a fan structure* Λ *of*

$$\operatorname{Trop}(\varphi(\operatorname{Spec} L/K)) = \operatorname{Trop}(\overline{\varphi(\operatorname{Spec} L)}),$$

we denote the composition

$$\operatorname{ZR}(L/K) \to \overline{\varphi(\operatorname{Spec} L)}^{\mathrm{ad}} \xrightarrow{\operatorname{Trop}_\Lambda^{\mathrm{ad}}} \Lambda,$$

by $\operatorname{Trop}_\Lambda^{\mathrm{ad}} \circ\varphi$*, where the first morphism is the canonical map. Taking all fan structures* Λ *of* $\operatorname{Trop}(\varphi(\operatorname{Spec} L/K))$*, we have a surjective map*

$$\operatorname{Trop}^{\mathrm{ad}} \circ\varphi\colon \operatorname{ZR}(L/K) \to \varprojlim \Lambda$$

called a tropicalization map *of the Zariski-Riemann space* $\operatorname{ZR}(L/K)$.

3.4.3. Tropicalizations of adic spaces over valuation fields of height 1. In this subsection, we shall define tropicalizations of adic space over a complete non-archimedean valuation field (L, v) of height 1.

We shall define the *tropicalization map*

$$\operatorname{Trop}_\Xi^{\mathrm{ad}}\colon U^{\mathrm{ad}} \to \Xi$$

for a closed subvarity $U \subset \mathbb{G}_m^n$ over L, and a $\sqrt{v(L^\times)}$-rational polyhedral complex structure Ξ of $\operatorname{Trop}(U)$.

We assume that Ξ is a polyhedral subcomplex of a polyhedral complex whose support is $N_\mathbb{R}$, and that any element of Ξ does not contains affine lines. For general polyhedral complex structure Ξ' of $\operatorname{Trop}(U)$, there exists a refinement Ξ of Ξ' satisfying the above assumptions, and we define $\operatorname{Trop}_{\Xi'}^{\mathrm{ad}}$ to be the composition of $\operatorname{Trop}_\Xi^{\mathrm{ad}}$ and the natural map $\Xi \to \Xi'$.

We construct the above map. As in [**Gub13**, Section 7], there exists a toric scheme $T_{c(\Xi)}$ associated with Ξ over the valuation ring $L^{\mathrm{alg},\circ}$ of L^{alg}. (The existence follows from [**BS10**, Corollary 3.11]. See [**Gub13**, Remark 7.6].) Then, by [**Gub13**, Proposition 11.12], the closure $\mathscr{U}$ of $U_{L^{\mathrm{alg}}}$ in the toric scheme $T_{c(\Xi)}$ is proper over $L^{\mathrm{alg},\circ}$. Hence for any $v \in U_{L^{\mathrm{alg}}}^{\mathrm{ad}}$, there is a natural morphism $\operatorname{Spec} \mathcal{O}_v \to \mathscr{U}$. We put $\chi(v)$ the image of the maximal ideal. This induces a map $\chi\colon U_{L^{\mathrm{alg}}}^{\mathrm{ad}} \to \mathscr{U}_s$, where $\mathscr{U}_s$ is the closed fiber of

$\mathscr{U}$. We consider the composition $U_{L^{\mathrm{alg}}}^{\mathrm{ad}} \to \Xi$ of this map and a map $\mathscr{U}_s \to \Xi$ defined by $\mathscr{U}_s \ni x \mapsto \xi_x \in \Xi$, where ξ_x is such that the orbit $O(\xi_x)$ contains x. By construction, this composition $U_{L^{\mathrm{alg}}}^{\mathrm{ad}} \to \Xi$ induces a map

$$\mathrm{Trop}_{\Xi}^{\mathrm{ad}} \colon U_L^{\mathrm{ad}} \to \Xi.$$

We extend definition of tropicalization maps to subvarieties of toric varieties by taking disjoint union of them on each orbits. Let X be a closed subvariety of a toric variety T_Σ over L. For a $\sqrt{v(L^\times)}$-rational polyhedral complex structure Ξ of $\mathrm{Trop}(X)$, we put

$$\mathrm{Trop}_{\Xi}^{\mathrm{ad}} \colon X^{\mathrm{ad}} \to \Xi$$

the disjoint union of $\mathrm{Trop}_{\Xi \cap \mathrm{Trop}(O(\sigma))}^{\mathrm{ad}}$ $(\sigma \in \Sigma)$.

Remark 3.4.21. *There is a canonical map* $\mathrm{Trop}(X^{\mathrm{Ber}}) \to \Xi$ *given by* $x \mapsto P_x$, *where* $P_x \in \Xi$ *is a polyhedron whose relative interior containing* x. *We have a commutative diagram*

$$\begin{tikzcd} X^{\mathrm{Ber}} \arrow[r, "\mathrm{Trop}"] \arrow[d] & \mathrm{Trop}(X^{\mathrm{Ber}}) \arrow[d] \\ X^{\mathrm{ad}} \arrow[r, "\mathrm{Trop}_{\Xi}^{\mathrm{ad}}"] & \Xi. \end{tikzcd}$$

CHAPTER 4

Tropical cohomology

In this chapter, we study *tropical cohomology*, (Subsection 4.1.1 and 4.1.2), *tropical cycle class maps* (Section 4.2), and show that the tropical analog of de Rham's theorem given by Jell-Shaw-Smacka [**JSS17**, Theorem 1] is given by integrations (Section 4.3).

Let K be a trivially valued field. Let M be a free $\mathbb{Z}$-module of finite rank. We put $N := \mathrm{Hom}(M, \mathbb{Z})$.

4.1. Tropical cohomology

4.1.1. Tropical cohomology of fan spaces. We remind tropical cohomology of fans introduced by Itenberg-Katzarkov-Mikhalkin-Zharkov [**IKMZ17**]. See also [**JSS17**, Section 3]. We define a slight modification, which is used to prove existance of retraction maps in Section 7.2 and 9.

Let T_Σ be the toric variety over K associated with a fan Σ in $N_\mathbb{R}$. Let Λ be a polyhedral complex in $\mathrm{Trop}(T_\Sigma)$. We put $A := |\Lambda|$. For a subset $B \subset \mathrm{Trop}(T_\Sigma)$, we put

$$\Lambda \cap B := \{P \cap B\}_{P \in \Lambda}.$$

We assume that Λ is a fan. Remind that for $P \in \Lambda$, we putted $\sigma_P \in \Sigma$ the cone such that $\mathrm{rel.int}(P) \subset \mathrm{Trop}(O(\sigma_P))$. Let $p \geq 0$ be a non-negative integer. We put

$$\mathrm{Span}(P) := \mathrm{Span}_\mathbb{Q}(P \cap \mathrm{Trop}(O(\sigma_P)))$$

the $\mathbb{Q}$-linear subspace of $\mathrm{Trop}(O(\sigma_P))$ spanned by $P \cap \mathrm{Trop}(O(\sigma_P))$,

$$F_p(P, \Lambda) := \sum_{\substack{P' \in \Lambda \cap \mathrm{Trop}(O(\sigma_P)) \\ \mathrm{rel.int}(P) \subset P'}} \wedge^p \mathrm{Span}(P') \subset \wedge^p \mathrm{Trop}(O(\sigma_P)),$$

and

$$F^p(P, \Lambda) := \wedge^p (M \cap \sigma_P^\perp)_\mathbb{Q} / \{ f \in \wedge^p (M \cap \sigma_P^\perp)_\mathbb{Q} \mid \alpha(f) = 0 \ (\alpha \in F_p(P, \Lambda))\}.$$

We have

$$F^p(P, \Lambda) \cong \mathrm{Hom}(F_p(P, \Lambda), \mathbb{Q}).$$

Since $F_p(P, \Lambda)$ and $F^p(P, \Lambda)$ depends only on the support $A = |\Lambda|$, we sometimes write $F_p(P, A)$ (resp. $F^p(P, A)$) instead of $F_p(P, A)$ (resp. $F_p(P, \Lambda)$). When there is no confusion, we simply write $F_p(P)$ (resp. $F^p(P)$) instead of $F_p(P, \Lambda)$ (resp. $F^p(P, \Lambda)$).

Remark 4.1.1. *Let $P_1, P_2 \in \Lambda$ with $P_2 \subset P_1$. In this case, we have $\sigma_{P_1} \subset \sigma_{P_2}$.*

- *When $\sigma_{P_1} = \sigma_{P_2}$, there exists a natural injection*

$$i_{P_2 \subset P_1} \colon F_p(P_1) \hookrightarrow F_p(P_2).$$

- *When $P_2 = P_1 \cap \overline{\mathrm{Trop}(O(\sigma_{P_2}))}$, the natural projection* $\mathrm{Trop}(O(\sigma_{P_1})) \twoheadrightarrow \mathrm{Trop}(O(\sigma_{P_2}))$ *induces a morphism*

$$i_{P_2 \subset P_1} \colon F_p(P_1) \to F_p(P_2).$$

- *In general, we put*

$$i_{P_2\subset P_1} := i_{P_2\subset Q}\circ i_{Q\subset P_1}\colon F_p(P_1)\to F_p(Q)\hookrightarrow F_p(P_2),$$

where $Q := P_1 \cap \mathrm{Trop}(O(\sigma_{P_2}))$.

DEFINITION 4.1.2. *Let* $B\subset A$ *be a subset, and* Λ *a fan structure of* A.

(1) *For every cone* $P\in\Lambda$, *we put* $C_q(B\cap P)$ *the free* $\mathbb{Q}$*-vector space generated by continuous maps* $\gamma\colon \Delta^q\to B\cap P$ *from the standard* q*-simplex* Δ^q *such that*

$$\gamma(\mathrm{rel.int}(\Delta^q))\subset B\cap \mathrm{rel.int}(P)$$

and the image of the relative interior of each face of Δ^q *is contained in the relative interior of a face of* P. *We put*

$$C_{p,q}(B,\Lambda) := \oplus_{P\in\Lambda}F_p(P,\Lambda)\otimes_{\mathbb{Q}} C_q(B\cap P).$$

We call its elements tropical (p,q)-chains *on* (B,Λ).

(2) *For* $\gamma\in C_q(B\cap P)$, *we denote the usual boundary by* $\partial(\gamma) := \sum_{i=0}^{q}(-1)^i\gamma^i$. *For each* $v\otimes\gamma\in F_p(P,\Lambda)\otimes C_q(B\cap P)$, *we put*

$$\partial(v\otimes\gamma) := \sum_{i=0}^{q}(-1)^i i_{P_{\gamma,i}\subset P}(v)\otimes\gamma^i\in C_{p,q-1}(B,\Lambda),$$

where $P_{\gamma,i}\in\Lambda$ *is a face of* P *such that*

$$\gamma^i(\mathrm{rel.int}(\Delta^{q-1}))\subset \mathrm{rel.int}(P_{\gamma,i}).$$

We obtain complexes $(C_{p,*}(B,\Lambda),\partial)$.

(3) *We define the* tropical homology groups *to be*

$$H^{\mathrm{Trop}}_{p,q}(B,\Lambda) := H_q(C_{p,*}(B,\Lambda),\partial).$$

We put $(C^{p,*}(B,\Lambda),\delta)$ *the dual complex of* $(C_{p,*}(B,\Lambda),\partial)$. *We call its cohomology groups*

$$H^{p,q}_{\mathrm{Trop}}(B,\Lambda) := H^q(C^{p,*}(B,\Lambda),\delta)$$

the tropical cohomology groups *of* (B,Λ).

REMARK 4.1.3. *We put*

$$Z^{p,q}(B,\Lambda) := \mathrm{Ker}(\delta\colon C^{p,q}(B,\Lambda)\to C^{p,q+1}(B,\Lambda)).$$

For a subset $D\subset B$, *we put*

$$C^{p,q}_D(B,\Lambda) := \mathrm{Ker}(C^{p,q}(B,\Lambda)\to C^{p,q}(D,\Lambda)).$$

We have

$$C^{p,q}(B,\Lambda)\cong \oplus_{P\in\Lambda}F^p(P)\otimes_{\mathbb{Q}}\mathrm{Hom}(C_q(B\cap P),\mathbb{Q}).$$

We identify the both hand sides.

For each $\alpha\in C^{p,q}(B,\Lambda)$ *and* $\gamma\in\bigsqcup_{P\in\Lambda}C_q(B\cap P)$, *we put* $\alpha_\gamma\in F^p(P_\gamma)$ *the restriction of* α *to*

$$\begin{array}{ccc} F_p(P_\gamma)\otimes_{\mathbb{Q}}\mathbb{Q}\cdot\gamma & \xrightarrow{\simeq} & F_p(P_\gamma) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ x\otimes\gamma & \longmapsto & x \end{array}$$

where $P_\gamma\in\Lambda$ *is the cone such that* $\gamma(\mathrm{rel.int}(\Delta^q))\subset\mathrm{rel.int}(P_\gamma)$. *We often write* $\alpha = (\alpha_\gamma)_\gamma$.

REMARK 4.1.4. *Tropical cohomology does not depend on the choice of the fan structure* Λ *of* $|\Lambda|$, *i.e., for a refinement* Λ' *of* Λ *and any* p, q, *the natural map*

$$H^{p,q}_{\mathrm{Trop}}(B, \Lambda) \to H^{p,q}_{\mathrm{Trop}}(B, \Lambda')$$

is an isomorphism. This follows from [**MZ13**, Proposition 2.8] *(see* [**JSS17**, Section 3] *for a proof of it). We write* $H^{p,q}_{\mathrm{Trop}}(A) := H^{p,q}_{\mathrm{Trop}}(A, \Lambda)$ *for a set* A *and a polyhedral complex* Λ *such that* $A = |\Lambda|$.

We define a slight modification, which is used to prove existance of retraction maps in Section 7.2 and 9.

DEFINITION 4.1.5. *Let* $B \subset A$ *be a subset, and* Λ *a fan structure of* A.

(1) For every cone $P \in \Lambda$, *we put* $C'_q(B \cap P)$ *the free* $\mathbb{Q}$*-vector space generated by continuous maps* $\gamma\colon \Delta^q \to B \cap P$ *from the standard* q*-simplex* Δ^q *such that the image* $\gamma(\Delta^q)$ *is not contained in any proper face of* P. *We put*

$$C'_{p,q}(B, \Lambda) := \oplus_{P \in \Lambda} F_p(P, \Lambda) \otimes_{\mathbb{Q}} C'_q(B \cap P).$$

(2) For $\gamma \in C'_q(B \cap P)$, *we denote the usual boundary by* $\partial(\gamma) := \sum_{i=0}^{q} (-1)^i \gamma^i$. *For each* $v \otimes \gamma \in F_p(P, \Lambda) \otimes C'_q(B \cap P)$, *we put*

$$\partial(v \otimes \gamma) := \sum_{i=0}^{q} (-1)^i i_{P_{\gamma,i} \subset P}(v) \otimes \gamma^i \in C'_{p,q-1}(B, \Lambda),$$

where $P_{\gamma,i} \in \Lambda$ *is the minimal face of* P *containing* $\mathrm{Im}(\gamma^i)$. *We obtain complexes* $(C'_{p,*}(B, \Lambda), \partial)$.

(3) We put

- $(C'^{p,*}(B, \Lambda), \delta)$ *the dual complex of* $(C'_{p,*}(B, \Lambda), \partial)$,
- $H'^{\mathrm{Trop}}_{p,q}(B, \Lambda) := H_q(C_{p,*}(B, \Lambda), \partial)$, *and*
- $H'^{p,q}_{\mathrm{Trop}}(B, \Lambda) := H^q(C^{p,*}(B, \Lambda), \delta)$.

LEMMA 4.1.6. *We assume that* $B \cap \Lambda := \{B \cap P\}_{P \in \Lambda}$ *is a polyhedral complex structure of* B. *Then, for any* p, q, *the natural morphism*

$$H'^{p,q}_{\mathrm{Trop}}(B, \Lambda) \to H^{p,q}_{\mathrm{Trop}}(B, \Lambda)$$

is an isomorphism.

PROOF. In the same way as [**JSS17**, Proposition 3.15], there is a natural isomorphism

$$H^{p,q}_{\mathrm{Trop}}(B, \Lambda) \cong H^q(B, \mathscr{F}^p|_B),$$

where $\mathscr{F}^p$ is a sheaf on A, see [**JSS17**, Proposition 3.6] for the definition. In a similar way, there is a natural isomorphism

$$H'^{p,q}_{\mathrm{Trop}}(B, \Lambda) \cong H^q(B, \mathscr{F}^p|_B).$$

Hence the Lemma holds. □

We give an easy lemma. Let $\psi\colon T_{\Sigma'} \to T_\Sigma$ be a toric morphism, Λ (resp. Λ') a fan in $\mathrm{Trop}(T_\Sigma)$ (resp. $\mathrm{Trop}(T_{\Sigma'})$), and $B \subset |\Lambda|$ and $B' \subset |\Lambda'|$ subsets such that, for $P \in \Lambda'$, there exists $Q \in \Lambda$ containing $\psi(P)$, and $\psi(B') \subset B$. Then, as usual, the map $\psi\colon \mathrm{Trop}(T_{\Sigma'}) \to \mathrm{Trop}(T_\Sigma)$ induces maps

$$\psi^*\colon C^{p,q}(B, \Lambda) \to C^{p,q}(B', \Lambda')$$

and

$$\psi^*\colon H^{p,q}_{\mathrm{Trop}}(B, \Lambda) \to H^{p,q}_{\mathrm{Trop}}(B', \Lambda').$$

LEMMA 4.1.7. *We assume that $\psi(B') = B$, $\psi(|\Lambda'|) = |\Lambda|$, and $\psi(P) \in \Lambda$ $(P \in \Lambda')$. Then, for any p, q, the map*

$$\psi^* \colon C^{p,q}(B, \Lambda) \to C^{p,q}(B', \Lambda')$$

is injective.

PROOF. The assertion follows from the surjectivity of $B' \to B$ and $|\Lambda'| \to |\Lambda|$. $\square$

4.1.2. Tropical cohomology of algebraic varieties over trivially valued fields. We define *tropical cohomology* of an algebraic variety X over K by tropical charts in a similar way to *tropical Dolbault cohomology* $H^{p,q}_{\mathrm{Trop,Dol}}(X)$. Tropical charts are first given by Chambert-Loir-Ducros [**CLD12**] over valued fields of height 1. Slightly different formulations were given by Gubler [**Gub16**] and Jell [**Jel16**]. Our formulations are Jell's ones in [**Jel16**, Section 3.2 and 3.3]. (Note that tropical cohomology and tropical Dolbeault cohomology are isomorphic by [**JSS17**, Theorem 1].) We also define tropical cohomology groups of the subset $X^\circ \subset X^{\mathrm{Ber}}$ (Definition 3.3.9), which are isomorphic to those of X.

We define tropical cohomology of X. We define a sheaf $\mathscr{C}^{p,q}_X$ on X^{Ber} by, for each open subset $V \subset X^{\mathrm{Ber}}$, putting $\mathscr{C}^{p,q}_X(V)$ the set of equivalence classes of $(U_i, V_i, \varphi_i, \Lambda_i, \alpha_i)_i$ consisting of

- open coverings $\{U_i\}_i$ of X and $\{V_i\}_i$ of V,
- closed immersions $\varphi_i \colon U_i \to \mathbb{A}^{n_i}$ such that $V_i = (\mathrm{Trop} \circ \varphi_i)^{-1}(\Omega_i) \subset U_i^{\mathrm{Ber}}$ for some open subset $\Omega_i \subset \mathrm{Trop}(\varphi_i(U_i))$,
- fan structures Λ_i of $\mathrm{Trop}(\varphi_i(U_i))$, and
- $\alpha_i \in C^{p,q}(\mathrm{Trop}(\varphi_i(V_i)), \Lambda_i)$

satisfying the following: for any i, j, there exists

- a covering $\{U_{i,j,k}\}_k$ of $U_i \cap U_j$,
- closed immersions $\varphi_{i,j,k} \colon U_{i,j,k} \to \mathbb{A}^{n_{i,j,k}}$,
- toric morphisms $\Psi_{(i,j,k),i} \colon \mathbb{A}^{n_{i,j,k}} \to \mathbb{A}^{n_i}$ and $\Psi_{(i,j,k),j} \colon \mathbb{A}^{n_{i,j,k}} \to \mathbb{A}^{n_j}$, and
- fan structures $\Lambda_{i,j,k}$ of $\mathrm{Trop}(\varphi_{i,j,k}(U_{i,j,k}))$

such that

- for each $P \in \Lambda_{i,j,k}$ and $l \in \{i, j\}$, there exists $Q \in \Lambda_l$ containing $\Psi_{(i,j,k),l}(P)$,
- the diagrams

$$\begin{tikzcd} & U_{i,j,k} \arrow[dl, "\varphi_i"'] \arrow[d, "\varphi_{i,j,k}"] \arrow[dr, "\varphi_j"] & \\ \mathbb{A}^{n_i} & \mathbb{A}^{n_{i,j,k}} \arrow[l, "\Psi_{(i,j,k),i}"] \arrow[r, "\Psi_{(i,j,k),j}"'] & \mathbb{A}^{n_j} \end{tikzcd}$$

  are commutative, and
- we have

$$\begin{aligned} \Psi^*_{(i,j,k),i}\alpha_i|_{\mathrm{Trop}(\varphi_{i,j,k}(V_i \cap V_j \cap U^{\mathrm{Ber}}_{i,j,k}))} &= \Psi^*_{(i,j,k),j}\alpha_j|_{\mathrm{Trop}(\varphi_{i,j,k}(V_i \cap V_j \cap U^{\mathrm{Ber}}_{i,j,k}))} \\ &\in C^{p,q}(\mathrm{Trop}(\varphi_{i,j,k}(V_i \cap V_j \cap U^{\mathrm{Ber}}_{i,j,k})), \Lambda_{i,j,k}). \end{aligned}$$

The equivalence relation is generated by

$$(U_i, V_i, \varphi_i, \Lambda_i, \alpha_i) \sim (U'_j, V'_j, \varphi'_j, \Lambda'_j, \alpha'_j),$$

satisfying the following: for each j, there exist $i(j)$ and a toric morphism $\psi_j \colon \mathbb{A}^{n'_j} \to \mathbb{A}^{n_{i(j)}}$ such that

- $U_{i(j)}$ (resp. $V_{i(j)}$) contains U'_j (resp. V'_j),

- the diagram

$$\begin{tikzcd} U_{i(j)} \arrow[r, "\varphi_{i(j)}"] & \mathbb{A}^{n_{i(j)}} \\ U'_j \arrow[u] \arrow[r, "\varphi'_j"] & \mathbb{A}^{n'_j} \arrow[u, "\psi_j"'] \end{tikzcd}$$

  is commutative,
- for $P \in \Lambda'_j$, there exists $Q \in \Lambda_{i(j)}$ containing $\psi_j(P)$, and
- $\alpha'_j = \psi_j^* \alpha_i|_{\mathrm{Trop}(\varphi'_j(V'_j))}$.

The coboundary map δ in Definition 4.1.2 induces a complex

$$\mathscr{C}_X^{p,0} \to \mathscr{C}_X^{p,1} \to \mathscr{C}_X^{p,2} \to \dots$$

of sheaves on X^{Ber}. The cohomology groups

$$H^{p,q}_{\mathrm{Trop}}(X) := \mathrm{Ker}(\mathscr{C}_X^{p,q}(X^{\mathrm{Ber}}) \to \mathscr{C}_X^{p,q+1}(X^{\mathrm{Ber}}))/\,\mathrm{Im}(\mathscr{C}_X^{p,q-1}(X^{\mathrm{Ber}}) \to \mathscr{C}_X^{p,q}(X^{\mathrm{Ber}}))$$

of global secitons of this complex are called the *tropical cohomology groups* of X. By [**JSS17**, Proposition 3.15], they are the sheaf cohomology groups of a sheaf $\mathscr{F}_X^p := \mathrm{Ker}(\mathscr{C}_X^{p,0} \to \mathscr{C}_X^{p,1})$ on X^{Ber}. For a closed subscheme $Z \subset X$, we put

$$\mathscr{C}^{p,q}_{Z\subset X} := \mathrm{Ker}(\mathscr{C}_X^{p,q} \to \pi_*\pi^*\mathscr{C}_X^{p,q}),$$

where $\pi\colon X \setminus Z \to X$ is the inclusion, and

$$H^{p,q}_{\mathrm{Trop},Z}(X) := \mathrm{Ker}(\mathscr{C}^{p,q}_{Z\subset X}(X) \to \mathscr{C}^{p,q+1}_{Z\subset X}(X))/\,\mathrm{Im}(\mathscr{C}^{p,q-1}_{Z\subset X}(X) \to \mathscr{C}^{p,q}_{Z\subset X}(X)).$$

We consider another expression of tropical cohomology by embeddings of X to toric varieties. In the rest of this subsection, we assume that X has a closed immersion to a toric variety. Then there are many closed immersions of X to toric varieties [**FGP14**, Theorem 1.2]. Jell used them to give another representation of tropical Dolbeault cohomology [**Jel16**, Section 3.2 and 3.3]. We imitate his difinition. We define a sheaf $\mathscr{C}^{p,q}_{T,X}$ on X^{Ber} in a similar way to $\mathscr{C}_X^{p,q}$ but the differences are using closed immersions $X \to T_{\Sigma_i}$ to a toric variety T_{Σ_i} instead of open subvarieties $U_i \subset X$ and closed immersions $U_i \to \mathbb{A}^{n_i}$. Then in a similar way to [**Jel16**, Proposition 3.3.4], we have the following.

LEMMA 4.1.8. *For any p, q, there exists a natural isomorphism $\mathscr{C}_X^{p,q} \cong \mathscr{C}^{p,q}_{T,X}$.*

In particular, the tropical cohomology $H^{p,q}_{\mathrm{Trop}}(X)$ is isomorphic to

$$H^{p,q}_{T,\mathrm{Trop}}(X) := \mathrm{Ker}(\mathscr{C}^{p,q}_{T,X}(X^{\mathrm{Ber}}) \to \mathscr{C}^{p,q+1}_{T,X}(X^{\mathrm{Ber}}))/\,\mathrm{Im}(\mathscr{C}^{p,q-1}_{T,X}(X^{\mathrm{Ber}}) \to \mathscr{C}^{p,q}_{T,X}(X^{\mathrm{Ber}})).$$

By [**JSS17**, Proposition 3.15], it is the sheaf cohomology of a sheaf $\mathscr{F}^p_{T,X} := \mathrm{Ker}(\mathscr{C}^{p,0}_{T,X} \to \mathscr{C}^{p,1}_{T,X})$ on X^{Ber}.

We define tropical cohomology groups of the set X° in the same way as those of X^{Ber}. (They are isomorphic by Corollary 7.2.11 and 7.2.12.) We only describe the differences. We define sheaves $\mathscr{C}^{p,q}_{X^\circ}$ and $\mathscr{C}^{p,q}_{T,X^\circ}$ on X° in a similar way to $\mathscr{C}^{p,q}_X$ and $\mathscr{C}^{p,q}_{T,X}$ by using an open subset Ω_i of $\mathrm{Trop}(\varphi_i(U_i^{\mathrm{Ber}} \cap X^\circ))$ instead of an open subset of $\mathrm{Trop}(\varphi(U_i))$. For a closed subscheme $Z \subset X$, we also define sheaves $\mathscr{C}^{p,q}_{Z^\circ\subset X^\circ}$ and $\mathscr{C}^{p,q}_{T,Z^\circ\subset X^\circ}$. We define cohomology groups $H^{p,q}_{\mathrm{Trop}}(X^\circ)$, $H^{p,q}_{T,\mathrm{Trop}}(X^\circ)$, $H^{p,q}_{\mathrm{Trop},Z^\circ}(X^\circ)$, and $H^{p,q}_{T,\mathrm{Trop},Z^\circ}(X^\circ)$. We have natural isomorphisms $\mathscr{C}^{p,q}_{X^\circ} \cong \mathscr{C}^{p,q}_{T,X^\circ}$ and $\mathscr{C}^{p,q}_{Z^\circ\subset X^\circ} \cong \mathscr{C}^{p,q}_{T,Z^\circ\subset X^\circ}$. The cohomology $H^{p,q}_{T,\mathrm{Trop}}(X^\circ)$ is the sheaf cohomology of the sheaf $\mathscr{F}^p_{T,X^\circ} := \mathrm{Ker}(\mathscr{C}^{p,0}_{T,X^\circ} \to \mathscr{C}^{p,1}_{T,X^\circ})$ on X°.

4.2. Tropical cycle class map, tropical homology, and intersection theory

In this section, we study tropical cycle class maps to tropical cohomology groups. In [**Liu20**], Liu defined them to tropical Dolbeault cohomology groups. We translate it to tropical cohomology groups by Jell-Shaw-Smacka's tropical analog of de Rham's theorem [**JSS17**, Theorem 1]. We also translate Liu's compatibility of tropical cycle class maps with intersection theory of algebraic cycles [**Liu20**, Theorem 1.1 and Remark 1.3 (1)]. (Theorem 4.2.3). For this, in Section 4.3, we will prove that the tropical analog of de Rham's theorem is given by integrations.

Let X be a d-dimensional smooth proper algebraic variety over K. We assume that there exists a closed immersion of X to a toric variety.

We define tropical homology. Since X is proper, i.e., X^{Ber} is compact, every element of $\mathscr{C}^{p,q}_{T,X}(X)$ has a representative of the form $(X, X^{\mathrm{Ber}}, \varphi\colon X \to T_\Sigma, \Lambda, \alpha)$. Hence we have the following.

LEMMA 4.2.1. *We have*

$$H^{p,q}_{\mathrm{Trop}}(X) \cong \varinjlim_{\varphi} H^{p,q}_{\mathrm{Trop}}(\mathrm{Trop}(\varphi(X))),$$

where $\varphi\colon X \to T_\Sigma$ runs through all closed immersion to toric varieties.

DEFINITION 4.2.2. *We put*

$$H_{p,q}^{\mathrm{Trop}}(X) := \varprojlim_{\varphi} H_{p,q}^{\mathrm{Trop}}(\mathrm{Trop}(\varphi(X))).$$

We call it the (rational) tropical homology group *of X.*

By basics on inductive and projective limits, we have

$$H_{p,q}^{\mathrm{Trop}}(X) \cong \mathrm{Hom}(H^{p,q}_{\mathrm{Trop}}(X), \mathbb{Q}).$$

There is a tropical analog $H^{p,q}_{\mathrm{Trop,Dol}}(X)$ of Dolbeault cohomology called *tropical Dolbeault cohomology.* (See [**Jel16**, Section 3.3 and 3.4]. Note that he denote it by $H^{p,q}_{d''}(X^{\mathrm{an}})$.) Jell, Shaw, and Smacka showed a tropical analog of de Rham's theorem [**JSS17**, Theorem 1]

$$H^{p,q}_{\mathrm{Trop,Dol}}(X) \cong H^{p,q}_{\mathrm{Trop}}(X) \otimes_{\mathbb{Q}} \mathbb{R}.$$

In Section 4.3, we will show that, as mentioned in [**JSS17**, Remark 3.25], the tropical analog of de Rham's theorem is given by integrations of (p, q)-forms.

We define the *tropical cycle class map* to the tropical cohomology group $H^{p,p}_{\mathrm{Trop}}(X) \otimes_{\mathbb{Q}} \mathbb{R}$ to be the composition

$$\mathrm{CH}^p(X) \to H^{p,p}_{\mathrm{Trop,Dol}}(X) \cong H^{p,p}_{\mathrm{Trop}}(X) \otimes_{\mathbb{Q}} \mathbb{R}$$

of Liu's tropical cycle class map $\mathrm{CH}^p(X) \to H^{p,p}_{\mathrm{Trop,Dol}}(X)$ ([**Liu20**, Definition 3.6]) and the isomorphism given by integration. In Section 7.2, we will show that tropical cycle class map induces an isomorphism $\mathrm{CH}^p(X)_{\mathbb{Q}} \cong H^{p,p}_{\mathrm{Trop}}(X)$.

There is also a natural *tropical cycle class map* to the tropical homology group. Let $Y \subset X$ be a closed subvariety of dimension p. Then, by [**MS15**, Theorem 3.4.14] and [**MZ13**, Proposition 4.3], its tropicalization $\mathrm{Trop}(\varphi(Y))$ with the weight defined in [**MS15**, Definition 3.4.3] determines an element of $H_{p,p}^{\mathrm{Trop}}(\mathrm{Trop}(\varphi(X)))$ which is compatible with the push-forward maps. Hence Y determines an element of $H_{p,p}^{\mathrm{Trop}}(X)$. This induces a morphism $Z_p(X) \to H_{p,p}^{\mathrm{Trop}}(X)$ from the set $Z_p(X)$ of algebraic cycles of dimension p to the tropical homology group.

By Corollary 4.3.10, Liu's Theorem [**Liu20**, Theorem 1.1] for tropical Dolbeault cohomology also holds for tropical cohomology. (He proved [**Liu20**, Theorem 1.1] only over complete non-archimedean valuation fields of height 1, but his proof also works over trivially valued fields.) Namely, we have the following.

THEOREM 4.2.3 (Liu [**Liu20**, Theorem 1.1, Remark 1.3(1)]). *We have a commutative diagram*

$$\begin{tikzcd} \mathrm{CH}^p(X)_{\mathbb{Q}} \arrow[d] & \times & \mathrm{Z}_p(X)_{\mathbb{Q}} \arrow[r] \arrow[d] & \mathbb{Q} \arrow[d, hook] \\ H^{p,p}_{\mathrm{Trop}}(X) \otimes_{\mathbb{Q}} \mathbb{R} & \times & H^{\mathrm{Trop}}_{p,p}(X) \arrow[r] & \mathbb{R}, \end{tikzcd}$$

where the first horizontal arrow is the intersection theory.

4.3. The tropical analog of de Rham's theorem

In this section, we shall show that the tropical analog of de Rham's theorem is given by integrations on semi-algebraic singular simplices (Corollary 4.3.10). See [**Jel16**] and [**JSS17**] for notations and basics on (p,q)-superforms and tropical Dolbeault cohomology.

Let X be an algebraic variety over a trivially valued field K. Let Σ be a fan in $N_{\mathbb{R}}$. We shall define semi-algebraic maps to tropicalizations of algebraic varieties.

DEFINITION 4.3.1. *A subset $A \subset \mathrm{Trop}(T_\Sigma)$ is said to be* semi-algebraic *if*

$$A \cap \mathrm{Trop}(O(\sigma)) \subset \mathrm{Trop}(O(\sigma)) = N_{\sigma,\mathbb{R}}$$

is semi-algebraic in the usual sense for any cone $\sigma \in \Sigma$.

REMARK 4.3.2. *For a closed immersion $\phi\colon X \to T_\Sigma$, the tropicalization* $\mathrm{Trop}(\phi(X))$ *is semi-algebraic.*

DEFINITION 4.3.3. *Let $S \subset \mathbb{R}^m$ be a semi-algebraic set. A map $\varphi\colon S \to \mathrm{Trop}(T_\Sigma)$ is said to be* semi-algebraic *if and only if the graph*

$$\Gamma_\varphi \subset \mathbb{R}^m \times \mathrm{Trop}(T_\sigma) = \mathbb{R}^m \times N_{\sigma,\mathbb{R}}$$

is semi-algebraic in the usual sense for any cone $\sigma \in \Sigma$.

EXAMPLE 4.3.4. *A map*

$$[0,1]^n \to [1,\infty]^n \subset (\mathbb{R} \cup \{\infty\})^n = \mathrm{Trop}(\mathbb{A}^n)$$

given by $0 \mapsto \infty$ and $t \mapsto \frac{1}{t}$ for $t \neq 0$ is a semi-algebraic homeomorphism.

We define the *semi-algebraic tropical cohomology group* $H^{p,q}_{\mathrm{Trop,semi-alg}}(X)$ in a similar way to the usual tropical cohomology group $H^{p,q}_{\mathrm{Trop}}(X)$ but using semi-algebraic singular simplices instead of usual singular simplices.

LEMMA 4.3.5. *We have a natural isomorphism $H^{p,q}_{\mathrm{Trop}}(X) \cong H^{p,q}_{\mathrm{Trop,semi-alg}}(X)$*

PROOF. By discussions in [**JSS17**, Section 3], what we need to show is that for any point x in the tropicalizations of open varieties of X there exist an open neighorhood U of x and a semi-algebraic strong deformation retractions of U onto x. By the existence of semi-algebraic triangulations [**BCR98**, Theorem 9.2.1] and a similar discussion to Lemma 7.2.6, it suffices to show that for a point y in the boundary $\partial\Delta$ of the singular simplex Δ and a suitable open neighorhood $V \subset \Delta$ of y, there exists a semi-algebraic strong deformation retractions of Vonto $V \cap \partial\Delta$. This assertion is trivial. ☐

We shall show that integrations of (p, q)-superforms on semi-algebraic singular simplices are well-defined. It suffices to show that for a cone $\sigma \in \Sigma$, a closed subvariety $Y \subset T_\sigma$, an open subset $V \subset \mathrm{Trop}(Y)$, a continuous semi-algebraic map $\varphi\colon \Delta^q \to V$, and a $(0, q)$-superform $w \in \mathscr{A}^{0,q}(V)$, the integration $\int_{\Delta^q} \varphi^* w \in \mathbb{R}$ is well-defined. Namely, then, for a (p, q)-superform $v \in \mathscr{A}^{p,q}(V)$ and a semi-algebraic tropical (p, q)-chain

$$\alpha = (\alpha_\varphi)_{\varphi\colon \Delta^q \to V} \in C^{\mathrm{semi-alg}}_{p,q}(V),$$

the integration

$$\int_\alpha v := \sum_\varphi \int_{\Delta^q} \varphi^* \langle v; \alpha_\sigma \rangle$$

is well-defined, here $\langle v; \alpha_\varphi \rangle \in \mathscr{A}^{0,q}(V)$ is the contraction. (See [**JSS17**, Section 2] for contractions.) By definition of $\mathscr{A}^{0,q}$, without loss of generality, we may assume that w is the pull-back of the restriction w_σ of w to an open subset of $\mathrm{Trop}(O(\sigma))$ under the projection $\Phi_\sigma\colon \mathrm{Trop}(T_\sigma) \to \mathrm{Trop}(O(\sigma))$ given by the natural one $N \twoheadrightarrow N_\sigma$. We deal w_σ as an usual smooth q-form. Since the restrictions of this projection to each orbits are semi-algebraic, the composition

$$\Phi_\sigma \circ \varphi\colon \Delta^q \to \mathrm{Trop}(T_\sigma) \to \mathrm{Trop}(O(\sigma))$$

is semi-algebraic. By [**HKT15**, Theorem 2.6], the integration $\int_{\Delta^q} (\Pi_\sigma \circ \varphi)^* w_\sigma$ is well-defined and takes value in $\mathbb{R}$. We define the integration of the $(0, q)$-form w by $\int_{\Delta^q} \varphi^* w := \int_{\Delta^q} (\Pi_\sigma \circ \varphi)^* w_\sigma$. Consequently, we have the following.

Proposition 4.3.6. *The integrations of (p, q)-superforms on semi-algebraic tropical (p, q)-chains defined by (4.3) are well-defined.*

The next Lemma, a version of the Stokes' theorem, is a slight generalization of [**HKT15**, Theorem 2.9].

Lemma 4.3.7. *Let $\varphi\colon \Delta^q \to \mathbb{R}^l$ be a continuous semi-algebraic map, and w a smooth $(q-1)$-form on an open neighorhood of $\varphi(\Delta^q)$. Then we have*

$$\int_{\Delta^q} \varphi^*(dw) = \int_{\partial \Delta^q} \varphi^* w.$$

Here, d is the usual derivation and $\partial \Delta^q$ is the boundary.

Proof. When φ is smooth on $\mathrm{rel.int}(\Delta^q)$, the Lemma is just [**HKT15**, Theorem 2.9]. (As mentioned in the proof of it, [**HKT15**, Theorem 2.9] works for continuous semi-algebraic map which is smooth on the interior.) In general, the Lemma follows from the fact that φ is smooth on a dense open semi-algebraic subset U of Δ^q ([**HKT15**, Corollary 1.12.1]) and the existence of a semi-algebraic triangulation $(\psi_i)_i\colon \bigsqcup_i \Delta^q \to \Delta^q$ of Δ^q such that the restriction of maps ψ_i to each dense open face is smooth and the images of dense open faces are contained in U ([**BCR98**, Theorem 9.21]). $\square$

Corollary 4.3.8. *The integrations give a morphism of complexes $\mathscr{A}^{p,*} \to \mathscr{C}^{p,*}_{\mathrm{semi-alg}}$.*

Remark 4.3.9. *Tropical cohomology $H^{p,q}_{\mathrm{Trop}}(X)_\mathbb{R}$ (resp. tropcial Dolbeault cohomology $H^{p,q}_{\mathrm{Trop,Dol}}(X)_\mathbb{R}$) is isomorphic to sheaf cohomology of a sheaf $\mathscr{F}^p_\mathbb{R}$ (resp.* $\mathrm{Ker}(\mathscr{A}^{p,0} \to \mathscr{A}^{p,1})$*) on X^{Ber}. Jell, Shaw, and Smacka's tropical analog of de Rham's theorem $H^{p,q}_{\mathrm{Trop,Dol}}(X) \cong H^{p,q}_{\mathrm{Trop}}(X)_\mathbb{R}$* [**JSS17**, Theorem 1] *is given by an isomorphism of the sheaves* $\mathrm{Ker}(\mathscr{A}^{p,0} \to \mathscr{A}^{p,1}) \cong \mathscr{F}^p$*, which is given by integration on points. See* [**JSS17**] *for details.*

Consequently, we have the following.

COROLLARY 4.3.10. *Jell, Shaw and Smacka's tropical de Rham's theorem* $H^{p,q}_{\mathrm{Trop,Dol}}(X) \cong H^{p,q}_{\mathrm{Trop}}(X)_{\mathbb{R}}$ *is given by integration of* (p,q)*-superforms on semi-algebraic singular simplicies.*

CHAPTER 5

A tropical characterization of algebraic subvarieties

5.1. Outline of this chapter

We shall give a characterization of algebraic subvarieties of normal toric varieties over complete non-achimedean valuation fileds in terms of their tropicalizations.

Let K be a complete non-archimedean valuation field with non-trivial valuation val. Let M be a free $\mathbb{Z}$-module of finite rank. Let T_Σ be the normal toric variety over K associated to a fan Σ in $N_\mathbb{R} := \mathrm{Hom}_\mathbb{Z}(M, \mathbb{R})$. Let $X \subset T_\Sigma^{\mathrm{Ber}}$ be an irreducivle Zariski closed Berkovich analytic subvariety.

Gubler showed that for any cone $\sigma \in \Sigma$, the intersection $\mathrm{Trop}(X) \cap N_{\sigma,\mathbb{R}}$ is a possibly infinite union of polyhedra [**Gub07**, Theorem 1.1]. On the other hand, if $X \subset T_\Sigma^{\mathrm{Ber}}$ is the Berkovich analytification of a Zariski closed algebraic subvariety of T_Σ, Bieri and Groves showed that the intersection $\mathrm{Trop}(X) \cap N_{\sigma,\mathbb{R}}$ is a *finite* union of polyhedra [**BG84**, Theorem A]. (See also [**EKL06**, Theorem 2.2.3].)

There is a unique cone $\sigma_X \in \Sigma$ such that $X \cap O(\sigma_X)^{\mathrm{Ber}}$ is a dense Zariski open analytic subvariety of X.

The main theorem of this chapter, a converse of the Bieri-Groves' theorem [**BG84**, Theorem A], is as follows:

THEOREM 5.1.1. *Let T_Σ, $X \subset T_\Sigma^{\mathrm{Ber}}$, and $\sigma_X \in \Sigma$ be as above. Assume that* $\mathrm{Trop}(X) \cap N_{\sigma,\mathbb{R}}$ *is a finite union of polyhedra. Then X is the Berkovich analytification of a Zariski closed algebraic subvariety of T_Σ.*

As a corollary, we have the following.

COROLLARY 5.1.2. *Let T_Σ and $X \subset T_\Sigma^{\mathrm{Ber}}$ be as above. Then X is the Berkovich analytification of a Zariski closed algebraic subvariety of T_Σ if and only if* $\mathrm{Trop}(X)$ *is a finite union of polyhedra.*

REMARK 5.1.3. *Chow's theorem over nonarchimedean fields follows from Theorem 5.1.1 as follows. When T_Σ is the projective space, the Berkovich analytic subvariety X is compact. By* [**Mar15**, Theorem 1]*, one can see that the tropicalization* $\mathrm{Trop}(X)$ *is a finite union of polyhedra. Hence, by Theorem 5.1.1, the analytic subvariety X is the Berkovich analytification of a Zariski closed algebraic subvariety of T_Σ.*

REMARK 5.1.4. *In* [**MNN14**]*, Madani, L. Nisse, and M. Nisse proved similar results for analytic subvarieties of tori over the field of complex numbers.*

This chapter is organized as follows. In Section 5.2, we recall initial forms. In Section 5.3, we prove Theorem 5.1.1 for Berkovich analytic hypersurfaces in tori by calculating their tropicalizations explicitly in terms of initial forms of analytic functions. In Section 5.4, we prove Theorem 5.1.1 by using surjective homomorphisms of tori to reduce to the case of hypersurfaces. This is the idea of the proof of Bieri-Groves' theorem [**BG84**, Theorem A], and is also used in [**Pay09-2**] to study fibers of tropicalization maps. In Section 5.5, we give examples of the tropicalizations of analytic and algebraic hypersurfaces in the 2-dimensional torus.

5.2. Initial forms

We shall introduce the initial forms of analytic functions on the analytification $\mathbb{G}_m^{r,\mathrm{Ber}}$ of the r-dimensional torus

$$\mathbb{G}_m^r := \mathrm{Spec}(K[T_1^{\pm 1}, \dots, T_r^{\pm 1}])$$

over K as follows. (See [**MS15**, Section 2.4] for the case of Laurent polynomials.) Let M' be the free abelian group generated by T_i ($1 \leq i \leq r$). We identify $\mathrm{Hom}_{\mathbb{Z}}(M', \mathbb{R})$ with $\mathbb{R}^r$ by sending $\phi \in \mathrm{Hom}_{\mathbb{Z}}(M', \mathbb{R})$ to $(\phi(T_1), \dots, \phi(T_r)) \in \mathbb{R}^r$. For $u = (u_1, \dots, u_r) \in \mathbb{Z}^r$, we put $T^u := T_1^{u_1} \cdots T_r^{u_r}$.

For a non-zero analytic function

$$f = \sum_{u \in \mathbb{Z}^r} a_u T^u \in \Gamma(\mathbb{G}_m^{r,\mathrm{Ber}}, \mathcal{O}) \setminus \{0\} \qquad (a_u \in K),$$

let

$$\mathrm{Trop}(f) \colon \mathbb{R}^r \to \mathbb{R}$$

be the piecewise linear function given by

$$\mathrm{Trop}(f)(w) := \min\{\, \mathrm{val}(a_u) + \langle w, u \rangle \mid u \in \mathbb{Z}^r, a_u \neq 0 \,\} \qquad (w \in \mathbb{R}^r),$$

where $\langle , \rangle$ is the standard inner product on $\mathbb{R}^r$.

We note that a formal power series

$$f = \sum_{u \in \mathbb{Z}^r} a_u T^u \in K[[T_1^{\pm 1}, \dots, T_r^{\pm 1}]]$$

is an analytic function on $\mathbb{G}_m^{r,\mathrm{Ber}}$ if and only if for any $w \in \mathbb{R}^r$, we have $\lim_{|u| \to \infty} \mathrm{val}(a_u) + \langle w, u \rangle = \infty$, where we put $|u| := \sum_{i=1}^r |u_i|$. Hence the function $\mathrm{Trop}(f)$ is well-defined.

Let K° (resp. $\kappa(\mathrm{val})$) be the valuation ring (resp. the residue field) of K. Let $\overline{\cdot} \colon K^\circ \to \kappa(\mathrm{val})$ be the projection. We fix an algebraic closure K^{alg} of K. We also denote the extension of the valuation val on K to K^{alg} by the same symbol. We put $\Gamma := \mathrm{val}((K^{\mathrm{alg}})^\times)$. There exists a group homomorphism $\varphi \colon \Gamma \to (K^{\mathrm{alg}})^\times$ such that $\mathrm{val} \circ \varphi = \mathrm{id}_\Gamma$, where id_Γ is the identity map on Γ [**MS15**, Lemma 2.1.15]. We fix such $\varphi \colon \Gamma \to (K^{\mathrm{alg}})^\times$ in this chapter.

Definition 5.2.1. *The* initial form *of f with respect to $w \in \mathbb{R}^r$ is the Laurent polynomial over the residue field k defined by*

$$\mathrm{in}_w(f) := \sum_{\substack{u \in \mathbb{Z}^r,\ a_u \neq 0 \\ \mathrm{val}(a_u) + \langle w, u \rangle = \mathrm{Trop}(f)(w)}} \overline{a_u \varphi(-\mathrm{val}(a_u))} T^u \in k[T_1^{\pm 1}, \dots, T_r^{\pm 1}].$$

Since $\lim_{|u| \to \infty} \mathrm{val}(a_u) + \langle w, u \rangle = \infty$, the Laurent polynomial $\mathrm{in}_w(f)$ is well-defined.

We also note that the initial form $\mathrm{in}_w(f)$ depends on the choice of $\varphi \colon \Gamma \to (K^{\mathrm{alg}})^\times$, but, in this paper, we focus only on whether the initial form $\mathrm{in}_w(f)$ is a monomial or not for each $w \in \mathbb{R}^r$. This does not depend on the choice of $\varphi \colon \Gamma \to (K^{\mathrm{alg}})^\times$.

Proposition 5.2.2. *For the Zariski closed Berkovich analytic subvarity $Z = V(\mathcal{I}) \subset \mathbb{G}_m^{r,\mathrm{Ber}}$ corresponding to a coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_{\mathbb{G}_m^{r,\mathrm{Ber}}}$, we have*

$$\mathrm{Trop}(Z) = \{\, w \in \mathbb{R}^r \mid \mathrm{in}_w(f) \textit{ is not a monomial for any } f \in \Gamma(\mathbb{G}_m^{r,\mathrm{Ber}}, \mathcal{I}) \setminus \{0\} \,\}.$$

Moreover, when $\mathcal{I}$ is generated by a non-zero analytic function $f \in \Gamma(\mathbb{G}_m^{r,\mathrm{Ber}}, \mathcal{O}) \setminus \{0\}$, we have

$$\mathrm{Trop}(V(f)) = \{\, w \in \mathbb{R}^r \mid \mathrm{in}_w(f) \textit{ is not a monomial} \}.$$

PROOF. For each affinoid domain U of $\mathbb{G}_m^{r,\mathrm{Ber}}$, $w \in \mathrm{Trop}(U)$, and $f \in \Gamma(U, \mathcal{I})$ with $\mathrm{Trop}^{-1}(w) \subset U$, when the initial form $\mathrm{in}_w(f)$ is not a monomial, the initial form $\mathrm{in}_w(g)$ is not a monomial for a function $g \in \Gamma(U, \mathcal{I})$ which is sufficiently close to f in $\Gamma(U, \mathcal{I})$. By [**Van75**, Theorem 3.1.1 and Example 3.1.3], the image of $\Gamma(\mathbb{G}_m^{r,\mathrm{Ber}}, \mathcal{I})$ in $\Gamma(U, \mathcal{I})$ is dense. Hence the first assertion follows from [**Rab12**, Theorem 7.8]. One can show that

$$\mathrm{in}_w(fg) = \mathrm{in}_w(f)\,\mathrm{in}_w(g)$$

for any $g \in \Gamma(\mathbb{G}_m^{r,\mathrm{Ber}}, \mathcal{O}) \setminus \{0\}$ and any $w \in \mathbb{R}^r$ in the same way as [**MS15**, Lemma 2.6.2 (3)], where the equality is proved for Laurent polynomials. Hence the second assertion holds. $\square$

5.3. The tropicalizations of analytic hypersurfaces in tori

In this section, we prove Theorem 5.1.1 for analytic hypersurfaces in the r-dimensional torus $\mathbb{G}_m^{r,\mathrm{Ber}}$.

First, we show that the tropicalization of an analytic hypersurface in $\mathbb{G}_m^{r,\mathrm{Ber}}$ is the $(r-1)$-skeleton (i.e., the union of cells of dimension less than or equal to $r-1$) of the polyhedral complex associated to the analytic function defining the analytic hypersurface. (See [**MS15**, Proposition 3.1.6 and Remark 3.1.7] for the case of algebraic hypersurfaces.) See [**MS15**, Section 2.3] for the terminology of polyhedral geometry used in this paper.

For a non-zero analytic function

$$f = \sum_{u \in \mathbb{Z}^r} a_u T^u \in \Gamma(\mathbb{G}_m^{r,\mathrm{Ber}}, \mathcal{O}) \setminus \{0\} \qquad (a_u \in K),$$

we write $\Sigma_{\mathrm{Trop}(f)}$ for the coarsest polyhedral complex in $\mathbb{R}^r$ containing

$$\sigma_u := \{\, w \in \mathbb{R}^r \mid \mathrm{Trop}(f)(w) = \mathrm{val}(a_u) + \langle w, u \rangle \,\}$$

for every $u \in \mathbb{Z}^r$ satisfying $a_u \neq 0$, where $\langle , \rangle$ is the standard inner product on $\mathbb{R}^r$; see [**MS15**, Definition 2.5.5]. The polyhedral complex $\Sigma_{\mathrm{Trop}(f)}$ is pure r-dimensional and its support is $\mathbb{R}^r$.

LEMMA 5.3.1. *For a non-zero analytic function*

$$f = \sum_{u \in \mathbb{Z}^r} a_u T^u \in \Gamma(\mathbb{G}_m^{r,\mathrm{Ber}}, \mathcal{O}) \setminus \{0\} \qquad (a_u \in K),$$

let $V(f) \subset \mathbb{G}_m^{r,\mathrm{Ber}}$ be the analytic hypersurface defined by f. Then the tropicalization $\mathrm{Trop}(V(f))$ *is the $(r-1)$-skeleton of $\Sigma_{\mathrm{Trop}(f)}$, i.e., the union of cells of $\Sigma_{\mathrm{Trop}(f)}$ of dimension less than or equal to $r-1$.*

PROOF. One can prove this lemma in the same way as in the case of Laurent polynomials; see [**MS15**, Proposition 3.1.6 and Remark 3.1.7]. $\square$

We shall now prove Theorem 5.1.1 for analytic hypersurfaces in $\mathbb{G}_m^{r,\mathrm{Ber}}$.

THEOREM 5.3.2. *Let $f \in \Gamma(\mathbb{G}_m^{r,\mathrm{Ber}}, \mathcal{O}) \setminus \{0\}$ be a non-zero analytic function. Assume that* $\mathrm{Trop}(V(f))$ *is a finite union of polyhedra. Then f is a Laurent polynomial. In particular, the Zariski closed Berkovich analytic subvarity $V(f) \subset \mathbb{G}_m^{r,\mathrm{Ber}}$ is the analytification of the algebraic hypersurface of $\mathbb{G}_m^r$ defined by f.*

Proof. We put

$$f = \sum_{u \in \mathbb{Z}^r} a_u T^u \in \Gamma(\mathbb{G}_m^{r,\mathrm{Ber}}, \mathcal{O}) \setminus \{0\}.$$

By Lemma 5.3.1, the $(r-1)$-skeleton of $\Sigma_{\mathrm{Trop}(f)}$ is a finite union of polyhedra. Since the polyhedral complex $\Sigma_{\mathrm{Trop}(f)}$ is pure r-dimensional and its support is $\mathbb{R}^r$, there are only finitely many maximal cells of $\Sigma_{\mathrm{Trop}(f)}$. We take a finite subset $\Lambda \subset \mathbb{Z}^r$ such that for each maximal cell $\sigma \in \Sigma_{\mathrm{Trop}(f)}$, there exists $u \in \Lambda$ satisfying $\sigma_u = \sigma$. Then we have $\bigcup_{u \in \Lambda} \sigma_u = \mathbb{R}^r$; in other words, for any $w \in \mathbb{R}^r$, there exists $u \in \Lambda$ such that

$$\mathrm{Trop}(f)(w) = \mathrm{val}(a_u) + \langle w, u \rangle.$$

We shall show that there are only finitely many $u \in \mathbb{Z}^r$ satisfying $a_u \neq 0$. Assume that there exist infinitely many $u \in \mathbb{Z}^r$ with $a_u \neq 0$. Then there exist $v = (v_1, \dots, v_r) \in \mathbb{Z}^r \setminus \Lambda$ and $1 \leq i \leq r$ such that $a_v \neq 0$ and $|v_i| > |u_i|$ for any $u = (u_1, \dots, u_r) \in \Lambda$. Take a real number $x_i \in \mathbb{R}$ such that

$$\mathrm{val}(a_v) + x_i v_i < \mathrm{val}(a_u) + x_i u_i$$

for any $u = (u_1, \dots, u_r) \in \Lambda$. Let $x := (0, \dots, 0, x_i, 0, \dots, 0) \in \mathbb{R}^r$ be the element such that the i-th entry is x_i and the j-th entry is 0 for $j \neq i$. Then we have

$$\mathrm{val}(a_v) + \langle x, v \rangle < \mathrm{val}(a_u) + \langle x, u \rangle$$

for any $u \in \Lambda$. Hence $x \in \mathbb{R}^r$ is not contained in $\bigcup_{u \in \Lambda} \sigma_u$, which contradicts $\bigcup_{u \in \Lambda} \sigma_u = \mathbb{R}^r$.

Consequently, there are only finitely many $u \in \mathbb{Z}^r$ satisfying $a_u \neq 0$. In other words, f is a Laurent polynomial. $\square$

Remark 5.3.3. *For an irreducible Zariski closed Berkovich analytic subvarity $Z \subset \mathbb{G}_m^{r,\mathrm{Ber}}$ of codimension 1, there exists a non-zero analytic function $f \in \Gamma(\mathbb{G}_m^{r,\mathrm{Ber}}, \mathcal{O}) \setminus \{0\}$ such that $Z = V(f)$. This easily follows from the fact that $Z \subset \mathbb{G}_m^{r,\mathrm{Ber}}$ is a Cartier divisor, and the line bundle on $\mathbb{G}_m^{r,\mathrm{Ber}}$ corresponding to Z is trivial; see* [**Lut16**, Lemma 2.7.4].

5.4. A proof

In this section, we shall prove Theorem 5.1.1 by using surjective homomorphisms of tori to reduce to the case of hypersurfaces.

Let M be a free $\mathbb{Z}$-module of finite rank. Let T_Σ be the normal toric variety over K associated to a fan Σ in $N_\mathbb{R} := \mathrm{Hom}_\mathbb{Z}(M, \mathbb{R})$, and X an irreducible Zariski closed Berkovich analytic subvarity of T_Σ^{Ber}. Take a unique cone $\sigma_X \in \Sigma$ such that $X \cap O(\sigma_X)^{\mathrm{Ber}}$ is a dense Zariski open Berkovich analytic subvarity of X. We fix a $\mathbb{Z}$-basis $m_1, \dots, m_n$ of $(\sigma_X)^\perp \cap M$. By using this $\mathbb{Z}$-basis, we identify $N_{\sigma_X,\mathbb{R}} := \mathrm{Hom}_\mathbb{Z}((\sigma_X)^\perp \cap M, \mathbb{R})$ with $\mathbb{R}^n$.

Assume that $\mathrm{Trop}(X) \cap N_{\sigma_X,\mathbb{R}}$ is a finite union of polyhedra.

Lemma 5.4.1. *Assume that the Zariski closed Berkovich analytic subvarity $X \cap O(\sigma_X)^{\mathrm{Ber}} \subset O(\sigma_X)^{\mathrm{Ber}}$ is algebraic. Then the Berkovich analytic subvarity $X \subset T_\Sigma^{\mathrm{Ber}}$ is algebraic.*

Proof. Let $X' \subset O(\sigma_X)$ be the algebraic subvariety such that $X'^{\mathrm{Ber}} = X \cap O(\sigma_X)^{\mathrm{Ber}}$. By [**Ber90**, Proposition 3.4.4], we have $(\overline{X'})^{\mathrm{Ber}} = \overline{(X'^{\mathrm{Ber}})} = X$, where $\overline{X'}$ is the algebraic Zariski closure of X' in T_Σ. Hence the Berkovich analytic subvarity $X \subset T_\Sigma^{\mathrm{Ber}}$ is algebraic. $\square$

By Lemma 5.4.1, it suffices to show that $X \cap O(\sigma_X)^{\mathrm{Ber}} \subset O(\sigma_X)^{\mathrm{Ber}}$ is algebraic.

We put $X' := X \cap O(\sigma_X)^{\mathrm{Ber}}$. Since $X' \subset X$ is Zariski open and X is irreducible, X' is irreducible. We put $d := \dim X' = \dim X$. By the $\mathbb{Z}$-basis $m_1, \dots, m_n$ of $(\sigma_X)^{\perp} \cap M$, we identify $O(\sigma_X)$ with $\mathbb{G}_m^n$. Let $\mathcal{I} \subset \mathcal{O}_{\mathbb{G}_m^{n,\mathrm{Ber}}}$ be the coherent ideal sheaf such that $V(\mathcal{I}) = X' \subset \mathbb{G}_m^{n,\mathrm{Ber}}$.

Let $\mathrm{Gr}(n-d-1, n)$ be the Grassmannian of $(n-d-1)$-dimensional subspaces of $\mathbb{Q}^n$, which is an integral variety over $\mathbb{Q}$; see [**LB15**, Section 5.3.2]. Let S be the set of surjective homomorphisms

$$\phi \colon \mathbb{G}_m^n \to \mathbb{G}_m^{d+1}$$

such that

$$\phi \colon \mathrm{Trop}(\mathbb{G}_m^n) = \mathbb{R}^n \to \mathrm{Trop}(\mathbb{G}_m^{d+1}) \cong \mathbb{R}^{d+1}$$

is injective on every d-dimensional polyhedron contained in $\mathrm{Trop}(X')$.

Since $\mathrm{Trop}(X')$ is a finite union of d-dimensional Γ-rational polyhedra, we get the following.

Lemma 5.4.2. *The subset*

$$\{ \ker(\phi) \cap \mathbb{Q}^n \subset \mathrm{Trop}(\mathbb{G}_m^n) \mid \phi \in S \}$$

is a dense Zariski open subset in $\mathrm{Gr}(n-d-1, n)(\mathbb{Q})$.

Proof. See [**M18**, Lemma 5.2]. □

Lemma 5.4.3. *For any homomorphism $\phi \in S$, the image $\phi(X') \subset \mathbb{G}_m^{d+1,\mathrm{Ber}}$ is a Zariski closed Berkovich analytic subvariety.*

Proof. For each d-dimensional polyhedron P contained in $\mathrm{Trop}(X')$, the $\mathbb{R}$-linear map

$$\phi|_P \colon P \to \mathbb{R}^{d+1}$$

is injective. Hence $\phi|_P$ induces a homeomorphism from P onto $\phi(P)$.

For any affinoid domain U of $\mathbb{G}_m^{d+1,\mathrm{Ber}}$, since $\mathrm{Trop}(U)$ is bounded, the intersection

$$\mathrm{Trop}(U) \cap \phi(P)$$

is bounded. Hence the intersection

$$\phi^{-1}(\mathrm{Trop}(U)) \cap P$$

is bounded. Since $\mathrm{Trop}(X')$ is a finite union of d-dimensional polyhedra, the intersection

$$\phi^{-1}(\mathrm{Trop}(U)) \cap \mathrm{Trop}(X')$$

is bounded. Hence

$$\begin{aligned} \mathrm{Trop}((\phi)^{-1}(U) \cap X') &\subset \mathrm{Trop}((\phi)^{-1}(U)) \cap \mathrm{Trop}(X') \\ &\subset \phi^{-1}(\mathrm{Trop}(U)) \cap \mathrm{Trop}(X') \end{aligned}$$

is bounded. Thus $\mathrm{Trop}((\phi)^{-1}(U) \cap X')$ is compact.

By [**Pay09-1**, Lemma 2.1], the tropicalization map

$$\mathrm{Trop} \colon \mathbb{G}_m^{n,\mathrm{Ber}} \to \mathbb{R}^n$$

is proper. Hence the closed subset

$$(\phi)^{-1}(U) \cap X' \subset \mathrm{Trop}^{-1}(\mathrm{Trop}((\phi)^{-1}(U) \cap X'))$$

is compact. It follows that for any compact subset $C \subset \mathbb{G}_m^{d+1,\mathrm{Ber}}$, the subset $(\phi)^{-1}(C) \cap X'$ is compact. By Lemma 2.2.2, the morphism $\phi|_{X'} \colon X' \to \mathbb{G}_m^{d+1,\mathrm{Ber}}$ is separated.

Hence the map of underlying topological spaces $|X'| \to |\mathbb{G}_m^{d+1,\mathrm{Ber}}|$ induced by $\phi|_{X'}$ is proper; see [**Ber90**, Section 3.1, p.50]. Moreover, the relative boundary of $\phi|_{X'}$ is empty by Lemma 2.2.2. Hence $\phi|_{X'}$ is a proper morphism of Berkovich analytic spaces; see [**Ber90**, Section 3.1, p.50] for the definition of proper morphisms. By [**Ber90**, Proposition 3.3.6], the image $\phi(X') \subset \mathbb{G}_m^{d+1,\mathrm{Ber}}$ is a Zariski closed Berkovich analytic subvarity. □

By Lemma 5.4.3, for each $\phi \in S$, we consider $\phi(X')$ as a Zariski closed Berkovich analytic subvarity of $\mathbb{G}_m^{d+1,\mathrm{Ber}}$. Since X' is irreducible, its image $\phi(X')$ is also irreducible. Since

$$\mathrm{Trop}(\phi(X')) = \phi(\mathrm{Trop}(X'))$$

and $\phi \in S$, the tropicalization $\mathrm{Trop}(\phi(X'))$ is a finite union of d-dimensional polyhedra. By [**Gub07**, Theorem 1.1], the irreducible Zariski closed Berkovich analytic subvarity $\phi(X') \subset \mathbb{G}_m^{d+1,\mathrm{Ber}}$ is d-dimensional. By Remark 5.3.3, it is an analytic hypersurface in $\mathbb{G}_m^{d+1,\mathrm{Ber}}$. Hence, by Theorem 5.3.2, it is the analytification of an algebraic subvariety of $\mathbb{G}_m^{d+1}$.

Therefore, for each $\phi \in S$, the Zariski closed Berkovich analytic subvarity

$$W_\phi := (\phi)^{-1}(\phi(X')) \subset \mathbb{G}_m^{n,\mathrm{Ber}}$$

is algebraic. We put

$$W := \bigcap_{\phi \in S} W_\phi.$$

Then W contains X'. Since $W_\phi \subset \mathbb{G}_m^{n,\mathrm{Ber}}$ is algebraic for every $\phi \in S$, the intersection W is algebraic. One can deduce that $X' \subset \mathbb{G}_m^{n,\mathrm{Ber}}$ is algebraic from the following:

LEMMA 5.4.4. *The dimension of W is less than or equal to $d = \dim X' = \dim X$.*

PROOF. Assume that there exists an irreducible component V of W of dimension greater than d. By [**Gub07**, Theorem 1.1], the tropicalization $\mathrm{Trop}(V)$ is a locally finite union of polyhedra of dimension greater than d. We take a polyhedron P of dimension $d+1$ contained in $\mathrm{Trop}(V)$.

Let S' be the set of surjective homomorphisms

$$\phi \colon \mathbb{G}_m^n \to \mathbb{G}_m^{d+1}$$

such that the $\mathbb{R}$-linear map

$$\phi \colon \ \mathrm{Trop}(\mathbb{G}_m^n) = \mathbb{R}^n \to \mathrm{Trop}(\mathbb{G}_m^{d+1}) \cong \mathbb{R}^{d+1}$$

is injective on P. By [**LB15**, Theorem 5.2.3], the subset

$$\{ \ker(\phi) \cap \mathbb{Q}^n \subset \mathrm{Trop}(\mathbb{G}_m^n) \mid \phi \in S' \}$$

is dense Zariski open in $\mathrm{Gr}(n-d-1, n)(\mathbb{Q})$. Since S is also Zariski open dense in $\mathrm{Gr}(n-d-1, n)(\mathbb{Q})$, the intersection $S \cap S'$ is non-empty.

We take an element $\phi \in S \cap S'$. Since $\phi(W_\phi) = \phi(X')$, we have

$$\phi(\mathrm{Trop}(W_\phi)) = \mathrm{Trop}(\phi(W_\phi)) = \mathrm{Trop}(\phi(X')) = \phi(\mathrm{Trop}(X')).$$

Since $\phi \in S$, the image $\phi(\mathrm{Trop}(W_\phi))$ is a finite union of d-dimensional polyhedra. Since $\phi \in S'$, the polyhedron $\phi(P)$ is $(d+1)$-dimensional. Hence we have

$$\phi(P) \not\subset \phi(\mathrm{Trop}(W_\phi)).$$

Since $P \subset \mathrm{Trop}(V)$, we have

$$\phi(\mathrm{Trop}(V)) \not\subset \phi(\mathrm{Trop}(W_\phi)),$$

which contradicts $V \subset W \subset W_\phi$.

Consequently, the dimension of W is less than or equal to d. □

PROOF OF THEOREM 5.1.1. Recall that $X' := X \cap O(\sigma_X)^{\mathrm{Ber}}$ and $d := \dim X' = \dim X$. By Lemma 5.4.4, the dimension of W is less than or equal to d. Since $X' \subset W$, the Berkovich analytic subvarity X' is an irreducible component of W. Since W is algebraic, by [**Duc18**, Proposition 2.7.16], the Zariski closed Berkovich analytic subvarity $X' \subset O(\sigma_X)^{\mathrm{Ber}}$ is algebraic. Therefore, by Lemma 5.4.1, the Berkovich analytic subvarity $X \subset T_\Sigma^{\mathrm{Ber}}$ is algebraic.

The proof of Theorem 5.1.1 is complete. □

5.5. Examples of the tropicalizations of algebraic and analytic hypersurfaces

In this section, we give two examples of the tropicalizations of hypersurfaces in the 2-dimensional torus $\mathbb{G}_m^{2,\mathrm{Ber}}$. The first example is analytic and not algebraic. It is a locally finite union of polyhedra, but the number of polyhedra is infinite. The second example is algebraic, and it is a finite union of polyhedra.

Let

$$f = \sum_{(i,j)\in\mathbb{Z}_{\geq 0}^2} a_{i,j} X^i Y^j \in \mathcal{O}(\mathbb{G}_m^{2,\mathrm{Ber}}) \setminus \{0\} \qquad (a_{i,j} \in K)$$

be a non-zero analytic function on $\mathbb{G}_m^{2,\mathrm{Ber}} = (\operatorname{Spec} K[X^\pm, Y^\pm])^{\mathrm{Ber}}$ satisfying

$$\operatorname{val}(a_{i,j}) = i^2 + j^2 + ij - i - j$$

for any $(i,j) \in \mathbb{Z}_{\geq 0}^2$. For each $(s,t) \in \mathbb{Z}_{\geq 0}^2$, we put

$$f_{s,t} := \sum_{\substack{0\leq i\leq s \\ 0\leq j\leq t}} a_{i,j} X^i Y^j \in K[X,Y] \setminus \{0\}.$$

First, we consider the tropicalization $\operatorname{Trop}(V(f)) \subset \mathbb{R}^2$. Since f has infinitely many non-zero terms, the analytic hypersurface $V(f) \subset \mathbb{G}_m^{2,\mathrm{Ber}}$ is not algebraic. The tropicalization $\operatorname{Trop}(V(f)) \subset \mathbb{R}^2$ is not a finite union of polyhedra; see FIGURE.1.

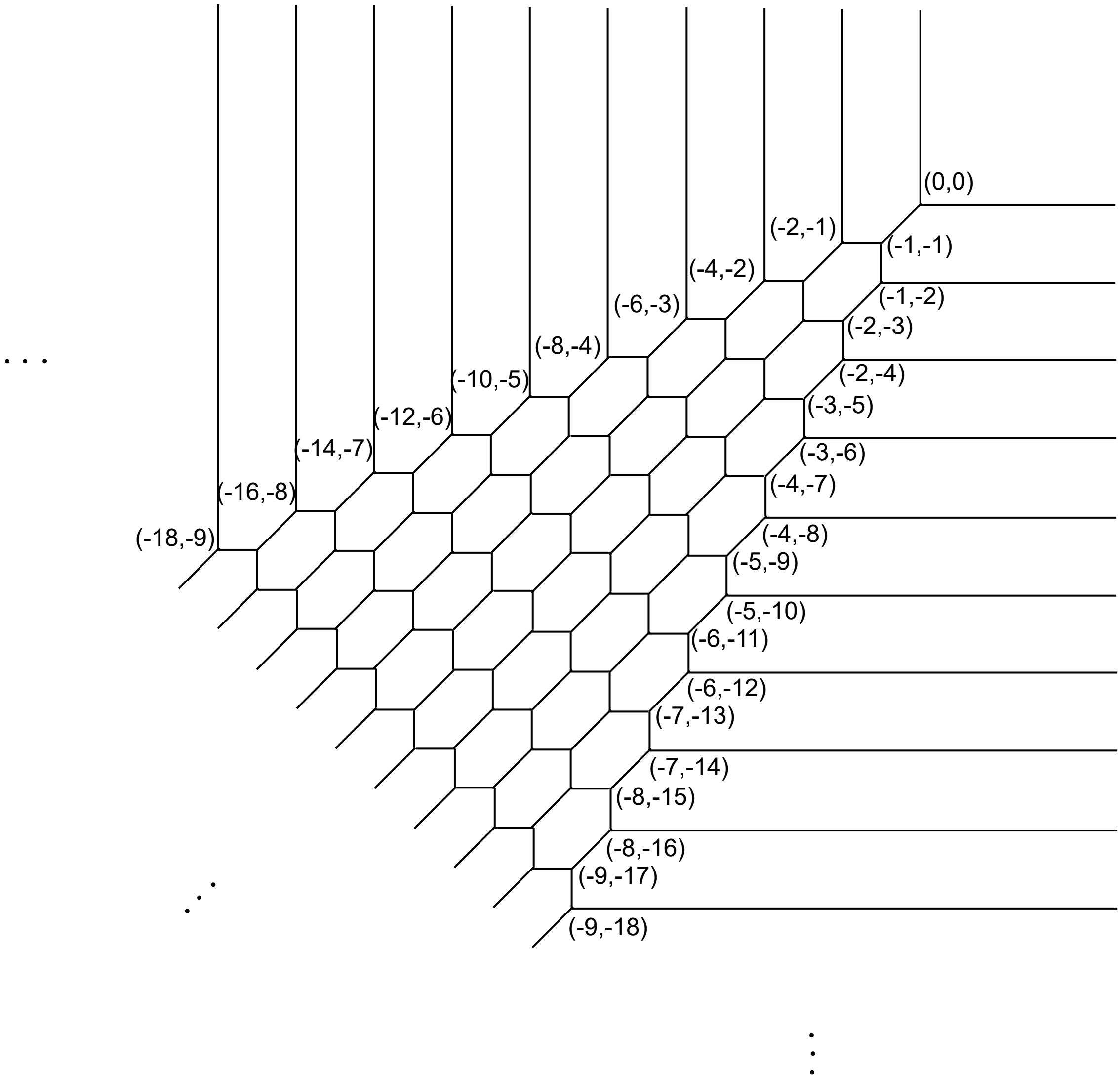


FIGURE.1. The tropicalization of the analytic hypersurface $V(f) \subset \mathbb{G}_m^{2,\mathrm{Ber}}$.

Next, we consider the tropicalization $\mathrm{Trop}(V(f_{s,t})) \subset \mathbb{R}^2$ for each $(s,t) \in \mathbb{Z}^2_{\geq 0}$. Since $f_{s,t}$ is a polynomial, the analytic hypersurface $V(f_{s,t}) \subset \mathbb{G}_m^{2,\mathrm{Ber}}$ is algebraic. The tropicalization $\mathrm{Trop}(V(f_{s,t})) \subset \mathbb{R}^2$ is a finite union of polyhedra; see Figure.2.

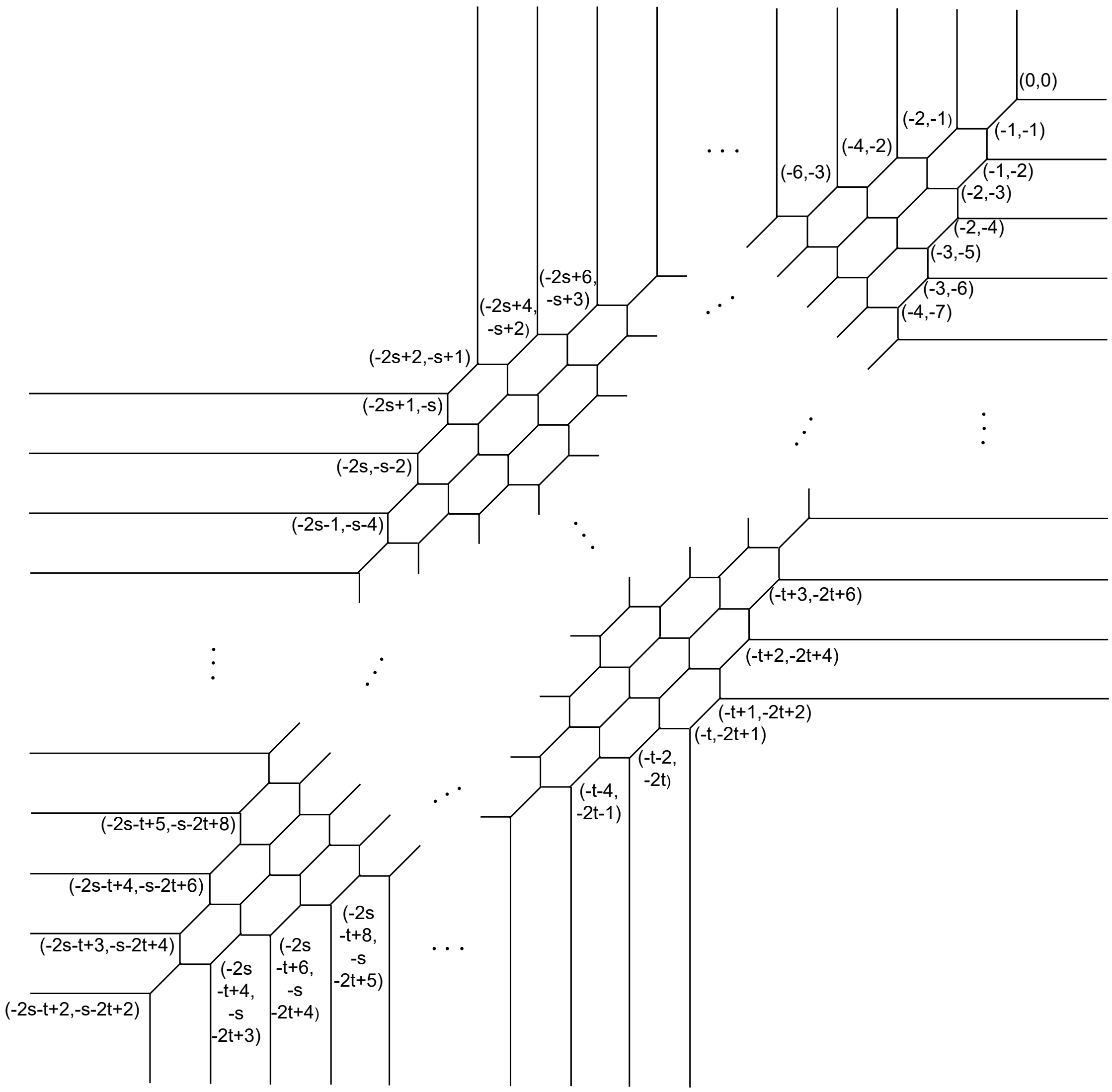


FIGURE.2. The tropicalization of the algebraic hypersurface $V(f_{s,t}) \subset \mathbb{G}_m^{2,\mathrm{Ber}}$.

CHAPTER 6

Tropical analogs of Milnor K-groups and their sheaf cohomology

6.1. Outline of this chapter

In this chapter, inspired by Liu's construction of *tropical cycle class maps* by *Milnor K-groups* ([**Liu20**, Definition 3.6]), we shall introduce a tropical analog K_T^* of (rational) Milnor K-groups, called *tropical K-groups* (Definition 6.2.2), and study its basic properties. The goal of this chapter is the following (Theorem 6.2.7 and Corollary 6.2.9).

THEOREM 6.1.1. *Tropical K-groups K_T^* form a cycle module in the sense of Rost* [**Ros96**, Definition 2.1]. *In particular, for any $p \geq 0$, their Zariski sheafification $\mathscr{K}_T^p$ on a smooth algebraic variety X over a trivially valued field has the Gersten resolution, i.e., a flasque resolution*

$$0 \to \mathscr{K}_T^p \xrightarrow{d} \bigoplus_{x \in X^{(0)}} i_{x*} K_T^p(k(x)) \xrightarrow{d} \bigoplus_{x \in X^{(1)}} i_{x*} K_T^{p-1}(k(x)) \xrightarrow{d} \bigoplus_{x \in X^{(2)}} i_{x*} K_T^{p-2}(k(x)) \xrightarrow{d} \dots$$
$$\xrightarrow{d} \bigoplus_{x \in X^{(p)}} i_{x*} K_T^0(k(x)) \xrightarrow{d} 0,$$

where $X^{(i)}$ is the set of points of codimension i, the residue field at $x \in X$ is denoted by $k(x)$, the maps $i_x \colon \operatorname{Spec}(k(x)) \to X$ are the natural morphisms, and we identify the groups $K_T^*(k(x))$ and the Zariski sheaf defined by them on $\operatorname{Spec} k(x)$. (See Section 4 for details.)

In particular, we have the following corollary (Corollary 6.2.9).

COROLLARY 6.1.2.

$$H^p_{Zar}(X, \mathscr{K}_T^p) \cong CH^p(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Corollary 6.1.2 is used to prove a tropical analog of the Hodge conjecture for smooth algebraic varieties over trivially valued fields in Chapter 7.

To prove Theorem 6.1.1, we use tropicalizations of Zariski-Riemann spaces (see Subsection 3.4.2), which give explicit descriptions of tropical K-groups (Corollary 6.2.4).

6.2. Tropical analogs of Milnor K-groups and their sheaf cohomology

Let K be a field. We equip it with the trivial valuation. Let M be a free $\mathbb{Z}$-module of finite rank. We put $N := \operatorname{Hom}(M, \mathbb{Z})$. Let L/K be a finitely generated extension of fields. Let $p \geq 0$ be a non-negative integer. For a morphism $\varphi \colon \operatorname{Spec} L \to \operatorname{Spec} K[M]$ over K and a fan structure Λ of $\operatorname{Trop}(\varphi(\operatorname{Spec} L/K))$, we put

$$F_p(0, \Lambda) := \sum_{P' \in \Lambda} \wedge^p \operatorname{Span}(P') \subset \wedge^p N_{\mathbb{R}},$$

where $\mathrm{Span}(P')$ is the $\mathbb{Q}$-vector space spanned by $P' \cap N_{\mathbb{Q}}$, and put

$$F^p(0,\Lambda) := \wedge^p M_{\mathbb{Q}} / \{ f \in \wedge^p M_{\mathbb{Q}} \mid \alpha(f) = 0 \ (\alpha \in F_p(0,\Lambda))\}.$$

Since $F^p(0,\Lambda)$ depends only on the support $\mathrm{Trop}(\varphi(\mathrm{Spec}\, L/K)) = |\Lambda|$ of Λ, we sometimes write $F^p(0, \mathrm{Trop}(\varphi(\mathrm{Spec}\, L/K)))$ instead of $F^p(0,\Lambda)$.

REMARK 6.2.1. *Consider two morphisms* $\varphi\colon \mathrm{Spec}\, L \to \mathbb{G}_m^r$ *and* $\varphi'\colon \mathrm{Spec}\, L \to \mathbb{G}_m^l$ *over* K *and a toric morphism* $\psi\colon \mathbb{G}_m^l \to \mathbb{G}_m^r$ *such that the diagram*

$$\begin{tikzcd} \mathrm{Spec}\, L \arrow[r, "\varphi"] \arrow[dr, "\varphi'"'] & \mathbb{G}_m^r \\ & \mathbb{G}_m^l \arrow[u, "\psi"'] \end{tikzcd}$$

is commutative. These induce a pull-back map

$$F^p(0, \mathrm{Trop}(\varphi(\mathrm{Spec}\, L/K))) \to F^p(0, \mathrm{Trop}(\varphi'(\mathrm{Spec}\, L/K))).$$

Note that since $\mathrm{Trop}(\varphi'(\mathrm{Spec}\, L/K)) \to \mathrm{Trop}(\varphi(\mathrm{Spec}\, L/K))$ *is surjective, this pull-back map is injective.*

DEFINITION 6.2.2. *We put*

$$K_T^p(L/K) := \varinjlim_{\varphi\colon \mathrm{Spec}(L) \to \mathbb{G}_m^r} F^p(0, \mathrm{Trop}(\varphi(\mathrm{Spec}\, L/K))),$$

where $\varphi\colon \mathrm{Spec}\, L \to \mathbb{G}_m^r$ *runs all* K*-morphisms to tori of arbitrary dimensions and morphisms are the pull-back maps*

$$F^p(0, \mathrm{Trop}(\varphi(\mathrm{Spec}\, L/K))) \to F^p(0, \mathrm{Trop}(\varphi'(\mathrm{Spec}\, L/K))).$$

We call $K_T^p(L/K)$ *the* p*-th (rational)* tropical K-group*. When there is no confusion, we denote it by* $K_T^p(L)$.

The following results give another expression of tropical K-groups.

LEMMA 6.2.3. *For a morphism*

$$\varphi\colon \mathrm{Spec}\, L \to \mathbb{G}_m^n = \mathrm{Spec}\, K[M]$$

over K*, we have*

$$F^p(0, \mathrm{Trop}(\varphi(\mathrm{Spec}\, L/K))) = \wedge^p M_{\mathbb{Q}}/J_M = \wedge^p M_{\mathbb{Q}}/J'_M,$$

where J_M *(resp.* J'_M*) is the* $\mathbb{Q}$*-vector subspace generated by* $f \in \wedge^p M_{\mathbb{Q}}$ *such that* $\wedge^p v(\varphi(f)) = 0$ *for* $v \in \mathrm{ZR}(L/K)$ *(resp. for* $v \in \mathrm{ZR}(L/K)$ *with*

$$\mathrm{ht}(v(\varphi(M))) = \mathrm{tr.deg}(k(\varphi(\mathrm{Spec}\, L))/K)).$$

Here, the element $\varphi(f) \in \wedge^p (L^\times)_{\mathbb{Q}}$ *is the image of* f *by the map* $\wedge^p\varphi\colon \wedge^p M_{\mathbb{Q}} \to \wedge^p (L^\times)_{\mathbb{Q}}$*, the map*

$$\wedge^p v\colon \wedge^p (L^\times)_{\mathbb{Q}} \to \wedge^p (\Gamma_v)_{\mathbb{Q}}$$
$$(\text{resp. } \wedge^p \varphi\colon \wedge^p M_{\mathbb{Q}} \to \wedge^p (L^\times)_{\mathbb{Q}})$$

is the wedge product of $v_{\mathbb{Q}}\colon (L^\times)_{\mathbb{Q}} \to (\Gamma_v)_{\mathbb{Q}}$ *(resp.* $\varphi_{\mathbb{Q}}\colon M_{\mathbb{Q}} \to (L^\times)_{\mathbb{Q}}$*) as a homomorphism of* $\mathbb{Q}$*-vector spaces, and* $k(\varphi(\mathrm{Spec}\, L))$ *is the residue field of the structure sheaf at* $\varphi(\mathrm{Spec}\, L) \in \mathbb{G}_m^n$.

PROOF. This follows from Remark 3.4.5, Lemma 3.4.7, Lemma 3.4.9, and surjectivity of tropicalization map $\mathrm{ZR}(L/K)^{\mathrm{ad}} \to \varprojlim \Lambda$,where Λ runs through all fan structures of $\mathrm{Trop}(\varphi((SpecL/K))$. Remind that $\mathrm{Trop}(\varphi(\mathrm{Spec}\, L/K))$ is a finite union of rational polyhedra of dimension $\mathrm{tr.deg}(k(\varphi(\mathrm{Spec}\, L)/K))$. □

COROLLARY 6.2.4. *We have*

$$K_T^p(L) \cong \wedge^p(L^\times)_{\mathbb{Q}}/J \cong \wedge^p(L^\times)_{\mathbb{Q}}/J',$$

where J (resp. J') is the $\mathbb{Q}$-vector subspace generated by $f \in \wedge^p(L^\times)_{\mathbb{Q}}$ such that $\wedge^p v(f) = 0$ for $v \in \mathrm{ZR}(L/K)$ (resp. for $v \in \mathrm{ZR}(L/K)$ with $\mathrm{ht}(v) = \mathrm{tr.deg}(L/K)$).

We shall show that tropical K-groups satisfy good properties, i.e., they define a cycle module in the sense of Rost [**Ros96**, Definition 2.1].

We recall the definition and several maps of Milnor K-groups. For a field E, its Milnor K-group is defined by

$$K_M^p(E) := T^p E^\times / J,$$

where $T^p E^\times$ is the p-th tensor of the multiplicative group $E^\times$ over $\mathbb{Z}$ and J is the subgroup of $T^p E^\times$ generated by

$$\{a_1 \otimes \cdots \otimes a_p \mid a_i = 1 - a_j \text{ for some } i, j\}.$$

In particular, we have $K_M^0(E) = \mathbb{Z}$ and $K_M^1(E) = E^\times$. The image of $a_1 \otimes \cdots \otimes a_p$ in $K_M^p(E)$ is denoted by $(a_1, \ldots, a_p)$.

- A morphism $\varphi \colon F \to E$ of fields induces a map

  $$\varphi_* \colon K_M^p(F) \to K_M^p(E)$$

  by

  $$\varphi_*((a_1, \ldots, a_p)) = (\varphi(a_1), \ldots, \varphi(a_p)).$$

- For a finite morphism $\varphi \colon F \to E$, there is the norm homomorphism

  $$\varphi^* \colon K_M^p(E) \to K_M^p(F)$$

  (Bass-Tate [**BT72**], Kato [**Kat80**]). It is a generalization of the multiplication

  $$\times [E : F] \colon K_M^0(E) = \mathbb{Z} \to \mathbb{Z} = K_M^0(F)$$

  and the usual norm map $E^\times \to F^\times$. This is defined by Bass and Tate [**BT72**] with respect to a choice of generators of E over F, and independence of such a choice was proved by Kato [**Kat80**].

- For a normalized discrete valuation $v \colon F^\times \to \mathbb{Z}$, there is the residue homomorphism (Milnor [**Mil70**])

  $$\partial_v \colon K_M^p(F) \to K_M^{p-1}(\kappa(v)).$$

  It is characterized by

  $$\partial_v((\pi, u_1, \ldots, u_{p-1})) = (\overline{u}_1, \ldots, \overline{u}_{p-1})$$
  $$\partial_v((u_1, \ldots, u_p)) = 0$$

  for a prime π of v and $u_i \in F$ with $v(u_i) = 0$ $(1 \leq i \leq p)$ and residue class $\overline{u}_i$. (Remind that $\kappa(v)$ is the residue field of v.)

LEMMA 6.2.5. *The canonical morphism*

$$\otimes^p L^\times \to K_T^p(L)$$

factors through

$$\otimes^p L^\times \to K_M^p(L) \to K_T^p(L).$$

PROOF. For any $a \in L$, there are no 2-rational-rank valuations of $K(a)$ which are trivial on K. Hence the assertion follows from Corollary 6.2.4. $\square$

LEMMA 6.2.6. *The maps φ_*, φ^*, and ∂_v for v which is trivial on K induce maps of tropical K-groups*

$$\begin{aligned}\varphi_* &\colon K_T^p(F) \to K_T^p(E),\\ \varphi^* &\colon K_T^p(E) \to K_T^p(F),\\ \partial_v &\colon K_T^p(F) \to K_T^{p-1}(\kappa(v)).\end{aligned}$$

PROOF. For φ_*, the assertion follows from Corollary 6.2.4. For ∂_v, the assertion follows from Corollary 6.2.4 and Remark 2.1.10.

We shall show that the norm homomorphism φ^* induces a map of tropical K-groups. It suffices to show that for a finite extension of fields $\varphi\colon F \to E$ and $\alpha \in K_M^p(E)$ such that $\wedge^p w(\alpha) = 0$ for any $w \in \mathrm{ZR}(E/K)$ with $\mathrm{ht}(w) = \mathrm{tr.deg}(E/K)$, we have

$$\wedge^p v(\varphi^* \alpha) = 0$$

for any $v \in \mathrm{ZR}(F/K)$ with $\mathrm{ht}(v) = \mathrm{tr.deg}(F/K)$. We shall show this assertion by induction on p. When $p = 0$, the assertion is trivial. We assume $p \geq 1$. Let $v_1 \in \mathrm{ZR}(F/K)$ be the unique generalization of v of height 1. Note that since $\mathrm{ht}(v) = \mathrm{tr.deg}(F/K)$, we have $\mathrm{rank}(v) = \mathrm{ht}(v)$. Hence we have $\mathrm{rank}\, v_1 = \mathrm{ht}(v_1) = 1$, i.e., the valuation v_1 is discrete. For an extension $w_i \in \mathrm{ZR}(E/K)$ of v_1 to E and any $u \in \mathrm{ZR}(\kappa(w_i)/K)$ with $\mathrm{ht}(u) = tr.\deg(\kappa(w_i)/K)$, we have

$$\wedge^{p-1} u(\partial_{w_i}(\alpha)) = \wedge^p \tilde{u}(\alpha) = 0,$$

where $\tilde{u} \in \mathrm{ZR}(E/K)$ is the specialization of w_i corresponding to u in the sense of Remark 2.1.10. Note that we have

$$\mathrm{ht}(\tilde{u}) = \mathrm{ht}(u) + 1 = \mathrm{tr.deg}(\kappa(w_i)/K) + 1 = \mathrm{tr.deg}(E/K).$$

Hence by applying the assumption of induction to $\varphi_i\colon \kappa(v_1) \to \kappa(w_i)$ and $\partial_{w_i}(\alpha) \in K_M^{p-1}(\kappa(w_i))$, we have

$$\wedge^{p-1} \overline{v}(\varphi_i^*(\partial_{w_i}(\alpha))) = 0,$$

where $\overline{v} \in \mathrm{ZR}(\kappa(v_1)/K)$ is the valuation corresponding to v in the sense of Remark 2.1.10. Hence we have

$$\begin{aligned}\wedge^p v(\varphi^* \alpha) &= \wedge^{p-1} \overline{v}(\partial_{v_1} \circ \varphi^* \alpha)\\ &= \wedge^{p-1} \overline{v}\left(\sum_{w_i} \varphi_i^* \circ \partial_{w_i}(\alpha)\right)\\ &= 0,\end{aligned}$$

where $w_i \in \mathrm{ZR}(E/K)$ runs through all extensions of $v_1 \in \mathrm{ZR}(F/K)$ to E, and the second equality follows from a basic property

$$\partial_{v_1} \circ \varphi^* = \sum_{w_i} \varphi_i^* \circ \partial_{w_i}$$

of Milnor K-groups. $\square$

We also denote the induced maps of tropical K-groups by $\varphi_*, \varphi^*, \partial_v$.

THEOREM 6.2.7. *The functor*

$$(\text{finitely generated fields over } K) \to (\mathbb{Z}_{\geq 0}\text{-graded abelian group})$$
$$L \mapsto \bigoplus_p K_T^p(L/K).$$

with $\varphi_*, \varphi^*, \partial_v$ and a natural morphism

$$K_M^p(L) \times K_T^q(L) \to K_T^{p+q}(L)$$

is a cycle module in the sense of Rost [**Ros96**, Definition 2.1].

PROOF. This follows from Lemma 6.2.5, Lemma 6.2.6, and the fact that Milnor K-groups form a cycle module [**Ros96**, Theorem 1.4 and Remark 2.4]. $\square$

EXAMPLE 6.2.8. *We give examples of tropical K groups.*

- *For any L/K, we have $K_T^0(L/K) = \mathbb{Q}$ by definition.*
- *For any L/K, by Corollary 6.2.4, we have $K_T^1(L/K) = (L^\times/(L \cap K^{\mathrm{alg}})^\times)_{\mathbb{Q}}$.*
- *For any L/K and any $p \geq \mathrm{tr.deg}(L/K) + 1$, we have $K_T^p(L/K) = 0$ since there are no valuations of L which are trivial on K and have rational ranks strictly greater than* $\mathrm{tr.deg}(L/K)$.
- *Let T be an indeterminate. For any L/K, we have*

$$K_T^{\mathrm{tr.deg}(L/K)+1}(L(T)/K) \overset{(\partial_x)_x}{\cong} \bigoplus_{x \in \mathbb{A}_L^{1,\mathrm{cl}}} K_T^{\mathrm{tr.deg}(L/K)}(k(x)/K),$$

where $\mathbb{A}_L^{1,\mathrm{cl}}$ is the set of closed points of $\mathbb{A}_L^1 = \operatorname{Spec} L[T]$, the field $k(x)$ is the residue field at $x \in \mathbb{A}_L^1$, and ∂_x is the residue homomorphism for the normalized discrete valuation of $L(T)$ corresponding to $x \in \mathbb{A}^{1,\mathrm{cl}}$. This follows from homotopy property of cycle modules for $\mathbb{A}^1$ [**Ros96**, Proposition 2.2 H] *and the above example. In particular, for L/K with* $\mathrm{tr.deg}(L/K) = 1$, *we have*

$$K_T^2(L(T)/K) \overset{(\partial_x)_x}{\cong} \bigoplus_{x \in \mathbb{A}_L^{1,\mathrm{cl}}} (k(x)^\times/(k(x) \cap K^{\mathrm{alg}})^\times)_{\mathbb{Q}}.$$

We give an explicit resolution of the Zariski sheaf of tropical K-groups on a smooth algebraic variety X over K. Let $X^{(i)}$ be the set of points of X of codimension i. For any i and points $x \in X^{(i)}, y \in X^{(i+1)}$, Rost defined a map

$$\partial_x^y \colon K_T^p(k(x)) \to K_T^{p-1}(k(y))$$

[**Ros96**, Section 2] as follows. When $y \notin \overline{\{x\}}$, we put $\partial_x^y = 0$. When $y \in \overline{\{x\}}$, we put

$$\partial_x^y := \sum_v \varphi_v^* \circ \partial_v \colon K_T^p(k(x)) \to K_T^{p-1}(k(y)),$$

where $v \in \mathrm{ZR}(k(x)/K)$ runs through all normalized discrete valuations of $k(x)$ whose center in $\overline{\{x\}}$ is y. Let $\eta \in X$ be the generic point. We denote by $\mathscr{K}_T^p$ the sheaf on the Zariski site X_{Zar} defined by

$$\mathscr{K}_T^p(U) := \mathrm{Ker}(K_T^p(K(X)) \xrightarrow{d} \oplus_{x \in U^{(1)}} K_T^{p-1}(k(x)),$$

where $d := (\partial_\eta^x)_{x \in U^{(1)}}$. We call it *the sheaf of tropical K-groups.*

COROLLARY 6.2.9. *For any $p \geq 0$, the sheaf $\mathscr{K}_T^p$ has the* Gersten resolution*, i.e., an exact sequence*

$$0 \to \mathscr{K}_T^p \xrightarrow{d} \bigoplus_{x \in X^{(0)}} i_{x*}(K_T^p(k(x))) \xrightarrow{d} \bigoplus_{x \in X^{(1)}} i_{x*}K_T^{p-1}(k(x)) \xrightarrow{d} \bigoplus_{x \in X^{(2)}} i_{x*}K_T^{p-2}(k(x)) \xrightarrow{d} \dots$$
$$\xrightarrow{d} \bigoplus_{x \in X^{(p)}} i_{x*}K_T^0(k(x)) \xrightarrow{d} 0,$$

where $i_x \colon \operatorname{Spec}(k(x)) \to X$ are the natural morphisms, we identify the groups $K_T^*(k(x))$ and the Zariski sheaf defined by them on $\operatorname{Spec} k(x)$, and $d := (\partial_x^y)_{\{x \in X^{(i)},\ y \in X^{(i+1)}\}}$. In particular, we have

$$H^p_{Zar}(X, \mathscr{K}_T^p) \cong CH^p(X)_{\mathbb{Q}}.$$

PROOF. The first assertion follows from Lemma 6.2.7 and [**Ros96**, Theorem 6.1]. The second assertion follows from the fact that for a normalized discrete valuation $v \in \mathrm{ZR}(E/K)$ of a field extension E/K, the residue homomorphism

$$\partial_v \colon K_T^1(E) \cong (E^\times)_{\mathbb{Q}}/((E \cap K^{alg})^\times)_{\mathbb{Q}} \to K_T^0(\kappa(v)) \cong \mathbb{Q}$$

coincides with the map induced by the valuation $v \colon E^\times \to \mathbb{Z}$. □

REMARK 6.2.10. *Note that residue homomorphisms $\partial_v \colon K_T^p(F) \to K_T^{p-1}(\kappa(v))$ of tropical K-groups lift to wedge products $\partial_v \colon \wedge^p F_{\mathbb{Q}}^\times \to \wedge^{p-1}\kappa(v)_{\mathbb{Q}}^\times$. For a field extension L/K, the direct sum of residue homomorphisms*

$$\wedge^p L(T)_{\mathbb{Q}}^\times \to \bigoplus_{x \in \mathbb{A}_L^{1,(1)}} \wedge^{p-1} k(x)_{\mathbb{Q}}^\times$$

is surjective (where T is the coordinate of $\mathbb{A}_L^1$). To see this, for any element $(a_x)_x$ of the right hand side, consider the number $\max\{[k(x) : L]\}$ *for x such that $a_x \neq 0$, then surjectivity easily follows from induction on this number.*

CHAPTER 7

A tropical analog of the Hodge conjecture

7.1. Outline of this chapter

We propose a tropical approach to problems on algeraic classes of cohomology groups such as the Hodge conjecture, the Tate conjecture, and Grothendieck's standard conjectures. The goal of this chapter is to prove the following tropical analog of the Hodge conjecture for smooth algebraic varieties over trivially valued fields.

THEOREM 7.1.1. *Let X be a smooth algebraic variety over a trivially valued field. Then, for any $p \geq 0$, the tropical cycle class map*

$$\mathrm{CH}^p(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to H^{p,p}_{\mathrm{Trop}}(X),$$

defined by Liu [**Liu20**, Definition 3.6]*, is an isomorphism. Here, $H^{p,p}_{\mathrm{Trop}}(X)$ is the tropical cohomology group.*

(Strictly speaking, Liu's tropical cycle class map is defined for tropial Dolbeault cohomology. It is isomorphic to tropical cohomology by [**JSS17**, Theorem 1]. See Section 4.2.)

By Corollary 6.1.2, Theorem 7.1.1 follows from the following (Theorem 7.2.2) (see Remark 7.2.4).

THEOREM 7.1.2. *Let X be a smooth algebraic variety over a trivially valued field. For any p, q, we have*

$$H^{p,q}_{\mathrm{Trop}}(X) \cong H^q(X_{\mathrm{Zar}}, \mathscr{K}^p_T),$$

where K^p_T is the Zariski sheaf of p-th tropical K-groups (see Chapter 6).

By the Gersten resolution (Theorem 6.1.1), we have the following.

COROLLARY 7.1.3. *We have $H^{p,q}_{\mathrm{Trop}}(X) = 0$ for $q \geq p + 1$.*

COROLLARY 7.1.4. *We assume X proper. For $p \geq 1$, we have $H^{p,0}_{\mathrm{Trop}}(X) = 0$.*

PROOF. This immediately follows from [**Ros96**, Lemma 12.8] and the theory of tropical compactifications. □

One of the most important corollary of the (original) Hodge conjecture is *Grothendieck's standard conjecture D* of algebraic cycles, i.e., kernels of the cycle class maps to singular homology groups equal the groups of algebraic cycles numerically equivalent to 0. The following (Corollary 7.2.5) is its tropical analog, which follows from Theorem 7.1.1 and Liu's theorem on compatibility of tropical cycle class maps and intersection theory [**Liu20**, Theorem 1.1] (see Section 4.2 and Remark 7.2.4). Note that tropical homology is the dual of tropical cohomology.

COROLLARY 7.1.5. *Let X be a smooth proper algebraic variety over a trivially valued field. We assume that there is a closed immersion of X to a normal toric variety. Then, for any $p \geq 0$, the kernel of the tropical cycle class map*

$$\mathrm{Z}_p(X) \otimes \mathbb{Q} \to H_{p,p}^{\mathrm{Trop}}(X)$$

from the $\mathbb{Q}$-vector space $\mathrm{Z}_p(X) \otimes \mathbb{Q}$ of algebraic cycles of dimension p to the tropical homology group is the $\mathbb{Q}$-vector subspace generated by algebraic cycles numerically equivalent to 0.

REMARK 7.1.6. *We define tropicalizations of algebraic varieties via non-archimedean geometry. When $K = \mathbb{C}$, tropicalizations can be defined as certain limits of complex manifolds. See* [**Jon16**].

REMARK 7.1.7. *We remark related works.*

- *There is an announcement by Katzarkov* [**Kat09**] *that he and Kontsevich used tropical geometry to study the Hodge conjecture for several types of abelian varieties.*
- *There is a paper by Zharkov* [**Zha20**] *on Kontsevich's idea to find a counter-example to the Hodge conjecture by finding a counter-example to a tropical analog of the Hodge conjecture. (This tropical analog of the Hodge conjecutre is slightly different from ours.)*
- *Babee-Huh gave a counter-example to a stronger version of the Hodge conjecture by tropical geometry* [**BH17**].
- *When X is curve, a generalization of Theorem 7.1.2 is already proved over (not necessary trivial) non-archimedean valuation fields by Jell-Wanner* [**JW18**] *for Mumford curves and Jell* [**Jel19**] *for general curves.*
- *Many properties (such as Poincare duality, hard Lefschetz property, and Hodge-Riemann relations) of tropical cohomology of smooth (i.e., locally matroidal) tropical varieties are known. See introduction of* [**AP20**] *on this topic.*

A character of our result is that it holds for general smooth algebraic varieties. Properties of thier tropical cohomology were, previously, not known so much. (Note that tropicalizations of smooth algebraic varieties are not necessary smooth tropical varieties, and properties of tropical cohomology groups of general non-smooth higher dimensional tropical varieties are still not known so much.)

To avoid difficulty of tropicalizations of higher dimensional algebraic varieties, we use a theorem for general "cohomology thoeries", which is first proved by Quillen [**Qui73**] for algebraic K-thoery, and developed by many mathmaticians including Bloch-Ogus [**BO74**], Gabber [**Gab94**], Rost [**Ros96**], and Collit-Thélène-Hoobler-Kahn [**CTHK97**]. By this theorem, we reduce Theorem 7.1.2 to (several easy computations and) $\mathbb{A}^1$-homotopy invariance, which is proved in Chapter 9.

Our main tool is *non-archimedean geometry.* In addition to usual tropicalizations of *Berkovich analytic spaces*, we use tropicalizations of *Huber's adic spaces*, introduced in Section 3.4.1. They are useful to comupute tropical cohomology and tropical K-groups.

Throughout this paper, we consider tropical cohomology and tropical K-groups of $\mathbb{Q}$-coefficients. Chapter 9 (a proof of $\mathbb{A}^1$-homotopy invariance) does not work for those of $\mathbb{Z}$-coefficients.

Organization of this chapter is as follows. In Section 7.2, we prove Theorem 7.1.2. The proof of key Proposition 7.2.1 is given in Section 7.3 (étale excision) and Chapter 9 ($\mathbb{A}^1$-homotopy invariance). In Section 7.3, we prove étale excision theorem for tropical cohomology over trivially valued fields. The proof is easy and short. In Section 7.4, we show the existance of corestriction maps for extensions of finite base fields. This is used to prove Theorem 7.1.2 over finite fields. In Section 7.5, we give an example, smooth toric varieties over $\mathbb{C}$. We show that their tropical cohomology groups are isomorphic to the weight graded pieces of their singular cohomology groups.

7.2. A proof of the main theorem

The aim of this section is to prove Theorem 7.2.2 (a tropical analog of Hodge conjecture). It follows from

- Proposition 7.2.1 (proved in Section 7.3 and Chapter 9),
- easy lemmas on tropical cohomology (Lemma 7.2.14 and 7.2.15 proved in Subsection 7.2.3), and
- a theorem on general "cohomology theories" [**CTHK97**, Corollary 5.1.11], which is first proved by Quillen [**Qui73**] for algebraic K-theory, and then developped by many mathematicians including Bloch-Ogus [**BO74**], Gabber [**Gab94**], Rost [**Ros96**], and Collit-Thélène-Hoobler-Kahn [**CTHK97**]
- (and the existence of corestriction maps (Proposition 7.4.5) when the base field is finite).

As a corollary of Theorem 7.2.2, a tropical analog of Grothendieck's standard conjecture D holds, i.e., tropical homological equivalence equals numerical equivalence (Corollary 7.2.5). In Subsecton 7.2.2, we will give basic lemmas used to prove Lemma 7.2.14 and 7.2.15.

Let K be a trivially valued field.

7.2.1. A proof of the main theorem. We prove the main theorem (Theorem 7.2.2).

The next Proposition is proved in Section 7.3 and Chapter 9.

PROPOSITION 7.2.1. *(1) (étale excision) Let $\Phi\colon X' \to X$ be an étale morphism of smooth quasi-projective varieties over K. Let $Z \subset X$ be a closed subscheme. We assume $Z' := \Phi^{-1}(Z) \to Z$ is an isomorphism. Then we have*

$$\Phi^*\colon H^{p,q}_{\mathrm{Trop},Z}(X) \cong H^{p,q}_{\mathrm{Trop},Z'}(X').$$

(2) ($\mathbb{A}^1$-homotopy invariance) Let $X \subset \mathbb{A}^{N_0}$ be an open subvariety over K. We put $\pi\colon X \times \mathbb{A}^1 \to X$ the first projection. Then the pullback map

$$\pi^*\colon H^{p,q}_{\mathrm{Trop}}(X) \to H^{p,q}_{\mathrm{Trop}}(X \times \mathbb{A}^1)$$

is an isomorphism.

Lemma 7.2.14 and 7.2.15, which are used in the proof of the following theorem, are proved in Section 7.2.3. We put $\mathscr{H}^{p,q}$ the Zariski sheaf on X associated to the presheaf $U \mapsto H^{p,q}_{\mathrm{Trop}}(U)$.

THEOREM 7.2.2. *Let X be a smooth algebraic variety over K. Then there exist natural isomorphisms*

$$H^q_{Zar}(X, \mathscr{H}^{p,0}) \cong H^{p,q}_{\mathrm{Trop}}(X),$$
$$\mathscr{K}^p_T \cong \mathscr{H}^{p,0}.$$

In particular, we have

$$H^{p,p}_{\mathrm{Trop}}(X) \cong CH(X)^p_{\mathbb{Q}}.$$

PROOF. The last assertion follows from the first one and Corollary 6.2.9.

For each r, by Proposition 7.2.1 and [**CTHK97**, Remarks 5.1.3, Corollary 5.1.11, Lemma 5.3.1 (a), and Proposition 5.3.2 (a)] (and Proposition 7.4.5 and a discussion in [**CTHK97**, Section 4] when K is finite (see also [**CTHK97**, Section 6])), there exists a spectral sequence

$$E_1^{p,q} = \coprod_{x \in X^{(p)}} H^{r,p+q}_{\mathrm{Trop},x}(X) \Rightarrow H^{r,p+q}_{\mathrm{Trop}}(X)$$

whose E_2-terms are $E_2^{p,q} = H^p_{Zar}(X, \mathscr{H}^{r,q})$, and we have

$$\mathscr{H}^{r,0}(V) \cong \mathrm{Ker}(d_1 \colon H^{r,0}_{\mathrm{Trop},\eta}(X) \to \oplus_{x \in V^{(1)}} H^{r,1}_{\mathrm{Trop},x}(X))$$

for each open subvariety $V \subset X$, where

- $H^{r,p+q}_{\mathrm{Trop},x}(X) := \varinjlim_{x \in U} H^{r,p+q}_{\mathrm{Trop},x}(U)$, where $U \subset X$ runs through all open neighborhoods of x,
- d_1 is the differential map of the spectral sequence, and
- $\eta \in X$ is the generic point.

(Note that we do not need to prove ètale excision for smooth algebraic varieties. That for smooth quasi-projective varieties are enough. See [**CTHK97**, Section 4].)

By Lemma 7.2.14, we have $E_1^{p,q} = 0$ for $q \geq 1$. By definition, we have $E_2^{p,q} = 0$ for $q \leq -1$. Hence

$$H^q_{Zar}(X, \mathscr{H}^{r,0}) = E_2^{q,0} = E_\infty^q = H^{r,q}_{\mathrm{Trop}}(X).$$

By Lemma 7.2.15 and Corollary 6.2.9, we have

$$\begin{aligned}
\mathscr{H}^{r,0}(V) &\cong \mathrm{Ker}(H^{r,0}_{\mathrm{Trop},\eta}(X) \xrightarrow{d_1} \oplus_{x \in V^{(1)}} H^{r,1}_{\mathrm{Trop},x}(X) \\
&\cong \mathrm{Ker}(K_T^r(K(X)) \xrightarrow{d=(\partial_\eta^x)_x} \oplus_{x \in V^{(1)}} K_T^{p-1}(k(x)) \\
&\cong \mathscr{K}_T^p(V).
\end{aligned}$$

(See Subsection 6.2 for definition of ∂_η^x.) $\square$

REMARK 7.2.3. *By Theorem 7.2.2, the complex of Zariski sheaves $\pi_* \mathscr{C}_X^{p,*}$ on X (where $\pi \colon X^{\mathrm{Ber}} \to X$ is the map taking supports) is a flasque resolution of $\pi_* \mathscr{F}_X^p \cong \mathscr{K}_T^p$.*

REMARK 7.2.4. *The isomorphism $CH^p(X)_{\mathbb{Q}} \cong H^{p,p}_{\mathrm{Trop}}(X)$ in Theorem 7.2.2 conincides with the tropical cycle class map in Section 4.2. This follows from the constructions of morphisms of sheaves which induce these two morphisms, see Section 4.2,* [**Liu20**, Definition 3.2, 3.6]*,* [**JSS17**, Lemma 3.21]*, and the proof of Theorem 7.2.2.*

By Theorem 4.2.3 and 7.2.2, we have the following corollary, which is a tropical analog of Grothendieck's standard conjecture D of algebraic cycles.

COROLLARY 7.2.5. *Let X, K be as in Theorem 7.2.2. When X is proper and has a closed immersion to a toric variety (e.g. projective), the kernel of the cycle class map $Z_p(X) \to H^{\mathrm{Trop}}_{p,p}(X)$ to the tropical homology is the set of algebraic cycles numerically equivalent to* 0.

7.2.2. Basic lemmas. In this subsection, we give several basic lemmas, which is used to prove Lemma 7.2.14 and 7.2.15 in Section 7.2.3.

LEMMA 7.2.6. *Let $\varphi\colon X \to T_\Sigma$ be a closed immersion of an algebraic variety X to the toric variety T_Σ over K associated with a fan Σ in $N_\mathbb{R}$ such that $\varphi(X)$ intersects with the dense open orbit, and Λ a fan structure of* $\mathrm{Trop}(\varphi(X))$. *For $r > 0$, we put*

$$C_r := \bigcup_{P \in \Lambda} \mathrm{rel.int}(P \cap N_{\sigma_P,\mathbb{R}} \cap \overline{\{n \in N_\mathbb{R} \mid \min_{n' \in \mathrm{Trop}(\varphi(X^\circ)) \cap N_\mathbb{R}} \mathrm{dist}(n, n') \leq r\}}),$$

where $\mathrm{dist}(n, n')$ *is the usual distance of the Euclid space $N_\mathbb{R} \cong \mathbb{R}^n$ (here we fix an identification $N_\mathbb{R} \cong \mathbb{R}^n$). (Remind that we put $\sigma_P \in \Sigma$ is the unique cone such that* $\mathrm{rel.int}(P) \subset N_{\sigma_P,\mathbb{R}}$.*) We assume that for any $P \in \Lambda$, the intersection $P \cap \mathrm{Trop}(\varphi(X^\circ))$ is a face of P. (Sufficiently fine Λ satisfies this assumption.)*

Then the pull-back map

$$H^{p,q}_{\mathrm{Trop}}(C_r, \Lambda) \to H^{p,q}_{\mathrm{Trop}}(\mathrm{Trop}(\varphi(X^\circ)), \Lambda)$$

is an isomorphism.

PROOF. Remind that by Lemma 4.1.6, usual tropical cohomology is isomorphic to the slight modification defined in Definition 4.1.5. Hence it suffices to show that there exists a strong deformation retraction $\psi\colon C_r \times [0,1] \to C_r$ of C_r onto $\mathrm{Trop}(\varphi(X^\circ))$ preserving the fan structure, i.e., for any cone $P \in \Lambda$, we have

$$\psi((P \cap C_r, [0,1])) \subset P \cap C_r.$$

We show the existence of such a ψ. Let $P \in \Lambda$ be a cone which is not contained in $\mathrm{Trop}(\varphi(X^\circ))$. Then since the intersection $P \cap \mathrm{Trop}(\varphi(X^\circ))$ is a non-empty face of P, there exists a strong deformation retraction

$$\psi_P\colon P \cap C_r \times [0,1] \to P \cap C_r$$

of $P \cap C_r$ onto

$$(P \cap C_r) \setminus \mathrm{rel.int}(P) = \bigcup_Q Q \cap C_r,$$

where $Q \in \Lambda$ runs through all proper faces of P. (This is a trivial fact on topology.) We put $C_{N,(p)}$ the intersection of C_r with the union of cones in Λ which are of dimension $\leq p$ or contained in $\mathrm{Trop}(\varphi(X^\circ))$. These ψ_P for cones $P \in \Lambda$ of dimension p which are not contained in $\mathrm{Trop}(\varphi(X^\circ))$ induce a strong deformation retraction $\psi_p\colon C_{N,(p)} \times [0,1] \to C_{N,(p)}$ of $C_{N,(p)}$ onto $C_{N,(p-1)}$ preserving the fan structure. These ψ_p ($p \geq 1$) induce a strong deformation retraction $\psi\colon C_r \times [0,1] \to C_r$ of C_r onto $\mathrm{Trop}(\varphi(X^\circ))$ preserving the fan structure. □

COROLLARY 7.2.7. *The pull-back map*

$$H^{p,q}_{\mathrm{Trop}}(\mathrm{Trop}(\varphi(X))) \to H^{p,q}_{\mathrm{Trop}}(\mathrm{Trop}(\varphi(X^\circ)), \Lambda)$$

is an isomorphism.

PROOF. Since $\mathrm{Trop}(\varphi(X)) = \bigcup_{r \in \mathbb{Z}_{\geq 1}} C_r$, every compact subset of $\mathrm{Trop}(\varphi(X))$ is contained in some C_r ($r \in \mathbb{Z}_{\geq 1}$). Hence, by Lemma 7.2.6, we have

$$H^{p,q}_{\mathrm{Trop}}(\mathrm{Trop}(\varphi(X^\circ)), \Lambda) \cong \varinjlim_{r \in \mathbb{Z}_{\geq 1}} H^{p,q}_{\mathrm{Trop}}(C_r, \Lambda) \cong H^{p,q}_{\mathrm{Trop}}(\mathrm{Trop}(\varphi(X))).$$

□

LEMMA 7.2.8. *Let T_Σ be a toric variety, Λ a fan in* $\mathrm{Trop}(T_\Sigma)$, *and A, B be subsets of $|\Lambda|$ such that $A \subset B$. We fix $p \in \mathbb{Z}_{\geq 0}$ and $q \in \mathbb{Z}_{\geq 1}$. We assume that the pull-back map*

$$H^{p,q-\epsilon}_{\mathrm{Trop}}(B, \Lambda) \to H^{p,q-\epsilon}_{\mathrm{Trop}}(A, \Lambda)$$

is injective for $\epsilon = 0$*, and surjective for* $\epsilon = 1$*. Then, for* $\alpha \in Z^{p,q}(B, \Lambda)$ *such that* $\alpha|_A = 0$*, there exists* $\beta \in C^{p,q-1}(B, \Lambda)$ *such that* $\beta|_A = 0$ *and* $\alpha = \delta(\beta)$*.*

PROOF. The proof is elementary. By injectivity of (7.2.8) for $\epsilon = 0$, there exists a $\beta' \in C^{p,q-1}(B, \Lambda)$ such that $\delta(\beta') = \alpha$. Since $\delta(\beta'|_A) = \alpha|_A = 0$, by surjectivity of (7.2.8) for $\epsilon = 1$, there exist $\beta'' \in Z^{p,q-1}(B, \Lambda)$ and $\gamma \in C^{p,q-2}(A, \Lambda)$ such that $\beta''|_A + \delta(\gamma) = \beta'|_A$. We put $\tilde{\gamma} \in C^{p,q-2}(B, \Lambda)$ the element such that its restriction to A is γ, and it is 0 for singular $(q-2)$-simplex whose image is not contained in A. We put $\beta := \beta' - \beta'' - \delta(\tilde{\gamma})$. By construction, this β satisfies the conditions in the Lemma. □

LEMMA 7.2.9. *Let* X *be an algebraic variety over* K*. Assume that* X *has a closed immersion to a toric variety. Then the pull-back maps induces an isomorphism*

$$H^{p,q}_{\mathrm{Trop},T}(X) \cong \varinjlim_{\varphi} H^{p,q}_{\mathrm{Trop}}(\mathrm{Trop}(\varphi(X^\circ))),$$

where φ *runs through all closed immersions of* X *to toric varieties.*

PROOF. Well-definedness of the morphism follows from compactness of X° (Remark 3.3.11). Surjectivity follows from Corollary 7.2.7. Injectivity follows from Lemma 7.2.6 and 7.2.8. □

REMARK 7.2.10. *Since* X° *is compact (Remark 3.3.11), for closed subscheme* $Z \subset X$*, we have*

$$H^{p,q}_{\mathrm{Trop},T,Z^\circ}(X^\circ) \cong \varinjlim_{\varphi} H^{p,q}_{\mathrm{Trop},\mathrm{Trop}(Z^\circ)}(\mathrm{Trop}(\varphi(X^\circ))).$$

COROLLARY 7.2.11. *We have*

$$H^{p,q}_{\mathrm{Trop},T}(X) \cong H^{p,q}_{\mathrm{Trop},T}(X^\circ).$$

COROLLARY 7.2.12. *For a closed subscheme* $Z \subset X$*, we have*

$$H^{p,q}_{\mathrm{Trop},T,Z}(X) \cong H^{p,q}_{\mathrm{Trop},T,Z^\circ}(X^\circ) \cong \varinjlim_{\varphi} H^{p,q}_{\mathrm{Trop},\mathrm{Trop}(Z^\circ)}(\mathrm{Trop}(\varphi(X^\circ))).$$

PROOF. In a similar way to Lemma 7.2.9, the pull back map

$$H^q(\Gamma(X^\circ \setminus Z^\circ, \mathscr{C}^{p,*}_{T,X^\circ})) \to H^{p,q}_{\mathrm{Trop},T}((X \setminus Z)^\circ)$$

is an isomorphism. The assertion follows from long exact sequences of realtive cohomology for pairs. □

LEMMA 7.2.13. *Let* $\varphi\colon X \to T_\sigma$ *be a closed immersion of an affine algebraic variety* X *to the affine toric variety* T_σ *associated with a cone* $\sigma \subset N_{\mathbb{R}}$*. Let* $x \in X$*. We assume that* $X \cap \varphi^{-1}(O(\sigma)) = \overline{\{x\}}$*. Let* $p_x \subset \mathcal{O}(X)$ *the prime ideal corresponding to* x*. We put* p *the codimension of* $\overline{\{x\}} \subset X$*.*

Then there exist $f_1, \dots, f_r \in \mathcal{O}(X) \setminus p_x$ *such that*

(1)

$$\mathrm{Trop}((\varphi, (f_i)_i)(X)) \cap (\sigma \times \{0\}^r) \subset \mathrm{Trop}(T_\sigma) \times (\mathbb{R} \cup \{\infty\})^r$$

is a finite union of cones of dimension $\leq p$*, and*

(2)

$$\mathrm{Trop}((\varphi, (f_i)_i)(X)) \cap (\sigma \times \{0\}^r \setminus \mathrm{rel.int}(\sigma \times \{0\}^r)) \subset \mathrm{Trop}(T_\sigma) \times (\mathbb{R} \cup \{\infty\})^r$$

is a finite union of cones of dimension $\leq p-1$*,*

here we consider f_i *as a morphism* $f_i\colon X \to \mathbb{A}^1$*, and*

$$(\varphi, (f_i)_i)\colon X \to T_\sigma \times \mathbb{A}^r$$

is a closed immersion.

PROOF. Let Σ be a fan structure of σ such that a subfan $\Lambda \subset \Sigma$ is a fan structure of $\mathrm{Trop}(\varphi(X)) \cap \sigma$. This induces a birational toric morphism $\pi \colon T_\Sigma \to T_\sigma$. We put X' the closure of $\pi^{-1}(\varphi(X) \cap \mathbb{G}_m^n)$ in T_Σ. Then, by [**MS15**, Theorem 6.3.4], the subvariety X' is contained in the toric subvariety $T_\Lambda \subset T_\Sigma$. Moreover, by Lemma 3.3.6, for any $\tau \in \Lambda$, the intersection $X' \cap O(\tau) \subset X'$ is pure codimension $= \dim(\tau)$.

We put $B \subset X$ the union of the image of the generic points of the irreducible components of $X' \cap O(\tau)$ for cones $\tau \in \Lambda$ of dimension $\geq p$ under the natural morphism $X' \to X$. Then any $y \in B$ is of codimension $\geq p$. In particular, every $y \in B \setminus \{x\}$ is not a generalization of x. Hence there exist $f_1, \dots, f_r \in \mathcal{O}(X) \backslash p_x$ such that for any $y \in B \setminus \{x\}$, there exists f_i contained in p_y. By construction, these f_i satisfy the properties in Lemma 7.2.13. (Remind that for any τ and $v \in X^{\mathrm{Ber}} \cap (\mathrm{Trop} \circ \varphi)^{-1}(\mathrm{rel.int}(\tau))$, there is a natural morphism $\operatorname{Spec} \mathcal{O}_v to X'$ from the spectrum of valuation ring O_v whose center center(v) (the image of maximal ideal) is in $X' \cap O(\tau)$.) $\square$

7.2.3. Several computations on tropical cohomology. We show Lemma 7.2.14 and 7.2.15, which are used to prove Theorem 7.2.2. Let X be a smooth algebraic variety over K.

LEMMA 7.2.14. *Let $p \geq 0$, $x \in X^{(p)}$, and $q \geq 1$. Then*
$$H^{r,p+q}_{\mathrm{Trop},x}(X) = 0.$$

PROOF. This follows from Corollary 7.2.11, Lemma 7.2.13, the fact that as usual singular cohomology, tropical cohomology can be computed from cell complex instead of singular chains ([**MZ13**, Proposition 2.2]), and the long exact sequences of relative tropical cohomology of pairs. $\square$

LEMMA 7.2.15. *There are natural isomorphisms*
$$H^{r,0}_{\mathrm{Trop},\eta}(X) \cong K^r_T(K(X))$$
and
$$H^{r,1}_{\mathrm{Trop},x}(X) \cong K^{p-1}_T(k(x)) \quad (x \in X^{(1)})$$
such that
$$\begin{array}{ccc} H^{r,0}_{\mathrm{Trop},\eta}(X) & \xrightarrow{d_1} & H^{r,1}_{\mathrm{Trop},x}(X) \\ \downarrow & & \downarrow \\ K^r_T(K(X)) & \xrightarrow{\partial^x_\eta} & K^{r-1}_T(k(x)) \end{array}$$
is commutative. (See Subsection 6.2 and the proof of Theorem 7.2.2 for notations.)

PROOF. The first isomorphism is given by Corollary 7.2.12. The second one is given as follows. (By construction, the diagram is commutative.) By Corollary 7.2.12, Lemma 7.2.13, and Remark 7.2.10, the tropical cohomology $H^{r,1}_{\mathrm{Trop},x}(X)$ is isomorphic to
$$\varinjlim_{\varphi} H^{r,1}_{\mathrm{Trop},\overline{l_\varphi} \backslash l_\varphi}(\overline{l_\varphi}, \Lambda_\varphi),$$
where $\varphi \colon U_\varphi \to T_{l_\varphi}$ run through all closed immersions from open neighorhood $U_\varphi \subset X$ of x to the toric varieties T_{l_φ} associated with 1-dimensional cones $l_\varphi \in N_{\varphi,\mathbb{R}}$ such that $\varphi^{-1}(O(l_\varphi)) = U_\varphi \cap \overline{\{x\}}$. (Here, the closure $\overline{l_\varphi}$ is taken in $\mathrm{Trop}(T_{l_\varphi})$) and Λ_φ is a fan structure of $\mathrm{Trop}(\varphi(U_\varphi))$.) A map
$$\varinjlim_{\varphi} H^{r,1}_{\mathrm{Trop},\bar{l} \backslash \{0\}}(\bar{l}, \Lambda_\varphi) \to K^{p-1}_T(k(x))$$

given by

$$H^{r,1}_{\mathrm{Trop},\bar{l}\setminus\{0\}}(\bar{l},\Lambda_\varphi) \ni \alpha_\varphi \mapsto (\alpha_\varphi)_{\gamma_\varphi}(d_\varphi \wedge \cdot)$$

is an isomoprhism, where $d_\varphi \in l$ is the primitive element, and $\gamma_\varphi\colon [0,1] \to \overline{l_\varphi}$ is a fixed homeomorphism such that $\gamma_\varphi(0)$ is $0 \in N_{\varphi,\mathbb{R}}$. The composition gives the second isomorphism in Lemma 7.2.15. $\square$

7.3. Étale excision

We prove étale excision (Proposition 7.2.1 (1)). Let $\Phi\colon X' \to X$ be an étale morphism of smooth quasi-projective varieties over a trivially valued field K. Let $Z \subset X$ be a closed subscheme. We assume that Φ induces an isomorphism $Z' := \Phi^{-1}(Z) \cong Z$. By Corollary 7.2.12, it suffices to show that the natural morphism

$$\mathscr{C}^{p,q}_{Z^\circ \subset X^\circ} \to \Phi_* \mathscr{C}^{p,q}_{Z'^\circ \subset X'^\circ}$$

of sheaves on X° is an isomorphism. We consider an open neighorhood $X^\circ \setminus (X \setminus Z)^\circ$ (resp. $X'^\circ \setminus (X' \setminus Z')^\circ$) of Z° in X° (resp. of Z'° in X'°). Remind that by definition, $X^\circ \setminus (X \setminus Z)^\circ$ (resp. $X'^\circ \setminus (X' \setminus Z')^\circ$) is the set of valuations contained in X (resp. X') whose center is in Z (resp. Z'). Since $\Phi\colon X' \to X$ is étale and $Z' \cong Z$, by [**Fu15**, Lemma 2.8.5], the map

$$X^\circ \setminus (X \setminus Z)^\circ \to X'^\circ \setminus (X' \setminus Z')^\circ$$

is a homeomorphism. Hence it suffices to show that for any $v' \in X'^\circ \setminus (X' \setminus Z')^\circ$, the morphism

$$\mathscr{C}^{p,q}_{Z^\circ \subset X^\circ, \Phi(v')} \to \mathscr{C}^{p,q}_{Z'^\circ \subset X'^\circ, v'}$$

of stalks is an isomorphism. When $v' \in Z'^\circ$, the both hand sides are 0. We assume $v' \in Z'^\circ$. Injectivity follows from Lemma 4.1.7.

We shall show surjectivity. Let $f_1, \dots, f_r \in \mathcal{O}_{X',\mathrm{supp}(v')}$.

Lemma 7.3.1. *There exist $g_1, \dots, g_r \in \mathcal{O}_{X,\mathrm{supp}(v)}$ such that*

$$\frac{f_i}{g_i} \in \mathcal{O}_{X',\mathrm{supp}(w')}$$

and

$$w'(\frac{f_i}{g_i} - 1) > 0$$

for any w' in an open neighorhood W' of v' in X^{Ber}.

Proof. This follows from the fact that the above conditions are open conditions, the morphism $\Phi\colon X' \to X$ is étale, and $Z' \cong Z$. $\square$

Hence we have

$$\mathrm{Trop} \circ (f_i)_i|_{W'} = \mathrm{Trop} \circ (g_i)_i \circ \Phi|_{W'},$$

here $(f_i)_i$ and $(g_i)_i$ are considered as morphisms to the affine space $\mathbb{A}^r$. Consequently, since $\{f_i\}_i$ are arbitraly, the morphism

$$\mathscr{C}^{p,q}_{(Z \subset X)^\circ, \Phi(v')} \to \mathscr{C}^{p,q}_{(Z' \subset X')^\circ, v'}$$

is surjective.

7.4. The existence of corestriction maps

In this section, we show the existence of corestriction maps, which is used to prove the main theorem Theorem 7.2.2 over finite fields. Let K be a trivially valued finite field and L a trivially valued finite field which is an extension of K. Let X be a smooth algebraic variety over K.

We shall construct a $\mathbb{Q}$-linear map

$$\mathrm{cor}\colon \mathscr{C}^{p,q}_{X_L}(X_L^{\mathrm{Ber}}) \to \mathscr{C}^{p,q}_{X}(X^{\mathrm{Ber}})$$

(*corestrction map*) such that $\mathrm{cor}\circ\delta = \delta\circ\mathrm{cor}$ and $\mathrm{cor}\circ\mathrm{res} = [L:K]$, where $\mathrm{res}\colon \mathscr{C}^{p,q}_{X}(X^{\mathrm{Ber}}) \to \mathscr{C}^{p,q}_{X_L}(X_L^{\mathrm{Ber}})$ is the base change.

We may assume that X is quasi-projective. We use $\mathscr{C}^{p,q}_{T,X}$ instead of $\mathscr{C}^{p,q}_{X}$ (Lemma 4.1.8). Let $\varphi\colon X_L \to T_{\Sigma',L}$ be a closed immersion to a toric variety $T_{\Sigma',L}$ over L and $\alpha \in C^{p,q}_{\mathrm{Trop}}(\mathrm{Trop}(\varphi(X_L)))$. We shall construct $\mathrm{cor}(\alpha) \in \mathscr{C}^{p,q}_{T,X}(X^{\mathrm{Ber}})$. We put $\pi\colon X_L \to X$ the base change. For simplicity, we assume that X_L is connected and $\varphi(X_L)$ intersects with the dense torus orbit.

LEMMA 7.4.1. *There exists a morphism $\psi\colon X \to T_\Sigma$ to a toric variety T_Σ such that*

- *the projection*

$$\Phi\colon\ \mathrm{Trop}((\varphi,\psi_L)(X_L)) \to \mathrm{Trop}(\psi(X))$$

  is finite-to-one, and
- *for any r, sufficiently fine fan structures Λ of $\mathrm{Trop}(\psi(X))$ and Λ' of $\mathrm{Trop}((\varphi,\psi_L)(X_L))$ and cones $P \in \Lambda$ and $P' \in \Lambda'$ such that $\Phi(P') = P$, there is a unique $\mathbb{Q}$-linear map*

$$\overline{\mathrm{cor}}_{P'}\colon \overline{F}^r(P', \mathrm{Trop}((\varphi,\psi_L)(X_L))) \to \overline{F}^r(P, \mathrm{Trop}(\psi(X))),$$

  such that for a valuation $\mu \in X^{\mathrm{ad}}$ with $\mathrm{Trop}^{\mathrm{ad}}_\Lambda(\psi(\mu)) = P$, the diagram

$$\begin{tikzcd}
\overline{F}^r(P', \mathrm{Trop}((\varphi,\psi_L)(X_L))) \arrow[r] \arrow[d] & \overline{F}^r(P, \mathrm{Trop}(\psi(X))) \arrow[d] \\
\bigoplus_\nu K^r_T(\kappa(\nu)) \arrow[r] & K^r_T(k(\mu))
\end{tikzcd}$$

  is commutative, where ν runs through all the elements in the finite set $\pi^{-1}(\mu)\cap(\mathrm{Trop}^{\mathrm{ad}}_{\Lambda'})^{-1}(P')$. the vertical arrows are canonical ones, and the last horizontal arrow is the sum of norm homomorphism.

PROOF. Uniqueness is trivial.

We claim that for any q, there exist a closed subscheme $Z_q \subset X$ of codimension q and a morphism $\psi_q\colon X \to T_{\Sigma_q}$ to a toric variety T_{Σ_q} satisfying the assertion of the Lemma for $X \setminus Z_q$ instead of X.

When $q = 1$, this follows from compactness of the Zariski Riemann space $\mathrm{ZR}(K(X)/K)$ and Lemma 7.4.2. (Note that for v and ψ_v as in Lemma 7.4.2 and any $\psi'\colon X \to T_{\Sigma_1}$, Lemma 7.4.2 holds for v and (ψ_v,ψ') by projection formula.)

When $q \geq 2$, the claim follows in a similar way to $q = 1$, by induction on q. $\square$

LEMMA 7.4.2. *For any $v \in \mathrm{ZR}(K(X)/K)$, there exists a morphism $\psi_v\colon X \to T_{\Sigma_v}$ such that for sufficiently fine fan structures Λ_v and Λ'_v, there exists $\overline{\mathrm{cor}}_{P'}$ as in Lemma 7.4.1 for the cones $P' \in \Lambda'_v$ such that $P' \mapsto \mathrm{Trop}^{\mathrm{ad}}_{\Lambda_v}(\psi_v(v)) \in \Lambda_v$.*

PROOF. There exists a commutative diagram

$$\begin{tikzcd}
 & T_{\Sigma',L} \times T_{\Lambda_{v,0},L} \arrow[r] & T_{\Lambda_{v,0}} & \\
X_L \arrow[ur, dotted, "{(\varphi,\psi_{v,0,L})}"] \arrow[r, dotted, "\psi'_{v,0}"] \arrow[dr, dotted, "\psi'_v"'] & T_{\Lambda'_{v,0},L} \arrow[u] & T_{\Sigma_{v,1}} \arrow[u] & X \arrow[ul, dotted, "\psi_{v,0}"'] \arrow[l, dotted, "\psi_{v,1}"'] \arrow[dl, dotted, "\psi_{v,2}"] \arrow[d, "\psi_v"] \\
 & T_{\Lambda'_v,L} \arrow[u] \arrow[r] & T_{\Lambda_v} \arrow[u] \arrow[r] & T_{\Sigma_v}
\end{tikzcd}$$

such that

- for any $*$, the variety T_* is a toric variety, the arrows between them are toric morphisms,
- the usual arrows are morphisms, the dotted arrows are rational maps,
- the toric morphisms $T_{\Lambda_v} \to T_{\Sigma_v}$ and $T_{\Lambda'_{v,0},L} \to T_{\Sigma',L} \times T_{\Lambda_{v,0},L}$ are birational,
- for any $*$, the closure of the image of suitable open subvariety of X in T_{Λ_*} or X_L in $T_{\Lambda'_*,L}$ is a tropical compactification,
- the toric variety $T_{\Lambda'_v}$ is an open toric subvariety of the toric variety $T_{\Lambda'_{v,0}} \times_{T_{\Lambda_{v,0}}} T_{\Lambda_v}$ (Lemma 7.4.3),
- the variety $\overline{\psi_{v,1}(U)}$ is the normalization of $\overline{\psi_{v,0}(U)}$, where we fix a sufficiently small open subvariety $U \subset X$,
- the morphism $\overline{\mathrm{Trop}(\psi'_{v,0}(U_L))} \to \overline{\mathrm{Trop}(\psi_{v,0}(U))}$ is finite-to-one,
- the fan $\Lambda'_{v,0}$ is sufficiently fine so that for $w \in \pi(v)$, the number $\dim(\mathrm{Trop}^{\mathrm{ad}}_{\Lambda'_{v,0}}(w))$ archives its minimum,
- we have
$$\overline{\{x_0\}} = \overline{\psi_{v,1}(U)} \cap \overline{\mathcal{O}(\sigma_{x_0})}$$
for some $\sigma_{x_0} \in \Sigma_{v,1}$, where $x_0 := \mathrm{center}(v) \in \overline{\psi_{v,1}(U)}$,
- for any p, the image of the set
$$\wedge^p \mathrm{Im}(\Gamma(\mathcal{O}(\sigma_{x_0}), \mathcal{O}) \to \kappa(v))$$
contains the image of the restriction
$$\pi_w^*|_{\varphi(M') \cap w^{-1}(\{0\})}$$
of $\pi_w^* \colon K_T^p(\kappa(w)) \to K_T^p(\kappa(v))$ for $w \in \pi^{-1}(v)$, where $\pi_w \colon \kappa(v) \to \kappa(w)$ is the natural finite morphism of the residue fields and $\mathcal{O}$ is the structure sheaf of the torus $\mathcal{O}(\sigma_{x_0})$.

Then there is a natural morphism
$$\overline{\psi_{v,1}(U)}_L \to \overline{\psi'_{v,0}(U_L)}.$$
This is because $\overline{\psi'_{v,0}(U_L)} \to \overline{\psi_{v,0}(U)}$ is finite, and $\overline{\psi_{v,1}(U)}_L$ is normal. (Remind that L/K is separable.) In particular,

- for any $w, \nu \in \mathrm{ZR}(L(X_L/K))$ such that $\pi(w) = v$ and
$$\mathrm{Trop}^{\mathrm{ad}}_{\Lambda_v}(\psi'_v(w)) = \mathrm{Trop}^{\mathrm{ad}}_{\Lambda_v}(\psi'_v(\nu)),$$
the local ring $\mathcal{O}_y$ of the structure sheaf at the center $y \in \overline{\psi_{v,1}(U)}_L$ of ν contains $\varphi(M') \cap w^{-1}(0)$, and

- for $w, \nu \in \mathrm{ZR}(L(X_L/K))$ such that $\pi(w) = v$,
$$\mathrm{Trop}^{\mathrm{ad}}_{\Lambda_v}(\psi_{v,2}(\pi(\nu))) = \mathrm{Trop}^{\mathrm{ad}}_{\Lambda_v}(\psi_{v,2}(v)),$$
and $y_\nu \in \overline{\{y_w\}}$, where $y_* \in \overline{\psi_{v,1}(U)}_L$ is the center of $*$ for $* = w, \nu$, we have
$$\mathrm{Trop}^{\mathrm{ad}}_{\Lambda'_v}(\psi'_v(\nu)) = \mathrm{Trop}^{\mathrm{ad}}_{\Lambda'_v}(\psi'_v(w)).$$
(Note that for each ν such that
$$\mathrm{Trop}^{\mathrm{ad}}_{\Lambda_v}(\psi_{v,2}(\pi(\nu))) = \mathrm{Trop}^{\mathrm{ad}}_{\Lambda_v}(\psi_{v,2}(v)),$$
there exists w as above.)

Consequently, there is a unique required $\mathbb{Q}$-linear map $\overline{\mathrm{cor}_{P'}}$ by projection formula ([**Ros96**, Definition 1.1 R2c]), two types of compatibility [**Ros96**, Defintion 1.1 R1c and Propositon 4.6 (1)], and the fact that for any $x, x' \in X$ with $x' \in \overline{\{x\}}$ and $v \in \mathrm{ZR}(k(x')/K) \subset X^{\mathrm{ad}}$, there exists a horizontal generalization $v' \in \mathrm{ZR}(k(x)/K)$ (see [**HK94**] for horizontal generalizations), and we have $\kappa(v) = \kappa(v')$ and
$$\bigoplus_{w' \in \pi^{-1}(v')} \kappa(w') \cong \bigoplus_{w \in \pi^{-1}(v)} \kappa(w).$$
(The last fact is because we have
$$\deg(\bigoplus_{w' \in \pi^{-1}(v')} \kappa(w')/\kappa(v')) = \deg(\kappa(v') \otimes_K L/\kappa(v)) = [L : K]$$
and the same holds for v instead of v'.) $\square$

By computations of dimension and the balancing condition ([**MS15**, Theorem 3.4.14]), we have the follwoing.

LEMMA 7.4.3. *Let T_{Σ_i} $(i = 1, 2, 3)$ be a toric variety and $\varphi_j \colon T_{\Sigma_j} \to T_{\Sigma_1}$ $(j = 2, 3)$ be a toric morphism. Let $X_{\Sigma_i} \subset T_{\Sigma_i}$ $(i = 1, 2, 3)$ be an algebraic variety which is a tropical compactification such that $\varphi_j(X_{\Sigma_j}) = X_{\Sigma_1}$ $(j = 2, 3)$ and $\mathrm{Trop}(X_{\Sigma_2}) \to \mathrm{Trop}(X_{\Sigma_1})$ is finite-to-one. Then there exists an open toric subvariety T_{Σ_4} of $T_{\Sigma_2} \times_{T_{\Sigma_1}} T_{\Sigma_3}$ such that*
$$\mathrm{Trop}(X_{\Sigma_2} \times_{X_{\Sigma_1}} X_{\Sigma_3}) = \overline{|\Sigma_4|} \subset \mathrm{Trop}(T_{\Sigma_2} \times_{T_{\Sigma_1}} T_{\Sigma_3}).$$

We put M the lattice such that $\mathrm{Spec}\, K[M] \subset T_\Sigma$ is the dense torus orbit.

For each cone $P \in \Lambda$, we take $f_1, \dots, f_{\dim(P)} \in M \cap \sigma_P^\perp$ such that the natural map
$$\langle f_1, \dots, f_{\dim(P)} \rangle_{\mathbb{Q}} \to \mathrm{Hom}_{\mathbb{Q}}(\mathrm{Span}(P), \mathbb{Q})$$
is an isomorphism, where $\sigma_P \in \Sigma$ is the cone such that $\mathrm{rel.int}(P) \subset \mathrm{Trop}(\mathcal{O}(\sigma_P))$. This induces decompositions
$$F^p(*) \cong \oplus_{j=0}^p \wedge^j \langle f_i \rangle_{\mathbb{Q}} \otimes \overline{F}^{p-j}(*)$$
for $* = P \in \Lambda$ or $P' \in \Lambda'$ such that $\Phi(P') = P$. For each continuous map $\gamma \colon \Delta^q \to P' \in C_q(P')$, we write the image of α_γ by
$$\sum_{j=0}^p \sum_l b_{j,l} \otimes \alpha_{\gamma,j,l}$$
$(b_{j,l} \in \wedge^j \langle f_i \rangle_{\mathbb{Q}}$ and $\alpha_{\gamma,j,i} \in \overline{F}^{p-j}(P'))$.

DEFINITION 7.4.4. *For each continuous map* $\gamma\colon \Delta^q \to P \in C_q(P)$*, we put*

$$\operatorname{cor}(\alpha)_\gamma := \sum_{P'} \sum_{j=0}^{p} \sum_{l} b_{j,l} \otimes \overline{\operatorname{cor}}_{P'}(\alpha_{(\Phi|_{P'})^{-1}\circ\gamma,j,l}),$$

where $P' \in \Lambda'$ *runs through all cones such that* $\Phi(P') = P$.

We put

$$\operatorname{cor}(\alpha) := (\operatorname{cor}(\alpha)_\gamma)_\gamma \in C^{p,q}_{\mathrm{Trop}}(\operatorname{Trop}(\psi(X))).$$

This $\operatorname{cor}(\alpha)_{P'}$ *is independent of the choices by uniqueness of* $\overline{\operatorname{cor}}$ *and projection formula (*[**Ros96**, Definition 1.1 R2c]*).*

Consequently, by Lemma 4.1.8, we have the following.

PROPOSITION 7.4.5. *Let* X *be a smooth algebraic variety over* K*. Then there is a* $\mathbb{Q}$*-linear map*

$$\operatorname{cor}\colon \mathscr{C}^{p,q}_{X_L}(X_L^{\mathrm{Ber}}) \to \mathscr{C}^{p,q}_{X}(X^{\mathrm{Ber}})$$

such that $\operatorname{cor}\circ\delta = \delta\circ\operatorname{cor}$ *and* $\operatorname{cor}\circ\operatorname{res} = [L:K]$.

(The compatibility $\operatorname{cor}\circ\delta = \delta\circ\operatorname{cor}$ follows from [**Ros96**, Definition 1.1 R3b].)

7.5. Example: smooth toric varieties

In this section, we shall show that the tropical cohomology groups of smooth toric varieties over $\mathbb{C}$ are isomorphic to the weight-graded pieces of their singular cohomology groups.

Let M be a free $\mathbb{Z}$-module of finite rank. Let T_Σ be the smooth toric variety over $\mathbb{C}$ associated with a smooth fan Σ in $N_{\mathbb{R}} := \operatorname{Hom}(M,\mathbb{R})$. We define the tropical cohomology groups $H^{p,q}_{\mathrm{Trop}}(T_\Sigma)$ of T_Σ by considering $\mathbb{C}$ as a trivially valued field.

We recall a spectral sequence whose E_2-terms are isomorphic to the weight graded pieces $\operatorname{gr}^W_{2(q-p)} H^{p+q}_{\mathrm{sing}}(T_\Sigma(\mathbb{C}),\mathbb{Q})$. We put $T_\Sigma^{(p)} \subset T_\Sigma$ the closure of the union of p-codimensional orbits. This induces a filtration

$$F^{(p)}C^r := C^r_{\mathrm{sing},T_\Sigma^{(p)}(\mathbb{C})}(T_\Sigma(\mathbb{C}),\mathbb{Q}) := \operatorname{Ker}(C^r_{\mathrm{sing}}(T_\Sigma(\mathbb{C}),\mathbb{Q}) \to C^r_{\mathrm{sing}}(T_\Sigma(\mathbb{C})\setminus T_\Sigma^{(p)}(\mathbb{C}),\mathbb{Q}))$$

on the group of singular cochains $C^r := C^r_{\mathrm{sing}}(T_\Sigma(\mathbb{C}),\mathbb{Q})$. Totaro showed that the spectral sequence

$$E_1^{p,q} := E_{\Sigma,1}^{p,q} := H^{q-p}_{\mathrm{sing}}(T_\Sigma^{(p)}(\mathbb{C})\setminus T_\Sigma^{(p+1)}(\mathbb{C}),\mathbb{Q}) \Rightarrow H^{p+q}_{\mathrm{sing}}(T_\Sigma(\mathbb{C}),\mathbb{Q})$$

induced from this filtration degenerates at E_2-terms, and $E_2^{p,q} := E_{\Sigma,2}^{p,q}$ is isomorphic to the weight-graded pieces $\operatorname{gr}^W_{2(q-p)} H^{p+q}_{\mathrm{sing}}(T_\Sigma(\mathbb{C}),\mathbb{Q})$ [**Tot14**, Theorem 4].

We give a natural morphism $E_2^{p,q} \to H^{q,p}_{\mathrm{Trop}}(T_\Sigma)$. There are natural isomorphisms

$$H^l_{\mathrm{sing}}(O(\sigma)(\mathbb{C}),\mathbb{Q}) \cong \wedge^l(M\cap\sigma^\perp)_{\mathbb{Q}} \quad (l \in \mathbb{Z}_{\geq 0}, \sigma\in\Sigma),$$

see [**CLS11**, Example 9.0.12]. We consider composition of this isomorphism and a natural map $\wedge^l(M\cap\sigma^\perp)_{\mathbb{Q}} \to K_T^l(k(\eta_\sigma))$, where for each $\sigma\in\Sigma$, we put $\eta_\sigma \in O(\sigma)$ the generic point. By [**Jor98**, Theorem 2.4.1] and Poincaré duality, we have a commutative diagram

$$\begin{tikzcd}
H^{q-p}_{\mathrm{sing}}(T_\Sigma^{(p)}(\mathbb{C})\setminus T_\Sigma^{(p+1)}(\mathbb{C}),\mathbb{Q}) \arrow[r, "d_1"] \arrow[d] & H^{q-p-1}_{\mathrm{sing}}(T_\Sigma^{(p+1)}(\mathbb{C})\setminus T_\Sigma^{(p+2)}(\mathbb{C}),\mathbb{Q}) \arrow[d] \\
\bigoplus_{x\in T_\Sigma^{(p)}} K_T^{q-p}(k(x)) \arrow[r, "d"] & \bigoplus_{x\in T_\Sigma^{(p+1)}} K_T^{q-p-1}(k(\sigma)),
\end{tikzcd}$$

where $d_1\colon E_1^{p,q} \to E_1^{p+1,q}$ is the differential, the map d is as in Corollary 6.2.9, and the vertical map are defined above. (Here, we identify $H^{q-p}_{\mathrm{sing}}(T_\Sigma^{(p)}(\mathbb{C}) \setminus T_\Sigma^{(p+1)}(\mathbb{C}))$ and $\bigoplus_{\sigma\in\Sigma_{(p)}} H^{q-p}_{\mathrm{sing}}(O(\sigma)(\mathbb{C}))$, where $\Sigma_{(p)} \subset \Sigma$ is the subset of p-dimensional cones.) By Corollary 6.2.9 and Theorem 7.2.2, this commutative diagram induce a morphism

$$E_2^{p,q} \to H^{q,p}_{\mathrm{Trop}}(T_\Sigma).$$

We shall show that this is an isomorphism.

REMARK 7.5.1. *For a smooth toric variety T_Σ, a closed toric subvariety $T_\Lambda \subset T_\Sigma$ of codimension 1, we put Ξ the fan associated with the smooth toric variety $T_\Sigma \setminus T_\Lambda$. We shall construct a morphism of long exact sequence*

$$\begin{tikzcd}
\dots \arrow[r] & E_{\Lambda,2}^{p-1,q-1} \arrow[r] \arrow[d] & E_{\Sigma,2}^{p,q} \arrow[r] \arrow[d] & E_{\Xi,2}^{p,q} \arrow[r] \arrow[d] & E_{\Lambda,2}^{p,q-1} \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & H^{q-1,p-1}_{\mathrm{Trop}}(T_\Lambda) \arrow[r] & H^{q,p}_{\mathrm{Trop}}(T_\Sigma) \arrow[r] & H^{q,p}_{\mathrm{Trop}}(T_\Xi) \arrow[r] & H^{q-1,p}_{\mathrm{Trop}}(T_\Lambda) \arrow[r] & \dots.
\end{tikzcd}$$

The projection gives an short exact sequence

$$0 \to E_{\Lambda,1}^{p-1,q-1} \to E_{\Sigma,1}^{p,q} \to E_{\Xi,1}^{p,q} \to 0.$$

(Here, we identify $E_{*,1}^{p,q}$ and $\bigoplus_{\sigma\in *_{(p)}} H^{q-p}_{\mathrm{sing}}(O(\sigma)(\mathbb{C}))$ for $* \in \{\Lambda, \Sigma, \Xi\}$.) This gives a long exact sequence of E_2-terms

$$\dots \to E_{\Lambda,2}^{p-1,q-1} \to E_{\Sigma,2}^{p,q} \to E_{\Xi,2}^{p,q} \to E_{\Lambda,2}^{p,q-1} \to \dots.$$

Theorem 7.2.2 and the Gersten resolutions of sheaves of tropical K-groups induces a natural isomorphism

$$H^{p,q}_{\mathrm{Trop},T_\Lambda}(T_\Sigma) \cong H^{p-1,q-1}_{\mathrm{Trop}}(T_\Lambda).$$

Consequently, by the long exact sequence of relative tropical cohomology groups for the pair $(T_\Sigma, T_\Lambda$, we have the above morphism of long exact sequences.

THEOREM 7.5.2. *We have*

$$E_2^{p,q} \cong H^{q,p}_{\mathrm{Trop}}(T_\Sigma).$$

PROOF. First, we show Theorem 7.5.2 for the torus $T_\Sigma = \mathbb{G}_m^n$ by induction on n. When $T_\Sigma = \mathbb{G}_m^0 = \{\mathrm{pt}\}$, Theorem 7.5.2 is trivial. When $T_\Sigma = \mathbb{G}_m^n$ $(n \geq 1)$, Theorem 7.5.2 follows from the assumption on induction, $\mathbb{A}^1$-homotopy invariance (Proposition 7.2.1 (2)), and the morphism of long exact sequences in Remark 7.5.1 for $\mathbb{G}_m^{n-1} \times \{0\} \subset \mathbb{G}_m^{n-1} \times \mathbb{A}^1$.

Second, we show Theorem 7.5.2 for general smooth toric variety T_Σ by induction on the integer $\#\Sigma$. When $\#\Sigma = 1$, the toric variety T_Σ is the algebraic torus $\mathbb{G}_m^n$, and Theorem 7.5.2 is already proved above. We assume $\#\Sigma \geq 2$. Then there is a codimension 1 closed toric subvariety $T_\Lambda \subsetneq T_\Sigma$. We put Ξ the fan associated with the smooth toric variety $T_\Sigma \setminus T_\Lambda$. We have $\#\Lambda < \#\Sigma$ and $\#\Xi < \#\Sigma$. Hence, by the assumption of induction and the morphism of long exact sequences in Remark 7.5.1 for $T_\Lambda \subset T_\Sigma$, Theorem 7.5.2 holds for T_Σ. $\square$

CHAPTER 8

The analytifications and tropicalizations of the projective line

In this chapter, we give explicit descriptions of Berkovich's and Huber's analytifications of the projective line $\mathbb{P}^1$ over complete valuation fields of height 1 (Subsection 8.0.1) and of their tropicalizations (Subsection 8.0.2). (See Chapter 2 and 3 for basics on analytic spaces and thier tropicalizations.) In Subsection 8.0.3, we compare tropicalizations $\mathbb{A}^1$ and those of fibers of $(X \times \mathbb{A}^1)^{\mathrm{Ber}} \to X^{\mathrm{Ber}}$ and $(X \times \mathbb{A}^1)^{\mathrm{ad}} \to X^{\mathrm{ad}}$ for algebraic variety X over trivially valued fields.

Let $(L, v_L \colon L^\times \to \mathbb{R})$ be a valuation field of height 1. We denote its completion by $(\hat{L}, v_L \colon \hat{L}^\times \to \mathbb{R})$. We fix an extension of v_L to L^{alg}. We also denote it by v_L. We denote the residue field of (L, v_L) by $\kappa(L)$. Let T be the affine coordinate of $\mathbb{P}^1$.

8.0.1. The analytifications of the projective line. We describe the structure of the two analytifications of the projective line. See [**Ber90**, 1.4.4] and [**Sch11**, Example 2.20].

As a set, Huber's analytification $\mathbb{P}^{1,\mathrm{ad}}_{\hat{L}}$ is the set of equivalence classes of continuous valuations. There is a canonical inclusion $\mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}} \subset \mathbb{P}^{1,\mathrm{ad}}_{\hat{L}}$ as sets. Huber's adic space $\mathbb{P}^{1,\mathrm{ad}}_{\hat{L}}$ consists of points of 5 types. The Berkovich analytic space $\mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}}$ is identified with the set of points of type $1-4$ by the inclusion $\mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}} \subset \mathbb{P}^{1,\mathrm{ad}}_{\hat{L}}$. See a picture in [**Sch11**, Example 2.20].

Definition 8.0.1.
- *$x \in \mathbb{P}^{1,\mathrm{ad}}_{\hat{L}}$ is said to be of* type 1 *when there exists $a \in \mathbb{P}^1_{\hat{L}}(\hat{L^{\mathrm{alg}}})$ such thaassociated with t x is the composition*

$$\mathcal{O}_{\mathbb{P}^1_{\hat{L}}} \xrightarrow{a} \hat{L^{\mathrm{alg}}} \xrightarrow{v_L} v_L(\hat{L^{\mathrm{alg}}}).$$

  We identify type 1 points and closed points of $\mathbb{P}^1_{\hat{L}}$.
- *$x \in \mathbb{P}^{1,\mathrm{ad}}_{\hat{L}}$ is said to be of* type 2 *(resp. of* type 3*) when there exists $a \in \mathbb{P}^1_{\hat{L}}(\hat{L^{\mathrm{alg}}}) \setminus \{\infty\}$ and $r \in v_L((\hat{L^{\mathrm{alg}}})^\times)$ (resp. $r \in \mathbb{R} \setminus v_L((\hat{L^{\mathrm{alg}}})^\times)$) such that x is defined by the restriction of*

$$\hat{L^{\mathrm{alg}}}[T] \ni \sum_i a_i (T-a)^i \mapsto \min_i \{v(a_i) + ir\}$$

  In this case, we say that x corresponds *to (a, r). Note that r is unique, but a is not unique. In fact, for $a' \in \mathbb{P}^1_{\hat{L}}(\hat{L^{\mathrm{alg}}})$ such that $v(a-a') > r$, the valuation x also corresponds to (a', r). In particular, we can take $a \in L^{\mathrm{alg}}$.*
- *$x \in \mathbb{P}^{1,\mathrm{ad}}_{\hat{L}}$ is said to be of* type 5 *when there exists $a \in \mathbb{P}^1_{\hat{L^{\mathrm{alg}}}}(\hat{L^{\mathrm{alg}}}) \setminus \{\infty\}$ and $r \in v_L((\hat{L^{\mathrm{alg}}})^\times)$ and $\epsilon \in \{+, -\}$ such that x is defined by the restriction of*

$$\hat{L^{\mathrm{alg}}}[T]^\times \ni \sum_i a_i (T-a)^i \mapsto \min_i \{(v(a_i) + ir, \epsilon i) \in \mathbb{R} \times \mathbb{Z}\},$$

where we consider $\mathbb{R}\times\mathbb{Z}$ *as an ordered group with the rexicographic order. In this case, we say that* x corresponds *to* (a,r,ϵ)*. It is a specialization of the type* 2 *point corrsponding to* (a,r)*. Type* 5 *points are height* 2 *valuations.*

A point $x\in\mathbb{P}^{1,\mathrm{ad}}_{\hat{L}}$ is said to be of type 4 when it is not type $1-3$, nor 5. Type 4 points are not important in this paper. We do not explain them anymore.

Remind that for each $x\in\mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}}$, we put $\kappa(x)$ the residue field of the valuation x. Note that it is different from the residue field of the stalk of the structure sheaf. We denote $[\kappa(x)\cap\kappa(L)^{\mathrm{alg}}:\kappa(L)]$ by $\mathrm{res.alg.deg}(x)$, which is always finite. This notation is not standard.

For $x,y\in\mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}}$, we put $[x,y]\subset\mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}}$ the minimal connected subset containing both x and y, which is homeomorphic to $[0,1]\subset\mathbb{R}$. We also put $[x,y):=(y,x]:=[x,y]\setminus\{y\}$ and $(x,y):=[x,y]\setminus\{x,y\}$.

Remark 8.0.2. *For a point* $x\in\mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}}\setminus\{\infty\}$ *of type* $1,2,$ *or* 3 *corresponding to* $a\in\mathbb{P}^1_{\hat{L}}(\hat{L^{\mathrm{alg}}})$ *or* $(a,r)\in(\mathbb{P}^1_{\hat{L}}(\hat{L^{\mathrm{alg}}})\setminus\{\infty\},\mathbb{R})$*, then* $(x,\infty)\subset\mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}}$ *is the set of valuations defined by*

$$\hat{L^{\mathrm{alg}}}[T]\ni\sum_i a_i(T-a)^i\mapsto\min_i\{v(a_i)+ir'\}\ (r'\in\mathbb{R}_{<r}).$$

(Here, we put $r:=\infty$ *when* x *is of type* 1*.) The map*

$$(x,\infty)\ni(\text{the valuation corresponding to }(a,r'))\mapsto r'\in\mathbb{R}_{<r}$$

is a homeomorphism.

Remark 8.0.3. *For a type* 1 *point* $a\in\mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}}\setminus\{\infty\}$ *and a point* $x\in\mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}}\setminus\{\infty\}$*, we have*

$$[x,\infty]\cap[a,\infty]=[(\text{the valuation corresponding to }(a,\min_{\tilde{x}}\tilde{x}(T-\tilde{a}))),\infty],$$

where $\tilde{x}$ *runs through all extensions of* x *to* $\mathbb{P}^{1,\mathrm{Ber}}_{\hat{L^{\mathrm{alg}}}}$*, and the closed point* $\tilde{a}\in\mathbb{P}^{1,cl}_{\hat{L^{\mathrm{alg}}}}$ *is a fixed extension of* a*.*

Remark 8.0.4. *For a type* 2 *point* $w\in\mathbb{P}^{1,\mathrm{ad}}_{\hat{L}}$*, there is a bijection*

$$\{\textit{non-trivial specializations of } w \textit{ in } \mathbb{P}^{1,\mathrm{ad}}_{\hat{L}}\}\cong\{\textit{connected components of }\mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}}\setminus\{w\}\}$$

defined by, for each (a,r) *to which* w *corresponds,*

$$\text{the valuation corresponding to }(a,r,+)\mapsto\text{the connected component containing }a$$

$$\text{the valuation corresponding to }(a,r,-)\mapsto\text{the connected component containing }\infty.$$

We put $u_{w,\infty}$ *(resp.* $u_{w,0}$*) the specialization of* w *correspoding to the connected component containing* ∞ *(resp.* 0*).*

Remark 8.0.5. *When* $w\notin[0,\infty]$*, we have* $u_{w,0}=u_{w,\infty}$*.*

Note that for a specialization u of $w\in\mathbb{P}^{1,\mathrm{ad}}_{\hat{L}}$, we have $\kappa(w)\cap\kappa(L)^{\mathrm{alg}}\subset\kappa(u)$.

Lemma 8.0.6. *Let* $w\in\mathbb{P}^{1,\mathrm{ad}}_{\hat{L}}$ *be a type* 2 *point, and* $u\in\mathbb{P}^{1,\mathrm{ad}}_{\hat{L}}$ *a non-trivial specialization of it.*

(1) [**APZ88**, Corollary 2.1] *The residue field $\kappa(w)$ as a valuation is isomorphic to the one variable rational function field over $\kappa(w) \cap \kappa(L)^{\mathrm{alg}}$. In particular, we have bijections*

$$\begin{aligned} &\{\textit{specializations of } w\} \\ \cong &\{\textit{equivalence classes of valuations of } \kappa(w) \textit{ which are trivial on } \kappa(L)\} \\ \cong &\mathbb{P}^1_{\kappa(w)\cap\kappa(L)^{\mathrm{alg}}}. \end{aligned}$$

(2) *We have $\kappa(w) \cap \kappa(L)^{\mathrm{alg}} = \kappa(u_{w,\infty}) = \kappa(u_{w,0})$.*
(3) *When $u \neq u_{w,\infty}$, for a type $1, 2,$ or 3 point $x \in \mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}}$ which is contained in the connected component of $\mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}} \setminus \{w\}$ corresponding to u, we have a canoncal inclusion $\kappa(u) \subset \kappa(x)$.*
(4) *For type $1, 2,$ or 3 points $x_1, x_2 \in \mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}}$ such that $x_2 \in (x_1, \infty)$, we have a canonical inclusion*

$$\kappa(x_2) \cap \kappa(L)^{\mathrm{alg}} \subset \kappa(x_1) \cap \kappa(L)^{\mathrm{alg}}.$$

(5) *There exists $w' \in [\infty, w)$ such that $\kappa(w') \cap \kappa(L)^{\mathrm{alg}} = \kappa(w) \cap \kappa(L)^{\mathrm{alg}}$.*

Proof. (2) follows from [**APZ88**, Theorem 2.1 and Corollary 2.1].

(3) and (4) follow from a discussion in the proof of [**APZ88**, Theorem 2.1 (d)] when x and x_1 are of type 1. They follow from the type 1 cases, [**APZ88**, Corollary 2.1], and [**APZ90**, Theorem 5.3], when x and x_1 are of type 2 or 3.

(5) easily follows from [**PP91**, Corollary 1.6] and the definitions of type 2 and 5 points. □

8.0.2. Tropicalizations and skeletons of the projective line. We shall give explicit descriptions of tropicalizations and tropical skeletons of the projective line.

We fix notations. Let M be a free $\mathbb{Z}$-module of finite rank. We put $N := \mathrm{Hom}(M, \mathbb{Z})$. Let $\varphi\colon \mathbb{P}^1_L \to T_{\Sigma,L}$ be a closed immersion to the toric variety T_Σ over L associated with a fan Σ in $N_\mathbb{R}$ such that $\varphi(\mathbb{P}^1_L)$ intersects with the dense open orbit $\mathbb{G}_m^n = \mathrm{Spec}\, L[M] \subset T_{\Sigma,L}$. Let $h_1, \dots, h_n \in L(T)$ be the image of a fixed generator $m_1, \dots, m_n$ of M under $\varphi\colon K[M] \to L(T)$. Let $a_{i,j} \in \mathbb{P}^{1,\mathrm{cl}}_{\hat{L}}$ be the zeros and poles of h_i in the base change $\mathbb{P}^1_{\hat{L}}$.

In this subsection, we assume the following.

Assumption 8.0.7. *There exists $r \leq n$ such that $f_i := h_i$ $(1 \leq i \leq r)$ is a polynomial and h_l $(l \geq r+1)$ is a quotient of products of copies of f_i $(1 \leq i \leq r)$.*

Remark 8.0.8. *For a closed point $a \in \mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}} \setminus \{\infty\}$, there exists a strong deformarion retraction of $\mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}}$ onto $[a, \infty]$. Let $g \in \hat{L}[T]$ be an irreducible polynomial whose zero is a. We consider it as a morphism $g\colon \mathbb{P}^1_{\hat{L}} \to \mathbb{P}^1_{\hat{L}}$. Then $\mathrm{Sk}_g\, \mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}} = [a, \infty]$. We have a commutative diagram*

$$\begin{tikzcd} \mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}} \arrow[d] \arrow[dr, "\mathrm{Trop}\circ g"] & \\ {[a,\infty]} \arrow[r, "\cong", "\mathrm{Trop}\circ g"'] & {[-\infty,\infty].} \end{tikzcd}$$

This commutative diagram gives a description of the reduction of "g". Let $w \in \mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}}$ be a type 2 point, $c \in \hat{L}$ and $d \in \mathbb{Z}_{\geq 1}$ such that $w(c \cdot g^d) = 0$. We put $u \in \mathbb{P}^{1,\mathrm{ad}}_{\hat{L}}$ the specialization of w corresponding to the connected component of $\mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}} \setminus \{w\}$ containing a. When $w \notin [a,\infty]$, we have $\overline{c \cdot g^d} \in \kappa(w) \cap \kappa(L)^{\mathrm{alg}}$. When $w \in [a,\infty]$, the zero (resp. the pole) of the reduction $\overline{c \cdot g^d} \in \kappa(w)$ in $\mathbb{P}^1_{\kappa(w)\cap\kappa(L)^{\mathrm{alg}}}$ is $\overline{u}$ (resp. $\overline{u_{w,\infty}}$), where $\overline{u} \in \mathbb{P}^{1,\mathrm{cl}}_{\kappa(w)\cap\kappa(L)^{\mathrm{alg}}}$ (resp. $\overline{u_{w,\infty}} \in \mathbb{P}^{1,\mathrm{cl}}_{\kappa(w)\cap\kappa(L)^{\mathrm{alg}}}$) is a closed point correspondng to u (resp. $u_{w,\infty}$) by the bijection in Lemma 8.0.6 (1), in other words, the element $\overline{c \cdot g^d} \in \kappa(w)$ is a product of a constant and copies of the monic irreducible polynomial corresponding to $\overline{u} \in \mathbb{P}^{1,\mathrm{cl}}_{\kappa(w)\cap\kappa(L)^{\mathrm{alg}}}$.

REMARK 8.0.9. *By definition, for $f = f_1 \cdot f_2 \in L(T)$ and $v \in \mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}}$, we have*

$$\mathrm{Trop}(f(v)) = \mathrm{Trop}(f_1(v)) + \mathrm{Trop}(f_2(v)).$$

LEMMA 8.0.10. *We assume Assumption 8.0.7. Then we have*

$$\mathrm{Sk}_{\varphi_{\hat{L}}}(\mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}}) = \bigcup_{i,j}[a_{i,j},\infty].$$

and there is a strong deformation retraction of $\mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}}$ to $\mathrm{Sk}_{\varphi_{\hat{L}}}(\mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}})$ such that

$$\begin{tikzcd}
\mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}} \arrow[d] \arrow[dr, "\mathrm{Trop}\circ\varphi_{\hat{L}}"] & \\
\mathrm{Sk}_{\varphi_{\hat{L}}}(\mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}}) \arrow[r, "\cong", "\mathrm{Trop}\circ\varphi_{\hat{L}}"'] & \mathrm{Trop}\circ\varphi_{\hat{L}}(\mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}})
\end{tikzcd}$$

is commutative.

PROOF. What we need to compute is the fiber $(\mathrm{Trop}\circ\varphi_{\hat{L}})^{-1}(a)$ for $a \in \mathrm{Trop}(\varphi_{\hat{L}}(\mathbb{P}^1_{\hat{L}})) \cap N_{\mathbb{R}}$. By Assumption 8.0.7, Remark 8.0.8, and Remark 8.0.9, we have

$$(\mathrm{Trop}\circ\varphi_{\hat{L}})^{-1}(a) = \bigsqcup_w D_w (:= \{w\} \cup \bigcup_C C),$$

where $w \in \mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}}$ runs through all elements in the finite set

$$(\mathrm{Trop}\circ\varphi_{\hat{L}})^{-1}(a) \cap \bigcup_{i,j}[a_{i,j},\infty]$$

and C runs through all connected component of $\mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}} \setminus \{w\}$ not intersecting with $\bigcup_{i,j}[a_{i,j},\infty]$. By [**GRW17**, Lemma 4.4], we may assume that $\hat{L}$ is algebraically closed and $\Gamma_{v_L} = \mathbb{R}$. Then the affinoid subdomain D_w is isomorphic to $\mathscr{M}(\hat{L}\langle T,(T-b_l)^{-1}\rangle_l)$ for some $b_l \in \hat{L}^\circ$. (Here, the ring $\hat{L}^\circ$ is the valuation ring of $\hat{L}$. See [**BGR84**, Section 6.1] for definition of the affinoid algebra $\hat{L}\langle T,(T-b_l)^{-1}\rangle_l$.) It has the formal model $\mathrm{Spf}\,\hat{L}\langle T,(T-b_l)^{-1}\rangle_l^\circ$. Here, the topological ring $\hat{L}\langle T,(T-b_l)^{-1}\rangle_l^\circ$ is the subring of $\hat{L}\langle T,(T-b_l)^{-1}\rangle_l$ consisting of bounded elements. In particular, the set $\{w\}$ is the Shilov boundary of D_w. Hence

$$B((\mathrm{Trop}\circ\varphi_{\hat{L}})^{-1}(a)) = \bigsqcup_w B(D_w) = \bigcup_w \{w\}.$$

Hence the first assertion of Lemma 8.0.10 holds. The second assertion also follows from the above discussion and the fact that there is a strong deformation retraction of $\mathscr{M}(\hat{L}\langle T, (T-b_l)^{-1}\rangle_l)$ to its Shilov boundary. $\square$

COROLLARY 8.0.11. *We assume Assumption 8.0.7 and $h_1 = T$. Let Ξ be a $\sqrt{v(L^\times)}$-rational polyhedral structure of $\mathrm{Trop}(\varphi_{\hat{L}}(\mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}}))$. Let $w \in \mathrm{Sk}_{\varphi_{\hat{L}}}(\mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}})$ be a type 2 point such that $\mathrm{Trop}(\varphi(w))$ is contained in Ξ and $u \in \mathbb{P}^{1,\mathrm{ad}}_{\hat{L}}$ a non-trivial specialization of w. We assume*

$$\mathrm{Trop}^{\mathrm{ad}}_{\Xi}(u) = \mathrm{Trop}^{\mathrm{ad}}_{\Xi}(u_{w,0}).$$

Then $u = u_0$.

COROLLARY 8.0.12. *We assume Assumption 8.0.7. Let Ξ be a $\sqrt{v(L^\times)}$-rational polyhedral structure of $\mathrm{Trop}(\varphi_{\hat{L}}(\mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}}))$. Let $l, e \in \Xi$ be such that $l \subsetneq e$. Let $w \in \mathrm{Sk}_{\varphi_{\hat{L}}}(\mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}})$ be a type 2 point such that $\mathrm{Trop}(\varphi(w)) = l$ and $u \in \mathbb{P}^{1,\mathrm{ad}}_{\hat{L}}$ a non-trivial specialization of w. We put C_u the connected component of $\mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}} \setminus \{w\}$ corresponding to u.*

Then $\mathrm{Trop}^{\mathrm{ad}}_{\Xi}(u) = e$ if and only if for any subset $D \subset C_u$ such that $w \in \overline{D}$, we have

$$\mathrm{Trop}(D) \cap \mathrm{rel.int}(e) \neq \emptyset.$$

LEMMA 8.0.13. *We assume Assumption 8.0.7. There exists a morphism $\psi \colon \mathbb{P}^1_L \to (\mathbb{P}^1_L)^l$ over L such that $\mathrm{Trop} \circ (\varphi_{\hat{L}} \times \psi_{\hat{L}})$ is injective on $\mathrm{Sk}_{(\varphi_{\hat{L}} \times \psi_{\hat{L}})}(\mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}})$.*

PROOF. We take monic irreducible polynomial $g_{i,j} \in L[T]$ which is irreducible over $\hat{L}$ and has zero $b_{i,j}$ sufficiently near to $a_{i,j}$. We put $\psi := (g_{i,j})_{i,j}$. By Remark 8.0.8 and 8.0.9, for any i_1, i_2, j_1, j_2, the map $\mathrm{Trop} \circ (\varphi_{\hat{L}} \times \psi_{\hat{L}})$ is injective on

$$\bigcup_{k=1,2} [a_{i_k,j_k}, \infty] \cup [b_{i_k,j_k}, \infty].$$

Hence, by Lemma 8.0.10 for $\varphi_{\hat{L}} \times \psi_{\hat{L}}$, Lemma 8.0.13 holds. $\square$

LEMMA 8.0.14. *Let $w \in \mathbb{P}^{1,\mathrm{ad}}$ be a type 2 point and*

$$V \subset (\kappa(w) \cap \kappa(L)^{\mathrm{alg}})^\times_{\mathbb{Q}} / \kappa(L)^\times_{\mathbb{Q}}$$

a finite dimensional $\mathbb{Q}$-vector subspace. Then there exist monic polynomials $g_i \in L[T]$ irreducible over $\hat{L}$, $a_i \in L$, and $r_i \in \mathbb{Z}$ such that

- *$w(a_i \cdot g_i^{r_i}) = 0$,*
- *the $\mathbb{Q}$-vector subspace V is generated by the reductions $\overline{a_i \cdot g_i^{r_i}}$,*
- *$[\infty, b_i] \cap [\infty, w] = [\infty, w_i]$ for some type 2 point w_i such that*

  $$\mathrm{res.alg.deg}(w_i) < \mathrm{res.alg.deg}(w),$$

  where $b_i \in \mathbb{P}^{1,\mathrm{Ber}}$ is the zero of g_i, and
- *for any $i \neq j$, we have $[\infty, b_i] \cap [\infty, b_j] \subset [\infty, w]$.*

Moreover, when $[\infty, w] \subset \mathrm{Sk}_{\varphi'_{\hat{L}}}(\mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}})$, we can take g_i, a_i, r_i satisfying

$$[\infty, b_i] \cap \mathrm{Sk}_{\varphi'_{\hat{L}}}(\mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}}) = [\infty, w'_i]$$

for some type 2 point w'_i such that

$$\mathrm{res.alg.deg}(w'_i) < \mathrm{res.alg.deg}(w).$$

Proof. We prove the assertion by induction on $\mathrm{res}.\mathrm{alg}.\deg(w)$. When $\mathrm{res}.\mathrm{alg}.\deg(w) = 1$, the assertion is trivial. We assume $\mathrm{res}.\mathrm{alg}.\deg(w) > 1$.

Note that the function $\mathrm{res}.\mathrm{alg}.\deg$ satisfies

$$\mathrm{res}.\mathrm{alg}.\deg(w_1) \leq \mathrm{res}.\mathrm{alg}.\deg(w_2)$$

for $w_1, w_2 \in \mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}}$ such that $w_1 \in [\infty, w_2]$. By Lemma 8.0.6 (5), there exists an element $w_0 \in [\infty, w]$ such that

$$\mathrm{res}.\mathrm{alg}.\deg(w_0) < \mathrm{res}.\mathrm{alg}.\deg(w)$$

and $w' \in [\infty, w_0]$ for any $w' \in [\infty, w]$ satisfying

$$\mathrm{res}.\mathrm{alg}.\deg(w') < \mathrm{res}.\mathrm{alg}.\deg(w).$$

By the assumption of the induction for w_0, it suffices to show the assertion of Lemma 8.0.14 for a finite dimensional subspace

$$W \subset (\kappa(w) \cap \kappa(L)^{\mathrm{alg}})^{\times}_{\mathbb{Q}} / (\kappa(w_0) \cap \kappa(L)^{\mathrm{alg}})^{\times}_{\mathbb{Q}}$$

instead of V and w_0 instead of w_i for any i.

We shall construct g_i. We put $u_{w_0,w} \in \mathbb{P}^{1,\mathrm{ad}}_{\hat{L}}$ the non-trivial specialization of w_0 corresponding to the connected component of $\mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}} \setminus \{w_0\}$ containing w. Then, by definition of w_0, we have $\kappa(u_{w_0,w}) = \kappa(w) \cap \kappa(L^{\mathrm{alg}})$. By Lemma 8.0.6 (1), the residue field $\kappa(w_0)$ is isomorphic to the one variable rational function field $l(t)$ over a finite extension l of $\kappa(L)$. We consider $u_{w_0,w}$ as a valuation of $l(t)$, which is not t^{-1}-adic valuation. We take finitely many monic irreducible polynomials $G_i \in l[t]$ such that $u_{w_0,w}(G_i) = 0$ and their reductions generate W, here we consider W as a subspace of a quotient of $\kappa(u_{w_0,w})^{\times}_{\mathbb{Q}}$. We put $u_i \in \mathbb{P}^{1,\mathrm{ad}}_{\hat{L}}$ the specialization of w_0 which corresponds to the monic irreducible polynomial $G_i \in l[t]$. (See Lemma 8.0.6 (1).) We take closed point $b_i \in \mathbb{P}^1_{\hat{L}}$ contained in the connected component of $\mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}} \setminus \{w_0\}$ corresponding to the specialization u_i of w_0. We put $g_i \in \hat{L}[T]$ the monic irreducible polynomial whose zero is b_i. We can take b_i so that $g_i \in L[T]$. By construction, these g_i, suitable $a_i \in L$, and $r_i \in \mathbb{Z}$ satisfy the assertion for W instead of V and w_0 instead of w_i for any i.

When $[\infty, w] \subset \mathrm{Sk}_{\varphi'_{\hat{L}}}(\mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}})$, we can take G_i such that

$$\deg(G_i) < \deg(\kappa(u_{w_0,w})/\kappa(w_0) \cap \kappa(L^{\mathrm{alg}})).$$

We can take b_i such that $[\infty, b_i] \cap \mathrm{Sk}_{\varphi'_{\hat{L}}}(\mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}}) \setminus [\infty, w_0]$ is small enough. Then w'_i defined by $[\infty, b_i] \cap \mathrm{Sk}_{\varphi'_{\hat{L}}}(\mathbb{P}^{1,\mathrm{Ber}}_{\hat{L}}) = [\infty, w'_i]$ satisfies

$$\mathrm{res}.\mathrm{alg}.\deg(w'_i) = \mathrm{res}.\mathrm{alg}.\deg(w_0) \times \deg(G_i) < \mathrm{res}.\mathrm{alg}.\deg(\kappa(u_{w_0,w})) = \mathrm{res}.\mathrm{alg}.\deg(\kappa(w)).$$

□

8.0.3. Tropicalizations of $\mathbb{A}^1$-fibers. In this subsection, we describe tropicalizations of $\mathbb{A}^1$-fibers of non-archimedean analytic spaces over a trivially valued field K. We also describe their certain compact subsets. We use them in Section 9.

Let X be an algebraic variety over K. We put $\pi \colon X \times \mathbb{A}^1 \to X$ the first projection. Remind that we have

$$X^{\mathrm{Ber}}/(\text{the equivalence relations of valuations}) \cong X^{\mathrm{ad,ht}\leq 1} \subset X^{\mathrm{ad}}.$$

Let $v \in X^{\mathrm{Ber}}$. We put $[v] \in X^{\mathrm{ad}}$ its equivalence class.

Remark 8.0.15. *There is a natural inclusion*

$$\mathbb{A}^{1,\mathrm{ad}}_{k(\mathrm{supp}(v))_v} \hookrightarrow \pi^{-1}([v])$$

whose image is the subset consisting of specializations of height 1 *valuations* $w \in \pi^{-1}([v])$. *We have the following commutative diagram:*

$$\begin{tikzcd}
\pi^{-1}(v) \arrow[r, "\simeq"] \arrow[d, hook] & \mathbb{A}^{1,\mathrm{Ber}}_{k(\mathrm{supp}(v))_v} \arrow[d, hook] \\
\pi^{-1}([v]) & \mathbb{A}^{1,\mathrm{ad}}_{k(\mathrm{supp}(v))_v}, \arrow[l, hook']
\end{tikzcd}$$

see [**Ber90**, Section 3.1] *and* [**Tem15**, Definition/Exercise 4.1.7.1]. *We identify elements of the above sets under the above morphisms.*

We put

$$\iota\colon \mathbb{A}^1_{k(\mathrm{supp}(v))} \cong \pi^{-1}(\mathrm{supp}(v)) \to X \times \mathbb{A}$$

the natural inclusion. Let $\varphi'\colon X \times \mathbb{A}^1 \to T_\Sigma$ be a closed immersion. We consider a composition

$$\varphi_{k(\mathrm{supp}(v))_v} \circ \iota_{k(\mathrm{supp}(v))_v}\colon \mathbb{A}^1_{k(\mathrm{supp}(v))_v} \to (X \times \mathbb{A}^1) \times_K k(\mathrm{supp}(v))_v \to T_\Sigma \times_K k(\mathrm{supp}(v))_v.$$

Here, the morphsim $\varphi'_{k(\mathrm{supp}(v))_v}$ and $\iota_{k(\mathrm{supp}(v))_v}$ are induced from φ' and ι, respectively. For simplicity, we denote $\varphi'_{k(\mathrm{supp}(v))_v} \circ \iota_{k(\mathrm{supp}(v))_v}$ by φ'.

Remark 8.0.16. *Let* Λ *be a fan structure of* $\mathrm{Trop}(\varphi'((X \times \mathbb{A}^1)^{\mathrm{Ber}}))$ *and* Ξ *a* $\sqrt{v(k(\mathrm{supp}(v))^\times)}$*-rational polyhedral complex structure of* $\mathrm{Trop}(\varphi'(\pi^{-1}(v)))$ *such that each element* $\zeta \in \Xi$ *is contained in some* $P \in \Lambda$. *Then the following diagram is commutative:*

$$\begin{tikzcd}
 & & \pi^{-1}(v) \arrow[rr, "\cong"] \arrow[dll, hook] \arrow[dd] & & \mathbb{A}^{1,\mathrm{Ber}}_{k(\mathrm{supp}(v))_v} \arrow[dl, hook] \arrow[dddd, "\mathrm{Trop}\circ\varphi'"] \\
\pi^{-1}([v]) \arrow[dd] & & & \mathbb{A}^{1,\mathrm{ad}}_{k(\mathrm{supp}(v))_v} \arrow[lll, hook'] \arrow[dddd, "\mathrm{Trop}^{\mathrm{ad}}_{\Xi}\circ\varphi"'] & \\
 & & (X \times \mathbb{A}^1)^{\mathrm{Ber}} \arrow[dd, "\mathrm{Trop}\circ\varphi'"'] & & \\
(X \times \mathbb{A}^1)^{\mathrm{ad}} \arrow[dd, "\mathrm{Trop}^{\mathrm{ad}}_{\Lambda}\circ\varphi'"'] & & & & \\
 & & \mathrm{Trop}(\varphi'((X \times \mathbb{A}^1)^{\mathrm{Ber}})) \arrow[dll] & & \mathrm{Trop}(\varphi'(\pi^{-1}(v))) \arrow[ll, hook'] \arrow[dl] \\
\Lambda & & & \Xi \arrow[lll] & .
\end{tikzcd}$$

Next we describe a subset $(\pi^\circ)^{-1}(v)$ of the $\mathbb{A}^1$-fiber $\pi^{-1}(v)$. The projection $\pi\colon X \times \mathbb{A}^1 \to X$ induces a surjective map $\pi^\circ\colon (X \times \mathbb{A}^1)^\circ \to X^\circ$. The canonical homeomorphism $\pi^{-1}(v) \cong \mathbb{A}^{1,\mathrm{Ber}}_{k(\mathrm{supp}(v))_v}$ induces a homeomorphism

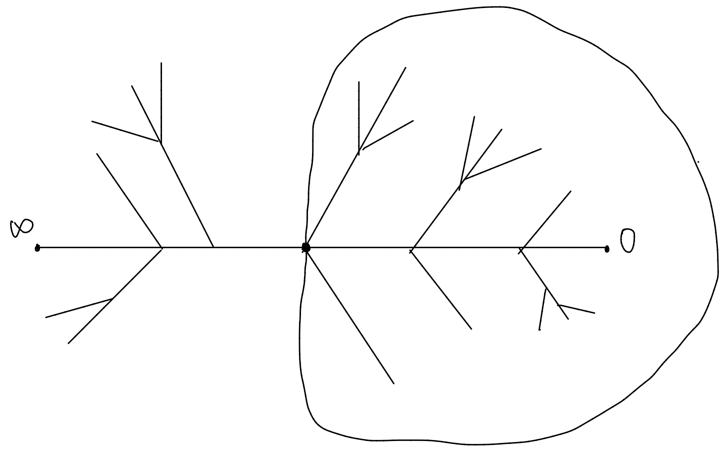
$$(\pi^\circ)^{-1}(v) \cong \mathscr{M}(k(\mathrm{supp}(v))_v\langle T\rangle).$$

(See [**BGR84**, Section 5.1] for Tate algebra $k(\mathrm{supp}(v)_v)\langle T\rangle$.) The affinoid subdomain $\mathscr{M}(k(\mathrm{supp}(v))_v\langle T\rangle)$ is a connected subset of $\mathbb{A}^{1,\mathrm{Ber}}_{k(\mathrm{supp}(v))_v}$. For any $a \in \mathscr{M}(k(\mathrm{supp}(v))_v\langle T\rangle)$, we have

$$[0,a] \subset \mathscr{M}(k(\mathrm{supp}(v))_v\langle T\rangle).$$

We put

$$\mathrm{Sk}_{\varphi'}((\pi^\circ)^{-1}(v) := \mathrm{Sk}_{\varphi'}(\pi^{-1}(v)) \cap (\pi^\circ)^{-1}(v).$$



A picture of $\mathscr{M}(k(\mathrm{supp}(v))_v\langle T\rangle) \subset \mathbb{P}^{1,\mathrm{Ber}}_{k(\mathrm{supp}(v))_v}$.

CHAPTER 9

$\mathbb{A}^1$-homotopy invariance

In this chapter, we shall prove Proposition 7.2.1(2), i.e., $\mathbb{A}^1$-homotopy invariance of tropical cohomology. By Leray spectral sequence, we reduce it to computations of sheaves (Proposition 9.1.1). By retractions (given in Subsection 9.3.1) and descriptions of F^p by tropical K-groups (Section 9.4), we reduce them to computations of tropicalizations of $\mathbb{A}^1$-fibers (Subsection 9.3.3) and those of tropical K-groups of residue fields of valuations in $\mathbb{A}^{1,\mathrm{ad}}$. Subsection 9.5.2 and 9.5.3 are the main parts of the proof. We use induction and the following facts:

- the residue fields of type 2 points of $\mathbb{A}^1$-fibers of height 1 valuations are isomorphic to the fields $k(s)$ of rational functions in one variable over fields k,
- there is a complex
$$\wedge^p k_{\mathbb{Q}}^{\times} \to \wedge^p k(s)_{\mathbb{Q}}^{\times} \xrightarrow{d} \bigoplus_{x \in \mathbb{A}_k^{1,\mathrm{cl}}} \wedge^{p-1} k(x)_{\mathbb{Q}}^{\times}$$
whose the last arrow is surjective (see Remark 6.2.10), and
- since tropical K-groups form a cycle module ([**M20-1**, Theorem 1.1]), we have an exact sequence
$$0 \to K_T^p(k) \to K_T^p(k(s)) \xrightarrow{d} \bigoplus_{x \in \mathbb{A}_k^{1,\mathrm{cl}}} K_T^{p-1}(k(x)) \to 0$$
[**Ros96**, Propostion 2.2 (H)].

In Subsection 9.3.2, we fix notations. In Subsection 9.5.1, we give a picture and give several notations used in Subsection 9.5.2 and 9.5.3. In Section 9.2, we give a rough explanation of the proof.

Let X be an open subvariety of an affine space over a trivially valued field K. We put $\pi \colon X \times \mathbb{A}^1 \to X$ the first projection.

9.1. The goal of this chapter

By Lemma 4.1.8 and Corollary 7.2.11, $\mathbb{A}^1$-homotopy invariance of tropical cohomology follows from
$$H_{\mathrm{Trop},T}^{p,q}((X \times \mathbb{A}^1)^\circ) \cong H_{\mathrm{Trop},T}^{p,q}(X^\circ).$$
Since tropical cohomology groups are isomorphic to the sheaf cohomology groups of $\mathscr{F}_T^*$ (see Section 4.1.2), the Leray spectral sequence
$$E_2^{p,q} = H^q(X^\circ, R^p\pi_*\mathscr{F}_{T,(X\times\mathbb{A}^1)^\circ}^r)$$
for $\pi^\circ \colon (X \times \mathbb{A}^1)^\circ \to X^\circ$ convergence to the tropical cohomology $H_{\mathrm{Trop},T}^{r,p+q}((X \times \mathbb{A}^1)^\circ)$.

The above isomorphisms of tropical homology groups follow from the following, which is the goal of this chapter.

Proposition 9.1.1. *Let $r \in \mathbb{Z}$.*

(1) *We have*

$$R^i\pi_*\mathscr{F}^r_{T,(X\times\mathbb{A}^1)^\circ}=0$$

for $i\geq 1$.

(2) *The canonical morphism*

$$\mathscr{F}^r_{T,X^\circ}\to\pi^\circ_*\mathscr{F}^r_{T,(X\times\mathbb{A}^1)^\circ}$$

is an isomorphism.

9.2. A rough explanation of the proof

We explain the essence of the proof of $\mathbb{A}^1$-homotopy invariance.

We compute stalks of sheaves at $v_0\in X^{\mathrm{Ber}}$. Since $\pi^{-1}(v_0)\cong\mathbb{A}^{1,\mathrm{Ber}}_{k(\mathrm{supp}(v_0))_{v_0}}$ is homeomorphic to the inverse limit of certain 1-dimensional polyhedral complexes (its tropicalizations), we have $R^i\pi_*\mathscr{F}^p_{(X\times\mathbb{A}^1)^{\mathrm{Ber}}}=0$ for $i\geq 2$ (Corollary 9.3.4).

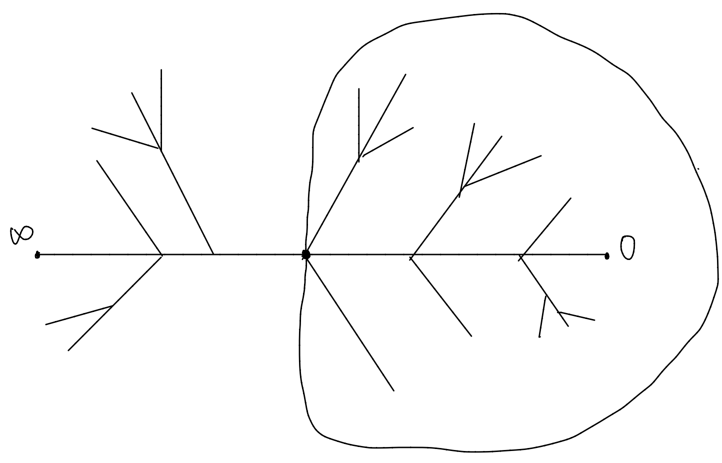
The rest of the proof is technical. We only prove Claim 9.2.8, which is essentially the same as the rest of the proof. Before we state Claim 9.2.8, we need preparations.

Let L be a complete valuation field of height 1 with valuation v_0. Let $\varphi_0'\colon\mathbb{A}^1\to T_{\Sigma_0'}$ be a closed immersion over L to a toric variety $T_{\Sigma_0'}$ associated with a fan Σ_0'. We put $T_{\Sigma'}$ the toric variety $\mathbb{A}^1\times T_{\Sigma_0'}$, which is associated with a fan Σ'. We put $\varphi':=(\mathrm{Id}^1_{\mathbb{A}},\varphi_0')\colon\mathbb{A}^1\to T_{\Sigma'}$. Let M be the lattice such that $\operatorname{Spec}K[M]$ is the dense torus orbit of $T_{\Sigma'}$. We put $N_: =\mathrm{Hom}(M,\mathbb{Z})$. We assume that $\mathrm{Trop}(\varphi'(\mathbb{A}^1))$ meets $\operatorname{Spec}K[M]$ and $\mathrm{Trop}\circ\varphi'$ is injective on the tropical skeleton $\mathrm{Sk}_{\varphi'}(\mathbb{A}^1)$. We fix a $\sqrt{v_0(L^\times)}$-rational polyhedral complex structure Ξ of

$$\mathrm{Trop}(\varphi'(\mathscr{M}(L\langle T\rangle))\cong\mathrm{Sk}_{\varphi'}(\mathbb{A}^1)\cap\mathscr{M}(L\langle T\rangle),$$

where T is the coordinate of $\mathbb{A}^1$ and $\mathscr{M}(L\langle T\rangle)$ is the affinoid subdomain of $\mathbb{A}^{1,\mathrm{Ber}}$ of radius 1 with center 0. We assume that Ξ is a subcomplex of a $\sqrt{v_0(L^\times)}$-rational polyhedral complex structure $\overline{\Xi}$ of $\mathrm{Trop}(\varphi'(\mathbb{A}^1))$.

REMARK 9.2.1. *By retractions, what we need to compute is certain cohomology on* $\mathrm{Trop}(\varphi'(\mathscr{M}(L\langle T\rangle)))$. *See Subsection 9.3.1.*



A picture of $\mathscr{M}(L\langle T\rangle)\subset\mathbb{P}^{1,\mathrm{Ber}}$.

NOTATION 9.2.2. *For each* 0*-dimensional polyhedron* $l \in \Xi$*, we put* $w_l \in \mathrm{Sk}_{\varphi'}(\mathbb{A}^1) \cap \mathscr{M}(L\langle T\rangle)$ *such that* $\mathrm{Trop}(\varphi'(w_l)) = l$*. The valuation* w_l *is of type* 1 *or* 2*. For* 0*-dimensional polyhedron* $l \in \Xi$ *and* 1*-dimensional polyhedron* $e \in \Xi$ *such that* $l \in e$ *and* w_l *is of type* 2*, we put* $u_{l,e} \in \mathbb{A}^{1,\mathrm{ad}}$ *be the specialization of* w_l *such that* $\mathrm{Trop}^{\mathrm{ad}}_{\Xi}(\varphi'(u_{l,e})) = e$*. For* 0*-dimensional polyhedron* $l \in \Xi$ *such that* w_l *is of type* 2*, we put* $u_{l,\infty} := u_{w_l,\infty} \in \mathbb{A}^{1,\mathrm{ad}}$*. For* $l \in e$ *with type* 2 *valuation* w_l*, we fix*

$$g_{l,e} \in M \cap \mathrm{Span}_{\mathbb{R}}(l)^{\perp} \setminus \mathrm{Span}_{\mathbb{R}}(e)^{\perp}.$$

Here, for $\xi \in \Xi$*, the set* $\mathrm{Span}_{\mathbb{R}}(\xi)$ *is the* $\mathbb{R}$*-vector subspace of* $N_{\mathbb{R}}$ *spanned by the elements in* ξ*.*

DEFINITION 9.2.3. *For an integer* $p \geq 0$ *and a* 0*-dimensional polyhedron* $l \in \Xi$ *(resp. a* 1*-dimensional polyhedron* $e \in \Xi$*), we put "*F^p*"*(l) *(resp. "*F^p*"*(e)*) the* $\mathbb{Q}$*-vector subspace of the tropical* K*-group* $K_T^p(\kappa(w_l))$ *of the residue field* $\kappa(w_l)$ *(resp.* $K_T^p(\kappa(u_{l,e}))$ *of the residue field* $\kappa(u_{l,e})$ *for some* $l \in \Xi$ *such that* $l \in e$ *and* w_l *is of type* 2*) generated by the image of the wedge product of* $M \cap \sigma_l'^{\perp} \cap w_l^{-1}(0)$ *(resp.* $M \cap u_{l,e}^{-1}(0)$*), where* $\sigma_l' \in \Sigma'$ *is the cone such that* $\mathrm{Trop}(\varphi'(w_l)) \in O(\sigma_l')^{\mathrm{Ber}}$ *(The orbit* $O(\sigma_l') \subset T_{\Sigma'}$ *corresponds to the cone* σ_l'*.)*

This "F^p"(l) (resp. "F^p"(e)) plays the role of $\overline{F}^p(P_l)$ (resp. $\overline{F}^p(P_e)$) in Section 9.

We assume the following.

ASSUMPTION 9.2.4. *The definition of "*F^p*"*(e) *does not depend on the choice of the* 0*-dimensional polyhedron* $l \in e$ *such that* w_l *is of type* 2*. For a* 1*-dimensional polyhedron* $e \in \Xi$ *and a* 0*-dimensional polyhedron* $l \in \Xi$ *such that* $l \in e$*, we have a natural isomorphism "*F^p*"*$(e) \cong$ *"*F^p*"*(l) *when* w_l *is of type* 1*, and*

$$\text{“}F^p\text{”}(e) \cong \text{“}F^p\text{”}(l) \cap K_T^p(\kappa(w_l) \cap \kappa(v_0)^{\mathrm{alg}})$$

when w_l *is of type* 2 *and* $e = \mathrm{Trop}^{\mathrm{ad}}_{\Xi}(u_{w_l,\infty})$*. (If we use* K_T^p *instead of "*F^p*", this assumption holds for sufficiently fine fan structure* Ξ*.)*

DEFINITION 9.2.5. *For* $p \geq 0$*, we put*

$$C^{p,0} := \bigoplus_{l \in \Xi \text{ polyhedra of dimension } 0} \text{“}F^p\text{”}(l)$$

$$C^{p,1} := \bigoplus_{e \in \Xi \text{ polyhedra of dimension } 1} \text{“}F^p\text{”}(e) \oplus \text{“}F^{p-1}\text{”}(e).$$

(We put "F^{-1}*"*$(e) := 0$*.)*

DEFINITION 9.2.6. *We define a map* $\delta \colon C^{p,0} \to C^{p,1}$ *as the sum of the following* $\delta_{l,e}$ *for* $l, e \in \Xi$ *such that* $l \in e$*. For* $l \in e$ *with a type* 2 *valuation* w_l*, we put*

$$\delta_{l,e} := (s_{\overline{u_{l,e}}}^{\overline{g_{l,e}}}, \partial_{\overline{u_{l,e}}}) \colon \text{“}F^p\text{”}(l) \to \text{“}F^p\text{”}(e) \oplus \text{“}F^{p-1}\text{”}(e),$$

where $\overline{u_{l,e}}$ *is the normalized discrete valuation on the residue field* $\kappa(w_l)$ *correspoding to the specialization* $u_{l,e} \in \mathbb{A}^{1,\mathrm{ad}}$ *of* w_l*,* $\partial_{\overline{u_{l,e}}}$ *is the residue homomorphism of tropical* K*-groups, see* [**M20-1**, Section 5]*), and the element* $\overline{g_{l,e}} \in \kappa(w_l)$ *is the reduction of* $g_{l,e} \in M \cap w_l^{-1}(0)$*, and* $s_{\overline{u_{l,e}}}^{\overline{g_{l,e}}}(a) := \partial_{\overline{u_{l,e}}}(\overline{g_{l,e}} \wedge a)$ *for* $a \in$ *"*F^p*"*(l)*. For* $l \in e$ *with a type* 1 *valuation* w_l*, we put*

$$\delta_{l,e} \colon \text{“}F^p\text{”}(l) \to \text{“}F^p\text{”}(e) \oplus \text{“}F^{p-1}\text{”}(e)$$

the map such that the composition of $\delta_{l,e}$ *and the first projection is the natural isomorphism in Assumption 9.2.4 and that of* $\delta_{l,e}$ *and the second projection is the* 0*-map.*

The following short exact sequence is an important tool to prove $\mathbb{A}^1$-homotopy invariance. For a type 2 point $w \in \mathbb{A}^{1,\mathrm{Ber}}$, its residue field $\kappa(w)$ is isomorphic to the field $k(s)$ of rational functions in one variable over a finite extension field k of the residue field $\kappa(v_0)$ of the base field L. Hence, for $p \geq 1$ and type 2 point $w \in \mathbb{A}^{1,\mathrm{Ber}}$, by [**?**, Theorem 1.1] and [**Ros96**, Proposition 2.2 (H)], we have a short exact sequence

$$0 \to K_T^p(\kappa(w) \cap \kappa(v_0)^{\mathrm{alg}}) \to K_T^p(\kappa(w)) \to^{(\partial_{\overline{u}})_{u \neq u_{w,\infty}}} \bigoplus_{u \in \overline{\{w\}} \setminus \{w, u_{w,\infty}\}} K_T^{p-1}(\kappa(u)) \to 0.$$

We assume the following analog.

Assumption 9.2.7. *For $p \geq 1$ and $l \in \Xi$ such that w_l is of type 2, the morphism*

$$\text{“}F^p\text{”}(\kappa(w_l)) \to^{(\partial_{\overline{u_{l,e}}})_{e \neq \mathrm{Trop}_{\Xi}^{\mathrm{ad}}(\varphi'(u_{w_l,\infty}))}} \bigoplus_{l \in e \in \Xi \setminus \{\mathrm{Trop}_{\Xi}^{\mathrm{ad}}(u_{w_l,\infty})\}} \text{“}F^{p-1}\text{”}(\kappa(u_{l,e}))$$

is surjective, where e runs through all 1-dimensional polyhedra in $\Xi \setminus \{\mathrm{Trop}_{\Xi}^{\mathrm{ad}}(u_{w_l,\infty})\}$ containing l.

Claim 9.2.8. *We have*

$$H^1 := C^{p,1}/\delta(C^{p,0}) = 0$$

amd the projection induces an injection

$$H^0 := \mathrm{Ker}(\delta \colon C^{p,0} \to C^{p,1}) \hookrightarrow \text{“}F^p\text{”}(\mathrm{Trop}(\varphi'(0))).$$

Proof of Claim. We prove Claim 9.2.8 by induction on the following n_l. For each 0-dimensional polyhedron $l \in \Xi$, we put C_l the subset of Ξ consisting of 0-dimensional $l' \in \Xi$ such that only one 1-dimensional polyhedron in Ξ contains l' and the minimal connected subset of $\mathrm{Trop}(\varphi'(\mathscr{M}(L\langle T\rangle)))$ containing both l and l' does not intersect with $\mathrm{Trop}(\varphi'([\infty, w_l]))$. We put n_l the maximum of the numbers $n_{l,l'}$ of 1-dimensional polyhedra in Ξ between l and $l' \in C_l$.

First, we show that $H^0 \to \text{“}F^p\text{”}(\mathrm{Trop}(\varphi'(0)))$ is injective. Let $\alpha = (\alpha_l)_{l \in \Xi} \in H^0$ (here $l \in \Xi$ runs through all 0-dimensional polyhedra). By induction on n_l in the ascending order, the condition $\delta\alpha = 0$, and the middle of the short exact sequence (9.2), we have $\alpha_l \in K_T^p(\kappa(w_l) \cap \kappa(v_0)^{\mathrm{alg}})$ for any 0-dimensional $l \in \Xi$. For any $l, e \in \Xi$ such that $l \subsetneq e$ and w_l is of type 2, the restriction of $s_{\overline{u_{l,e}}}^{\overline{g_{l,e}}}$ to $K_T^p(\kappa(w_l) \cap \kappa(v_0)^{\mathrm{alg}})$ coincides with the map coming from the natural injection $\kappa(w_l) \cap \kappa(v_0)^{\mathrm{alg}} \hookrightarrow \kappa(u_{l,e})$. Hence, by $\delta\alpha = 0$, the element α is uniquely determined by α_0.

Next, we show $H^1 = 0$. Let $\alpha \in H^1$. By induction on n_l in the ascending order and Assumption 9.2.7, there exists a representative $\tilde{\alpha} = (\tilde{\alpha}_e)_{e \in \Xi}$ of α (here $e \in \Xi$ runs through all 1-dimensional polyhedra) such that for any 1-dimensional $e \in \Xi$, we have

$$\tilde{\alpha}_e = (\tilde{\alpha}_{e,1}, 0) \in \text{“}F^p\text{”}(e) \oplus \text{“}F^{p-1}\text{”}(e)$$

($\tilde{\alpha}_{e,1} \in \text{“}F^p\text{”}(e)$). Then, by induction on n_l in the descending order and Assumption 9.2.4, the element 0 is a representative of $\alpha \in H^1$, i.e., $H^1 = 0$. □

Remark 9.2.9. *Claim 9.2.8 is essentially the same as the proof of $\mathbb{A}^1$-homotopy invariance, but there are small differences.*

- *We do not know whether there exist enough tropicalizations satisfying Assumption 9.2.4 and 9.2.7. Hence we exchange tropicalizations under the process of induction. See Subsection 9.5.3.*

- *We compute tropicalizations and the tropical K-groups of the residue fields of not only the inverse image $\pi^{-1}(v_0)$ of an elements v_0 but also the family of the inverse images $\pi^{-1}(v)$ of valuations v in a neighborhood of $v_0 \in X^{\mathrm{Ber}}$.*

9.3. Preliminaries

9.3.1. The existence of retractions. We shall show existance of retractions. This follows from properness of $\pi^\circ\colon (X \times \mathbb{A}^1)^\circ \to X^\circ$. We consider a commutative diagram

$$\begin{tikzcd} X \times \mathbb{A}^1 \arrow[r, "\pi"] \arrow[d, "\varphi'"'] & X \arrow[d, "\varphi"] \\ T_{\Sigma'} \arrow[r, "\Psi"] & T_{\Sigma}, \end{tikzcd}$$

φ and φ' are closed immersions to toric varieties $T_{\Sigma'}$ and T_Σ, and Ψ is a toric morphism. Let $v_0 \in X^{\mathrm{Ber}}$, and Λ and Λ' be a fan structure of $\mathrm{Trop}(\varphi(X))$ and $\mathrm{Trop}(\varphi'(X \times \mathbb{A}^1))$, respectively.

Lemma 9.3.1. *For a sufficiently small polyhedron $A \subset \mathrm{Trop}(T_\Sigma)$ which is a neighborhood of $\mathrm{Trop}(\varphi(v_0))$ and any cone $P \in \Lambda'$ intersecting with $\Psi^{-1}(A)$, the intersection*

$$P \cap \mathrm{Trop}(\varphi'((X \times \mathbb{A}^1)^\circ)) \cap \Psi^{-1}(\mathrm{Trop}(\varphi(v_0)))$$

is nonempty.

Proof. By a commutative diagram

$$\begin{tikzcd} (X \times \mathbb{A}^1)^\circ \arrow[r, "\pi^\circ"] \arrow[d, "\mathrm{Trop} \circ \varphi'"'] & X^\circ \arrow[d, "\mathrm{Trop} \circ \varphi"] \\ \mathrm{Trop}(\varphi'((X \times \mathbb{A}^1)^\circ)) \arrow[r, "\Psi|_{\mathrm{Trop}(\varphi'((X \times \mathbb{A}^1)^\circ))}"] & \mathrm{Trop}(\varphi(X^\circ)), \end{tikzcd}$$

the restriction

$$\Psi|_{\mathrm{Trop}(\varphi'((X \times \mathbb{A}^1)^\circ))}\colon \mathrm{Trop}(\varphi'((X \times \mathbb{A}^1)^\circ)) \to \mathrm{Trop}(\varphi(X^\circ))$$

of Ψ is proper. (Note that the π° and $\mathrm{Trop} \circ \varphi$ are proper, and $\mathrm{Trop} \circ \varphi'$ is continuous.) Hence $\mathrm{Trop}(\varphi'((X \times \mathbb{A}^1)^\circ)) \cap \Psi^{-1}(A)$ is compact. Hence the Lemma holds. $\square$

Lemma 9.3.2. *For a polyhedron A as in Lemma 9.3.1, the pull-back map*

$$H^{p,q}(\mathrm{Trop}(\varphi'((X \times \mathbb{A}^1)^\circ)) \cap \Psi^{-1}(\mathrm{rel.int}(A)), \Lambda') \to H^{p,q}(\mathrm{Trop}(\varphi'((\pi^\circ)^{-1}(v_0))), \Lambda')$$

is an isomorphism.

Proof. In a similar way to the proof of Lemma 7.2.6, it suffices to show that there exists a strong deformation retraction ψ of

$$\mathrm{Trop}(\varphi'((X \times \mathbb{A}^1)^\circ)) \cap \Psi^{-1}(\mathrm{rel.int}(A))$$

onto

$$\mathrm{Trop}(\varphi'((X \times \mathbb{A}^1)^\circ)) \cap \Psi^{-1}(\mathrm{Trop}(\varphi(v_0)))$$

which preserving the polyhedral structure, i.e., for any cone $P \in \Lambda'$ intersecting with $\Psi^{-1}(\mathrm{rel.int}(A))$, we have

$$\psi(P \cap \Psi^{-1}(\mathrm{rel.int}(A)), [0,1]) \subset P \cap \Psi^{-1}(\mathrm{rel.int}(A)).$$

Remind that $\{P \cap \mathrm{Trop}(\varphi'((X\times\mathbb{A}^1)^\circ)) \cap \Psi^{-1}(\mathrm{Trop}(\varphi(v_0)))\}_{P\in\Lambda'}$ is a polyhedral structure of $\mathrm{Trop}(\varphi'((X\times\mathbb{A}^1)^\circ)) \cap \Psi^{-1}(\mathrm{Trop}(\varphi(v_0)))$. Hence, by Lemma 9.3.1, in similar way to the proof of Lemma 7.2.6, there exists a strong deformation retraction of

$$\mathrm{Trop}(\varphi'((X\times\mathbb{A}^1)^\circ)) \cap \Psi^{-1}(\mathrm{rel.int}(A))$$

onto

$$\bigcup_{\xi\in\Xi} P_\xi \cap \Psi^{-1}(\mathrm{rel.int}(A))$$

preserving the polyhedral structure. For each $\xi \in \Xi$, there exists a deformation retraction of $P_\xi \cap \Psi^{-1}(\mathrm{rel.int}(A))$ onto ξ. These induce a required strong deformation retraction ψ. $\square$

LEMMA 9.3.3. [**JSS17**, Proposition 3.11] *For a sufficiently small polyhedron $A \subset \mathrm{Trop}(T_\Sigma)$ which is a neighborhood of* $\mathrm{Trop}(\varphi(v_0))$*, we have*

$$H^{p,0}(\mathrm{rel.int}(A)\cap\mathrm{Trop}(\varphi(X)),\Lambda) \cong H^{p,0}(\mathrm{Trop}(\varphi(v)),\Lambda) \cong F^p(\mathrm{Trop}(\varphi(v)),\Lambda).$$

9.3.2. Notations. In this subsection, we fix notations and give several assumptions used in the rest of this section.

We fix $v_0 \in X^{\mathrm{Ber}}$. What we need to compute are stalks $\mathscr{F}^r_{T,X^\circ,v_0}$, $R^i\pi_*\mathscr{F}^r_{T,(X\times\mathbb{A}^1)^\circ,v_0}$ of sheaves. In the rest of this section, we assume that v is of height 1. (When v is of height 0, the proof is similar and much easier. We omit it.) By Lemma 9.3.2 and Lemma 9.3.3, without loss of generality, we may assume that $\mathrm{supp}(v_0)$ is the generic point of X. Since we consider stalks, we can shrink X to open subvarieties. By Lemma 9.3.2 and Lemma 9.3.3, every element of these stalks is represented by a commutative diagram

$$\begin{tikzcd} X\times\mathbb{A}^1 \arrow[r, "\pi"] \arrow[d, "\varphi'"'] & X \arrow[d, "\varphi"] \\ T_{\Sigma'} \arrow[r, "\Psi"] & \mathbb{G}_m^L, \end{tikzcd}$$

and an element of $C^{p,q}(\mathrm{Trop}(\varphi(v_0)),\Lambda)$ or $C^{p,q}(\mathrm{Trop}(\varphi'(X\times\mathbb{A}^1)^\circ)\cap\Psi^{-1}(\mathrm{Trop}(\varphi(v_0))),\Lambda')$, where

- φ and φ' are closed immersions to toric varieties $T_{\Sigma'}$ and T_Σ such that $\varphi'(X\times\mathbb{A}^1)$ intersects with the dense open orbit,
- Ψ is a toric morphism,
- a fan structure Λ (resp. Λ') of $\mathrm{Trop}(\varphi(X))$ (resp. $\mathrm{Trop}(\varphi'(X\times\mathbb{A}^1))$) such that for any $P\in\Lambda'$, we have $\Psi(P)\subset Q$ for some $Q\in\Lambda$.

In the rest of this section, we fix $(\varphi,\varphi',\Psi,\Omega,\Lambda,\Lambda')$.

By [**MZ13**, Proposition 2.2] and Lemma 9.3.2, we have the following.

COROLLARY 9.3.4.

$$\pi^\circ_*\mathscr{C}^{p,q}_v = 0$$

for $q\geq 2$.

We put M the character lattice of the dense open torus of $T_{\Sigma'}$, $N := \mathrm{Hom}(M,\mathbb{Z})$, and

$$\Xi := \Lambda'\cap\mathrm{Trop}(\varphi'((\pi^\circ)^{-1}(v_0))) := \{P\cap\mathrm{Trop}(\varphi'((\pi^\circ)^{-1}(v_0)))\}_{P\in\Lambda'}.$$

The composition

$$\mathbb{A}^1_{K(X)} = \pi^{-1}(\mathrm{Spec}\,K(X)) \xrightarrow{\varphi'} T_{\Sigma'}$$

is generically defined by finitely many rational functions $h_i \in K(X)(T)$ $(1\leq i\leq n)$.

Without loss of generality, we may assume the following.

ASSUMPTION 9.3.5. *(1) There exist $a_i \in O(X)^\times$ such that $\{v(a_i)\}_i$ is a $\mathbb{Q}$-basis of $\mathbb{Q} \cdot \Gamma_{v_0}$,*
(2) $\varphi\colon X \to \mathbb{G}_m^L$ is of the form
$$(\varphi_0, (a_i)_i)\colon X \to \mathbb{G}_m^{L-\dim_{\mathbb{Q}} \mathbb{Q}\cdot\Gamma_{v_0}} \times \mathbb{G}_m^{\dim_{\mathbb{Q}} \mathbb{Q}\cdot\Gamma_{v_0}},$$
(3) the cone $P_{v_0} \in \Lambda$ whose relative interior containing $\mathrm{Trop}(\varphi(v_0))$ is of dimension $\dim_{\mathbb{Q}} \mathbb{Q} \cdot \Gamma_{v_0}$,
(4) $\varphi'\colon U \times \mathbb{A}^1 \to T_{\Sigma'}$ is of the form
$$(\varphi'_0, \mathrm{Id}_{\mathbb{A}^1})\colon U \times \mathbb{A}^1 \to T_{\Sigma'_0} \times \mathbb{A}^1,$$
(5) Assumption 8.0.7 holds for the above h_i, and
(6) the set Ξ is a polyhedral complex structure of $\mathrm{Trop}(\varphi'((\pi^\circ)^{-1}(v_0)))$.

(By Assemption 9.3.5 (1), a sufficiently fine fan Λ satisfies Assumption 9.3.5 (3).)

The polyhedral complex Ξ is a graph, i.e., consisting of only 0 and 1-dimensional polyhedra. To express an element of Ξ of dimension 0 (resp. of dimension 1, of any dimension), we use character l (resp. e, ξ) (sometimes, with subscripts). For simplicity, for example, we write "$l \in \Xi$" instead of "$l \in \Xi$ of dimension 0". Note that for $e \in \Xi$, we have $\mathrm{rel.int}(e) \subset N_{\mathbb{R}}$.

- For each $\xi \in \Xi$, we put $P_\xi \in \Lambda'$ the cone whose relative interior contains $\mathrm{rel.int}(\xi)$.
- For each $l, e \in \Xi$ such that $l \in e \cap N_{\mathbb{R}}$, we fix $g_{l,e} \in M \cap P_l^\perp$ such that $g_{l,e} \notin P_e^\perp$.

9.3.3. Technical conditions. In this subsection, we consider Conditions of a triple $(\varphi_c, \varphi'_c, \Psi_c)$ of closed immersions φ_c, φ'_c and a toric morphism Ψ_c such that a diagram

$$\begin{tikzcd} X \times \mathbb{A}^1 \arrow[r, "\pi"] \arrow[d, "\varphi'_c"'] & X \arrow[d, "\varphi_c"] \\ T_{\Sigma'_c} \arrow[r, "\Psi_c"] & \mathbb{G}_m^{n_c}, \end{tikzcd}$$

is commutative. In Subsection 9.5.3, we need to exchange tropicalizations under the process of induction. We discuss how to exchange tropicalizations with Conditions holding.

CONDITION 9.3.6. *The restriction of $\mathrm{Trop} \circ \varphi'_c$ to $\mathrm{Sk}_{\varphi'_c}((\pi^\circ)^{-1}(v_0))$ is injective.*

CONDITION 9.3.7. *The fibers of the map*
$$\Psi_c\colon \mathrm{Trop}(\varphi'_c((X \times \mathbb{A}^1)^\circ)) \to \mathrm{Trop}(\varphi_c(X^\circ))$$
are finite unions of 1-dimensional polyhedra.

We identify $(\mathrm{Spec}\, K(X)/K)^{\mathrm{Ber}}$ and its image under the natural map
$$(\mathrm{Spec}\, K(X)/K)^{\mathrm{Ber}} \hookrightarrow X^{\mathrm{Ber}}.$$
Let Λ'_c be a fan structure of $\mathrm{Trop}(\varphi'_c(X \times \mathbb{A}^1))$ such that
$$\Xi_c := \Lambda'_c \cap \mathrm{Trop}(\varphi'_c((\pi^\circ)^{-1}(v_0)))$$
is a polyhedral complex structure of $\mathrm{Trop}(\varphi'_c((\pi^\circ)^{-1}(v_0)))$. We put $O(\sigma'_{c,v_0}) \subset T_{\Sigma'_c}$ the orbit containing a dense subset of $\varphi'_c(\pi^{-1}(\mathrm{supp}(v_0)))$.

CONDITION 9.3.8. • *For any*

$$v \in (\mathrm{Trop} \circ \varphi_c)^{-1}(\mathrm{Trop} \circ \varphi_c)(v_0) \cap (\mathrm{Spec}\, K(X)/K)^{\mathrm{Ber}},$$

we have

$$\mathrm{Trop}(\varphi'_c((\pi^\circ)^{-1}(v))) = \mathrm{Trop}(\varphi'_c((\pi^\circ)^{-1}(v_0))).$$

• *For v as above and*

$$(l, w, w_0) \in (\Xi_c, \mathrm{Sk}_{\varphi'_c}((\pi^\circ)^{-1}(v)) \cap (\mathrm{Trop} \circ \varphi'_c)^{-1}(l), \mathrm{Sk}_{\varphi'_c}((\pi^\circ)^{-1}(v_0)) \cap (\mathrm{Trop} \circ \varphi'_c)^{-1}(l))$$

such that $\dim(l) = 0$*, both w and w_0 are*
- *of type 2 and correspond to the same $(a, r) \in (K(X)^{\mathrm{alg}}, \mathbb{Q} \cdot \Gamma_{v_0})$ when $l \in \mathrm{Trop}(O(\sigma'_{c,v_0}))$, and*
- *of type 1 and correspond to the same $a \in K(X)^{\mathrm{alg}} \cup \{\infty\}$ when $l \notin \mathrm{Trop}(O(\sigma'_{c,v_0}))$.*

• *For v as above and*

$$(l, e, w, u, w_0, u_0) \in (\Xi_c, \Xi_c, \mathrm{Sk}_{\varphi'_c}((\pi^\circ)^{-1}(v)) \cap (\mathrm{Trop} \circ \varphi'_c)^{-1}(l), \pi^{-1}([v]) \cap (\mathrm{Trop}^{\mathrm{ad}}_{\Xi_c} \circ \varphi'_c)^{-1}(e),$$
$$\mathrm{Sk}_{\varphi'_c}((\pi^\circ)^{-1}(v_0)) \cap (\mathrm{Trop} \circ \varphi'_c)^{-1}(l), \pi^{-1}([v_0]) \cap (\mathrm{Trop}^{\mathrm{ad}}_{\Xi_c} \circ \varphi'_c)^{-1}(e))$$

such that $u \in \overline{\{w\}}$, $u_0 \in \overline{\{w_0\}}$, and $l \subsetneq e \cap \mathrm{Trop}(O(\sigma'_{c,v_0}))$, both u and u_0 correspond to a same $(a, r, \epsilon) \in (K(X)^{\mathrm{alg}}, \mathbb{Q} \cdot \Gamma_{v_0}, \{+, -\})$.

By Lemma 9.3.11 and 9.3.14, without loss of generality, we may assume the following.

ASSUMPTION 9.3.9. *The triple $(\varphi, \varphi', \Psi)$ satisfies Condition 9.3.6, 9.3.7, and 9.3.8.*

REMARK 9.3.10. *By Condition 9.3.6, the tropicalization $\mathrm{Trop}(\varphi'((\pi^\circ)^{-1}(v_0)))$ is simply connected, i.e., the polyhderal complex Ξ is a tree.*

By Condition 9.3.8 and Lemma 2.2.1, we have

$$\mathrm{Trop}(\varphi'((\pi^\circ)^{-1}(v_0))) = \mathrm{Trop}(\varphi'((X \times \mathbb{A}^1)^\circ)) \cap \Psi^{-1}(\mathrm{Trop}(\varphi(v_0))).$$

In particular, we have $\dim(P_l) = \dim(P_{v_0})$ $(l \in \Xi)$ *and* $\dim(P_e) = \dim(P_{v_0}) + 1$ $(e \in \Xi)$*.*

LEMMA 9.3.11. *For any $\varphi'_{c,0} \colon X \times \mathbb{A}^1 \to T_{\Sigma_0}$, there exists $\varphi'_{c,1} \colon X \times \mathbb{A}^1 \to T_{\Sigma_1}$ such that $\varphi'_c := (\varphi'_{c,0}, \varphi'_{c,1})$ satisfies Condition 9.3.6.*

PROOF. This follows from Lemma 8.0.13. □

LEMMA 9.3.12. *Let $\varphi'_{c,0} \colon X \times \mathbb{A}^1 \to T_{\Sigma_0}$ satisfies Condition 9.3.6, $f \in O(X)[T]$ a polynomial irreducible over $K(X)_{v_0}$. We consider f as a morphism $X \times \mathbb{A}^1 \to \mathbb{A}^1$. Then $\varphi'_c := (\varphi'_{c,0}, f) \colon X \times \mathbb{A}^1 \to T_{\Sigma_0} \times \mathbb{A}^1$ satisfies Condition 9.3.6.*

PROOF. This follows from Remark 8.0.8. □

Let $\varphi'_{c,0} \colon X \times \mathbb{A}^1 \to T_{\Sigma'_{c,0}}$ be a closed immersion. The composition

$$\mathbb{A}^1_{K(X)} = \pi^{-1}(\mathrm{Spec}\, K(X)) \xrightarrow{\varphi'_{c,0}} T_{\Sigma'_{c,0}}$$

is generically defined by finitely many rational functions $h_{c,0,i} \in K(X)(T)$ $(1 \leq i \leq n_c)$.

By Lemma 8.0.10 and Remark 8.0.8, we have the following.

LEMMA 9.3.13. *We assume that Assumtion 8.0.7 holds for $h_{c,0,i}$. We assume that each f_i (as in Assumption 8.0.7) is a constant or a product of first degree polynomials in $K(X)[T]$. Let $a_{i,j}$ be the zeros of f_i. We put $c_i = f_i$ when f_i is a constant. Then, for $v \in (\operatorname{Spec} K(X)/K)^{\mathrm{Ber}}$ such that $v(c_i) = v_0(c_i)$ and $v(a_{i,j}) = v_0(a_{i,j})$ for any i, j, the natural indentifications of $[a_{i,j}, \infty]$ in $\mathbb{P}^{1,\mathrm{Ber}}_{K(X)_v}$ and those in $\mathbb{P}^{1,\mathrm{Ber}}_{K(X)_{v_0}}$ induce a bijection*

$$\mathrm{Sk}_{\varphi'_{c,0}}((\pi^\circ)^{-1}(v)) \cong \mathrm{Sk}_{\varphi'_{c,0}}((\pi^\circ)^{-1}(v_0)).$$

Moreover, the diagram

$$\begin{tikzcd}
\mathrm{Sk}_{\varphi'_{c,0}}((\pi^\circ)^{-1}(v)) \arrow[r, "\cong"] \arrow[d, "\mathrm{Trop}\circ\varphi'_{c,0}"'] & \mathrm{Sk}_{\varphi'_{c,0}}((\pi^\circ)^{-1}(v_0)) \arrow[d, "\mathrm{Trop}\circ\varphi'_{c,0}"] \\
\mathrm{Trop}(\varphi'_{c,0}((\pi^\circ)^{-1}(v))) \arrow[r, equal] & \mathrm{Trop}(\varphi'_{c,0}((\pi^\circ)^{-1}(v_0)))
\end{tikzcd}$$

is commutative.

LEMMA 9.3.14. *Under Assumption 8.0.7 for $h_{c,0,i}$, by exchanging X to an open subvariety, there exists $\varphi_c \colon X \to \mathbb{G}_m^{n_c}$ such that $(\varphi_c, \varphi'_c := \varphi_{c,0} \times \varphi_c, \mathrm{pr}_2)$ satisfies Condition 9.3.7 and 9.3.8, where pr_2 is the second projection.*

PROOF. When each f_i is a constant or a product of first degree polynomials, the assertion follows from Lemma 9.3.13. In general, the assertion (Condition 9.3.7 part (resp. Condition 9.3.8 part)) follows from Lemma 2.1.9, 3.2.1, and the above case (resp. Lemma 3.2.1, 2.2.6, and the above case). $\square$

9.4. Explicit descriptions of F^p by tropical K-groups

We describe F^p and maps between them by tropical K-groups.

First, we describe them by $\overline{F}^p$. For a cone $P \in \Lambda'$, we put

$$\overline{F}_p(P) := F_p(P)/\operatorname{Im}(\mathrm{Span}(P) \otimes F_{p-1}(P) \to F_p(P)),$$

where

$$\mathrm{Span}(P) \otimes F_{p-1}(P) \to F_p(P)$$

is a natural morphism, and $\overline{F}^p(P)$ the quotient of the p-th wedge product of

$$(M \cap \sigma_P'^{\perp} \cap P^{\perp})_{\mathbb{Q}} := \{m \in M \cap \sigma_P'^{\perp} \mid n(m) = 0 \ (n \in P \cap \mathrm{Trop}(O(\sigma'_P)))\}_{\mathbb{Q}}$$

by the $\mathbb{Q}$-vector subspace generated by elements which are trivial on $\overline{F}_p(P)$, where $\sigma'_P \in \Sigma'$ is the cone such that $\mathrm{rel.int}(P) \subset \mathrm{Trop}(O(\sigma'_P))$. We have $\overline{F}^p(P) \cong \mathrm{Hom}_{\mathbb{Q}}(\overline{F}_p(P), \mathbb{Q})$. For $l \in \Xi$, we have a natural isomorphism

$$\bigoplus_{k=0}^{p} \wedge^{p-k} \langle a_j \rangle_{\mathbb{Q}} \otimes \overline{F}^k(P_l) \cong F^p(P_l)$$

of $\mathbb{Q}$-vector spaces given by

$$\alpha \otimes \beta \mapsto \alpha \wedge \beta.$$

For $l, e \in \Xi$ such that $l \in e \cap N_{\mathbb{R}}$, we have a natural isomorphism

$$\bigoplus_{k=0}^{p} \wedge^{p-k} \langle a_j \rangle_{\mathbb{Q}} \otimes \overline{F}^k(P_e) \oplus \bigoplus_{k'=1}^{p} \wedge^{p-k'} \langle a_j \rangle_{\mathbb{Q}} \otimes \mathbb{Q} \cdot g_{l,e} \otimes \overline{F}^{k'-1}(P_e) \cong F^p(P_e)$$

of $\mathbb{Q}$-vector spaces given by

$$\alpha \otimes \beta \mapsto \alpha \wedge \beta, \ \alpha' \otimes g_{l,e} \otimes \beta' \mapsto \alpha' \wedge g_{l,e} \wedge \beta'.$$

Hence $F^p(P_l) \to F^p(P_e)$ is determined by the morphism

$$\iota_{l,e} \colon \overline{F}^k(P_l) \to \overline{F}^k(P_e) \oplus (\mathbb{Q} \cdot g_{l,e} \otimes \overline{F}^{k-1}(P_e)),$$

which is given by a decomposition

$$(M \cap P_l^\perp)_\mathbb{Q} = \mathbb{Q} \cdot g_{l,e} \oplus (M \cap P_e^\perp).$$

Next, we express $\overline{F}^p$ by tropical K-grpups. We put

$$\mathrm{Star}(P) := \bigcup_{\substack{P' \in \Lambda \cap \mathrm{Trop}(O(\sigma'_P)) \\ \mathrm{rel.int}(P) \subset P'}} P'.$$

We put $\mathrm{Star}(P)/P \cap \mathcal{O}(\sigma'_P)$ the quotient given by the equivalence relation that $n \sim n'$ if and only if $n = n' + r$ for some $r \in P \cap \mathcal{O}(\sigma'_P)$.

Remark 9.4.1. *The set* $\overline{F}_p(P)$ *is determined explicitly by* $\mathrm{Star}(P)/P \cap \mathcal{O}(\sigma'_P)$. *The set* $\mathrm{Star}(P)/\mathrm{Span}(P)$ *is canonically identified with the intersection*

$$\mathrm{Trop}(\overline{\varphi'(X \times \mathbb{A}^1) \cap O(\sigma'_P))} \cap O(P \cap N_{\sigma'_P,\mathbb{R}}),$$

where the closure $\overline{\varphi'(X \times \mathbb{A}^1) \cap O(\sigma'_P))}$ *of* $\varphi'(X \times \mathbb{A}^1) \cap O(\sigma'_P))$ *is taken in the affine toric variety* $T_{P \cap N_{\sigma'_P,\mathbb{R}}}$ *associated with the cone* $P \cap N_{\sigma'_\mathbb{R},\mathbb{R}}$. *In particular, for any* $x \in \mathrm{Trop}^{\mathrm{ad}}(\varphi'(X \times \mathbb{A}^1)^{\mathrm{ad}})$ *whose image in* Λ' *is* P, *by Corollary 3.4.14 and Remark 3.4.19, we have*

$$\begin{aligned} &\overline{F}^p(P) \\ &= \wedge^p(M \cap \sigma_P'^\perp \cap P^\perp)_\mathbb{Q} / \bigcap_{v \in (\mathrm{Trop}^{\mathrm{ad}} \circ \varphi')^{-1}(x)} \mathrm{Ker}(\wedge^p(M \cap \sigma_P'^\perp \cap P^\perp)_\mathbb{Q} \to K_T^p(k(\mathrm{supp}(v)))) \\ &= \wedge^p(M \cap \sigma_P'^\perp \cap P^\perp)_\mathbb{Q} / \bigcap_{v \in \mathrm{Sk}_{\varphi'}((X \times \mathbb{A}^1)^{\mathrm{ad}}) \cap (\mathrm{Trop}^{\mathrm{ad}} \circ \varphi')^{-1}(x)} \mathrm{Ker}(\wedge^p(M \cap \sigma_P'^\perp \cap P^\perp)_\mathbb{Q} \to K_T^p(k(\mathrm{supp}(v)))). \end{aligned}$$

Proposition 9.4.2. *For* $l \in \Xi$ *(resp.* $l, e \in \Xi$ *such that* $l \in e \cap N_\mathbb{R}$*), we have*

$$\overline{F}^k(P_l) \cong \wedge^k(M \cap \sigma_l'^\perp \cap P_l^\perp)_\mathbb{Q} / \bigcap_w \mathrm{Ker}(\wedge^k(M \cap \sigma_{P_l}'^\perp \cap P_l^\perp)_\mathbb{Q} \to K_T^k(\kappa(w)))$$

(resp.

$$\overline{F}^k(P_e) \cong \wedge^k(M \cap \sigma_{v_0}'^\perp \cap P_e^\perp)_\mathbb{Q} / \bigcap_{w,u} \mathrm{Ker}(\wedge^k(M \cap P_e^\perp)_\mathbb{Q} \to K_T^k(\kappa(u))),$$

where w *runs through*

$$w \in (\mathrm{Trop} \circ \varphi')^{-1}(l) \cap \bigcup_{v \in (\mathrm{Trop} \circ \varphi)^{-1}(\mathrm{Trop}(\varphi(v_0))) \cap (\mathrm{Spec}\, K(X)/K)^{\mathrm{Ber}}} \mathrm{Sk}_{\varphi'}(\pi^\circ)^{-1}(v)$$

(resp. w *runs through the above set and* u *runs through all non-trivial specializations of* w *in* $\mathbb{A}^{1,\mathrm{ad}}_{k(\mathrm{supp}(v))_v} \subset \pi^{-1}([v])$ *such that* $(\mathrm{Trop}^{\mathrm{ad}}_\Xi \circ \varphi')(u) = e \in \Xi$*).*

Proof. Proposition 9.4.2 for l follows from Proposition 3.4.15, 3.4.16, Lemma 2.2.1, the definition of Shilov boundaries, and Remark 9.4.1 for $x = l \in \mathrm{Trop}^{\mathrm{ad}}(\varphi'(X \times \mathbb{P}^1)^{\mathrm{ad}})$.

By Remark 9.4.1 for $x = (l, l_e) \in \mathrm{Trop}^{\mathrm{ad}}(\varphi'(X \times \mathbb{P}^1)^{\mathrm{ad}})$ (in the sense of [**M20-1**, Lemma 4.9]) (here we fix a point $l_e \in \mathrm{rel.int}(e)$), Proposition 9.4.2 for (l, e) follows from that every

$$u \in \mathrm{Sk}_{\varphi'}((X \times \mathbb{P}^1)^{\mathrm{ad}}) \cap (\mathrm{Trop}^{\mathrm{ad}} \circ \varphi')^{-1}(x)$$

is of the form in the assertion of Proposition 9.4.2. We shall show this. In the following, we identify elements as written in Remark 8.0.15 for suitable v. See also Remark 8.0.16. We put $w \in (X \times \mathbb{P}^1)^{\mathrm{ad}}$ generalization of u of height 1, then $w \in \mathrm{Sk}_{\varphi'}((X \times \mathbb{P}^1)^{\mathrm{ad}})$ and $\mathrm{Trop}^{\mathrm{ad}}(\varphi'(w)) = l$ by Proposition 3.4.16. Hence w is contained in the set in the assertion of Proposition 9.4.2. By Condition 9.3.7, the images $\pi(u), \pi(w)$ are also in $\mathrm{Sk}_{\varphi}(X^{\mathrm{ad}})$. Then since

$$\mathrm{Trop}^{\mathrm{ad}}(\varphi'(\pi(u))) = \mathrm{Trop}^{\mathrm{ad}}(\varphi'(\pi(w))),$$

by Proposition 3.4.16, we have $\mathrm{ht}(\pi(u)) = \mathrm{ht}(\pi(w))$ hence $\pi(u) = \pi(w)$, i.e., $u \in \pi^{-1}(\pi(w))$. Hence u is of the form in the assertion of Proposition 9.4.2. $\square$

By construction of natural morphisms $\overline{F}^k(P_l) \to K_T^k(\kappa(w))$ and $\overline{F}^k(P_e) \to K_T^k(\kappa(u))$, we have the following three Lemmas. Remind that we put $\overline{u}$ the normalized discrete valuation of $\kappa(w)$ corresponding to u in the sense of Remark 8.0.4. We denote the first (resp. the second) projection by pr_1 (resp. pr_2). Note that by definition, for $\zeta \in \Xi$ and $e \in \Xi$ such that $\zeta \in e \setminus N_{\mathbb{R}}$, the image of $\overline{F}^p(P_\zeta) \subset F^p(P_\zeta)$ in $F^p(P_e)$ is contained in $\overline{F}^p(P_e)$.

LEMMA 9.4.3. *For l, e, w, u as in Proposition 9.4.2, we have a commutative diagram*

$$\begin{tikzcd}
\overline{F}^k(P_l) \arrow[r, "\iota_{l,e}"] \arrow[d] & \overline{F}^k(P_e) \oplus (\mathbb{Q} \cdot g_{l,e} \otimes \overline{F}^{k-1}(P_e)) \arrow[r, "pr_1"] & \overline{F}^k(P_e) \arrow[d] \\
K_T^k(\kappa(w)) \arrow[rr, "\frac{1}{\bar{u}(\overline{g_{l,e}})} s_{\bar{u}}^{\overline{g_{l,e}}}"] & & K_T^k(\kappa(u)),
\end{tikzcd}$$

where the last horizontal arrow is defined by

$$\frac{1}{\bar{u}(\overline{g_{l,e}})} s_{\bar{u}}^{\overline{g_{l,e}}}(a) := \frac{1}{\bar{u}(\overline{g_{l,e}})} \partial_{\bar{u}}(\overline{g_{l,e}} \wedge a).$$

(Here, the element $\overline{g_{l,e}} \in \kappa(w)$ is the image of $g_{l,e}$ under $\varphi_w \colon M \cap P_l^\perp \to \kappa(w)$.)

LEMMA 9.4.4. *For l, e, w, u as in Proposition 9.4.2, we have a commutative diagram*

$$\begin{tikzcd}
\overline{F}^k(P_l) \arrow[r, "\iota_{l,e}"] \arrow[d] & \overline{F}^k(P_e) \oplus (\mathbb{Q} \cdot g_{l,e} \otimes \overline{F}^{k-1}(P_e)) \arrow[r, "pr_2"] & \mathbb{Q} \cdot g_{l,e} \otimes \overline{F}^{k-1}(P_e) \arrow[d] \\
K_T^k(\kappa(w)) \arrow[rr, "\partial_{\bar{u}}"] & & K_T^{k-1}(\kappa(u)),
\end{tikzcd}$$

where the second vertical arrow is defined by

$$1 \cdot g_{l,e} \otimes a \mapsto \bar{u}(\overline{g_{l,e}}) a.$$

Remind that by Assumption 9.3.5 (4), we have $[0, \infty] \subset \mathrm{Sk}_{\varphi'} \pi^{-1}(\pi(w))$.

LEMMA 9.4.5. *For l, e, w, u as in Proposition 9.4.2, $\zeta \in \Xi$ such that $\zeta \in e \setminus (e \cap N_{\mathbb{R}})$, and*

$$\mu \in (\mathrm{Trop} \circ \varphi')^{-1}(\zeta) \cap \mathrm{Sk}_{\varphi'} \pi^{-1}(\pi(w)),$$

we have a commutative diagram

$$\begin{tikzcd}
\overline{F}^k(P_\zeta) \arrow[r] \arrow[d] & \overline{F}^k(P_e) \arrow[d] \\
K_T^k(\kappa(\mu)) & K_T^k(\kappa(u)) \arrow[l, hook']
\end{tikzcd}$$

where the second horizontal arrow is induced from inclusion $\kappa(u) \subset \kappa(\mu)$. (See Lemma 8.0.6 (3) when $u \neq u_{w,0}$.)

COROLLARY 9.4.6. *For ζ, e as in Lemma 9.4.5, the map $\overline{F}^k(P_\zeta) \to \overline{F}^k(P_e)$ is injective.*

PROOF. This follows from Proposotion 9.4.2 and Lemma 9.4.5. ☐

LEMMA 9.4.7. *There exists an affinoid subdomain U of X^{Ber} containing v_0 such that for any*

$$v \in U \cap (\mathrm{Trop} \circ \varphi)^{-1}(\mathrm{Trop}(\varphi(v_0))) \cap (\mathrm{Spec}\, K(X)/K)^{\mathrm{Ber}}$$

and (l, w, w_0) (resp. (l, e, w, w_0, u, u_0)) as in Condition 9.3.8 such that l is contained in at least 3 edges or only 1 edge, we have

$$\mathrm{Ker}(\wedge^k(M \cap \sigma'_{P_l} \cap P_l^{\perp})_{\mathbb{Q}} \to \wedge^k(\kappa(w_0)^\times)_{\mathbb{Q}}) \subset \mathrm{Ker}(\wedge^k(M \cap \sigma'_{P_l} \cap P_l^{\perp})_{\mathbb{Q}} \to \wedge^k(\kappa(w)^\times)_{\mathbb{Q}})$$

(resp.

$$\mathrm{Ker}(\wedge^k(M \cap P_e^{\perp})_{\mathbb{Q}} \to \wedge^k(\kappa(u_0)^\times)_{\mathbb{Q}}) \subset \mathrm{Ker}(\wedge^k(M \cap P_e^{\perp})_{\mathbb{Q}} \to \wedge^k(\kappa(u)^\times)_{\mathbb{Q}}).$$

PROOF. It suffices to show that for any $f \in K(X)(T)$, there exists an affinoid subdomain U_f of X^{Ber} containing v_0 such that for

$$v \in U_f \cap (\mathrm{Trop} \circ \varphi)^{-1}(\mathrm{Trop}(\varphi(v_0))) \cap (\mathrm{Spec}\, K(X)/K)^{\mathrm{Ber}}$$

and (l, w, w_0) (resp. (l, e, w, w_0, u, u_0)) as in Condition 9.3.8 such that $f \in m_{w_0}$ (resp. $f \in m_{u_0}$), we have $f \in m_w$ (resp. $f \in m_u$). The existence of such a U_f follows from Lemma 3.2.1, 2.2.6, 9.3.13, and Definition 8.0.1, in a similar way to Lemma 9.3.14. (resp. from Lemma 3.2.1, 2.2.6, 9.3.13, Corollary 8.0.12, and Definition 8.0.1, in a similar way to Lemma 9.3.14.) ☐

Without loss of generality, by exchanging φ to new one φ_n and φ' to (φ', φ_n), we may assume the following.

ASSUMPTION 9.4.8.

$$U := (\mathrm{Trop} \circ \varphi)^{-1}(\mathrm{Trop}(\varphi(v_0)))$$

satisfies the assertion of Lemma 9.4.7.

The residue homomorphism $\partial_{\bar{u}}$ on tropical K-groups has a natural lifting

$$\partial_{\bar{u}} \colon \wedge^k (\kappa(w)^\times)_{\mathbb{Q}} \to \wedge^{k-1}(\kappa(u)^\times)_{\mathbb{Q}}.$$

COROLLARY 9.4.9. *For (l, e, w_0, u_0) as in Condition 9.3.8, we have a commutative diagram*

$$\begin{array}{ccc}
\mathrm{Im}(\wedge^k(M \cap P_l^{\perp})_{\mathbb{Q}} \to \wedge^k(\kappa(w_0)^\times)_{\mathbb{Q}}) & \xrightarrow{\frac{1}{\bar{u}(\overline{g_{l,e}})} s_{\bar{u}}^{\overline{g_{l,e}}}} & \mathrm{Im}(\wedge^k(M \cap P_e^{\perp})_{\mathbb{Q}} \to \wedge^k(\kappa(u_0)^\times)_{\mathbb{Q}}) \\
\downarrow & & \downarrow \\
\overline{F}^k(P_l) & \xrightarrow{\mathrm{pr}_1 \circ \iota_{l,e}} & \overline{F}^k(P_e),
\end{array}$$

where the first horizontal arrow is defined by

$$\frac{1}{\bar{u}(\overline{g_{l,e}})} s_{\bar{u}}^{\overline{g_{l,e}}}(a) := \frac{1}{\bar{u}(\overline{g_{l,e}})} \partial_{\bar{u}}(\overline{g_{l,e}} \wedge a).$$

PROOF. This follows from Corollary 9.4.2, Lemma 9.4.3, Assumption 9.4.8. ☐

COROLLARY 9.4.10. *For (l, e, w_0, u_0) as in Condition 9.3.8, we have a commutative diagram*

$$\begin{tikzcd}
\mathrm{Im}(\wedge^k(M\cap P_l^\perp)_\mathbb{Q} \to \wedge^k(\kappa(w_0)^\times)_\mathbb{Q}) \arrow[r, "\partial_{\bar{u}}"] \arrow[d] & \mathrm{Im}(\wedge^{k-1}(M\cap P_e^\perp)_\mathbb{Q} \to \wedge^{k-1}(\kappa(u_0)^\times)_\mathbb{Q}) \arrow[d] \\
\overline{F}^k(P_l) \arrow[r, "\mathrm{pr}_2\circ\iota_{l,e}"] & \mathbb{Q}\cdot g_e\otimes\overline{F}^{k-1}(P_e),
\end{tikzcd}$$

where the second vertical arrow is defined by

$$a \to \frac{1}{\bar{u}(\overline{g_{l,e}})}\cdot g_e\otimes a.$$

PROOF. This follows from Corollary 9.4.2, Lemma 9.4.4, and Assumption 9.4.8. $\square$

COROLLARY 9.4.11. *For (e,ζ,u,μ) as in Lemma 9.4.5 such that $\pi(\mu) = v_0$, we have a commutative diagram*

$$\begin{tikzcd}
\mathrm{Im}(\wedge^k(M\cap\sigma'^\perp_{P_\zeta}\cap P_\zeta^\perp)_\mathbb{Q} \to \wedge^k(\kappa(\mu)^\times)_\mathbb{Q}) \arrow[r, equal] \arrow[d] & \mathrm{Im}(\wedge^k(M\cap\sigma'^\perp_{P_\zeta}\cap P_\zeta^\perp)_\mathbb{Q} \to \wedge^k(\kappa(u)^\times)_\mathbb{Q}) \arrow[d] \\
\overline{F}^k(P_\zeta) \arrow[r] & \overline{F}^k(P_e).
\end{tikzcd}$$

PROOF. This follows from Corollary 9.4.2, Lemma 9.4.5, and Assumption 9.4.8,. $\square$

9.5. Proofs of $(\pi^\circ_* \mathscr{F}^p_{(X\times\mathbb{A}^1)^\circ})_{v_0} \cong \mathscr{F}^p_{X^\circ,v_0}$ and $(R^1\pi^\circ_*\mathscr{F}^p_{(X\times\mathbb{A}^1)^\circ})_{v_0} = 0$

9.5.1. A picture of $\mathrm{Trop}(\varphi'((\pi^\circ)^{-1}(v_0)))$ and several notations. In this subsection, we give a picture (of an example) of $\mathrm{Trop}(\varphi'((\pi^\circ)^{-1}(v_0)))$ and several notations used in the inductions in Subsection 9.5.2 and 9.5.3.

When there is no confusion, we denote by 0 the tropicalization $\mathrm{Trop}(\varphi'(0))$ of the type 1 point $0 \in \mathbb{A}^{1,\mathrm{Ber}}_{K(X)_{v_0}} = \pi^{-1}(v_0)$. To simplify notations, we put $B := B(K(X)_{v_0}\langle T\rangle)$, the Shilov boundary of the affinoid domain $\mathscr{M}(K(X)_{v_0}\langle T\rangle)$. It consists of only one point. We identify B and its unique element.

For $\mathrm{pt} \in \mathrm{Trop}(\varphi'((\pi^\circ)^{-1}(v_0)))$ and $l \in \Xi$, we put

- w_{pt} the unique element of
$$\mathrm{Sk}_{\varphi'}\,\pi^{-1}(v_0)\cap(\mathrm{Trop}\circ\varphi')^{-1}(\mathrm{pt}),$$
- $e_{l,0} \in \Xi$ the edge which contains l and is contained $\mathrm{Trop}(\varphi'([0, w_l]))$ when $l \neq 0$,
- $e_{l,\infty} \in \Xi$ the edge which contains l and is contained $\mathrm{Trop}(\varphi'((\infty, w_l]))$ when $l \neq \mathrm{Trop}(\varphi'(B))$,
- $\mathrm{res.alg.deg}(\mathrm{pt}) := \mathrm{res.alg.deg}(w_{\mathrm{pt}})$,
- $L_0(l)$ the number of edges $e \in \Xi$ contained in $\mathrm{Trop}(\varphi'([0, w_l]))$,
- $L_a(l)$ the number of edges $e \in \Xi$ contained in $\mathrm{Trop}(\varphi'((\infty, w_l]))$,
- $L'_b(\mathrm{pt})$ the number of $l' \in \Xi\cap\mathrm{Trop}(\varphi'((\infty, w_{\mathrm{pt}}]))$ such that
  - $\mathrm{res.alg.deg}(l') = \mathrm{res.alg.deg}(\mathrm{pt})$, and
  - l' is contained in at least 3 or only 1 edge,

  and

- $L_b(\mathrm{pt}) :=$

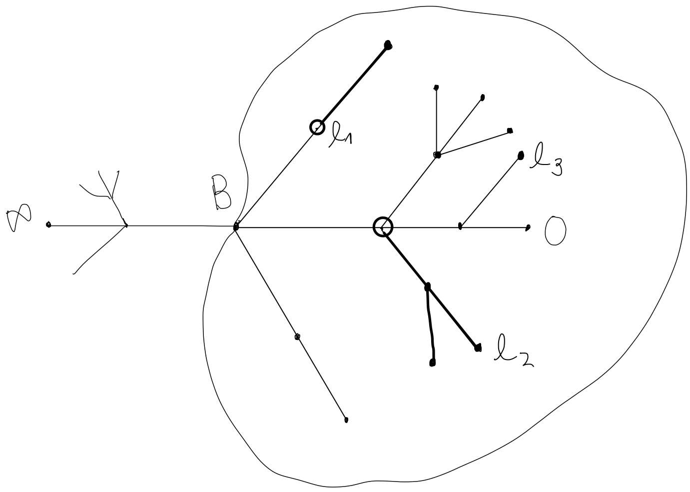
$$\begin{cases} L'_b(\mathrm{pt}) \text{ when } \mathrm{pt} \in \Xi \text{ and it is contained in at least 3 or only 1 edge in } \Xi, \\ L'_b(\mathrm{pt}) + \frac{1}{2} \text{ otherwise .} \end{cases}$$

To simplify notations, we assume that $\mathrm{Trop}(\varphi'(B))$ is contained in at least 3 edges in Ξ.

For each $e \in \Xi$, we put $\mathrm{Span}(e) := \mathbb{Q} \cdot (x_2 - x_1)$, where $x_1 \neq x_2 \in \mathrm{rel.int}(e)$. We denote the image of

$$\mathrm{Span}(e) \otimes F_{p-1}(P_e) \to F_p(P_e)$$

by $\mathrm{Span}(e) \wedge F_{p-1}(P_e)$.



Example 9.5.1.
A picture of an example of $\mathrm{Trop}(\varphi'((\pi^\circ)^{-1}(v_0)))$.

	l_1	l_2	l_3
L_0	4	4	2
L_a	1	3	3
$\mathrm{res.alg.deg}$	1	2	1
L_b	$1+\frac{1}{2}$	2	4

Here, we idetify $\mathrm{Trop}(\varphi'((\pi^\circ)^{-1}(v_0)))$ *and* $\mathrm{Sk}'_\varphi(\pi^\circ)^{-1}(v_0) \subset \mathbb{P}^{1,\mathrm{Ber}}_{K(X)_{v_0}}$, *and* $(\mathrm{res.alg.deg} = 1)$*-locus is expressed by fine lines, and* $(\mathrm{res.alg.deg} = 2)$*-locus is expressed by fat lines.*

9.5.2. A proof of $(\pi^\circ_* \mathscr{F}^p_{(X\times\mathbb{A}^1)^\circ})_{v_0} \cong \mathscr{F}^p_{X^\circ,v_0}$. In this subsection, we shall prove

$$\mathscr{F}^p_{X^\circ,v_0} \cong (\pi^\circ_* \mathscr{F}^p_{(X\times\mathbb{A}^1)^\circ})_{v_0}.$$

It is injective since the composition

$$\pi \circ s_0 \colon X \to X \times \mathbb{A}^1 \to X$$

is the identity map, where $s_0 \colon X \to X \times \mathbb{A}^1$ is the section at 0. We shall show surjectivity. It suffices to show that for

$$\alpha = (\alpha_{\mathrm{pt}})_{\mathrm{pt}} \in Z^{p,0}_{\mathrm{Trop}}(\mathrm{Trop}(\varphi'((\pi^\circ)^{-1}(v_0)))), \Lambda')$$

such that $\alpha_0 = 0$, we have $\alpha = 0$. We fix such a α.

LEMMA 9.5.2. *We fix $l \in \Xi \cap N_{\mathbb{R}} \setminus \operatorname{Trop}(\varphi'(B))$. We assume that*

$$\alpha_l(\operatorname{Span}(e) \wedge F_{p-1}(P_e)) = 0$$

for any $e \in \Xi \setminus \{e_{l,\infty}\}$ containing l.

Then we have

$$\alpha_l(\operatorname{Span}(e_{l,\infty}) \wedge F_{p-1}(P_{e_{l,\infty}})) = 0.$$

PROOF. This follows from Corollary 8.0.11, Assumption 9.3.5 (4), Proposition 9.4.2, Lemma 9.4.4, and an inclusion relation

$$\operatorname{Ker}(K^k_T(\kappa(w)) \xrightarrow{(\partial_{\overline{u}})_{u\neq u_{w,\infty}}} \oplus_{u\neq u_{w,\infty}} K^{k-1}_T(\kappa(u))) \subset \operatorname{Ker}(K^k_T(\kappa(w)) \xrightarrow{\partial_{\overline{u_{w,\infty}}}} K^{k-1}_T(\kappa(u_{w,\infty})))$$

for w as in Lemma 9.4.2 for l, where $u \in \mathbb{P}^{1,\mathrm{ad}}_{k(\operatorname{supp}(\pi(w)))_{\pi(w)}}$ runs through all non-trivial specializations of w except for $u_{w,\infty}$. (The inclusion follows from the fact that tropial K-groups form a cycle module [**?**, Theorem 1.1], Lemma 8.0.6 (1), and [**Ros96**, Proposition 2.2].) □

LEMMA 9.5.3. *We have*

$$\alpha_l(\operatorname{Span}(e) \wedge F_{p-1}(P_e)) = 0$$

for any $l, e \in \Xi$ with $l \subset e$.

PROOF. We shall show that for any $l \in \Xi \setminus \{\operatorname{Trop}(\varphi'(B))\}$, by induction on $L_a(l) \in \mathbb{Z}_{\geq 0}$ in the descending order, we have

$$\alpha_l(\operatorname{Span}(e_{l,\infty}) \wedge F_{p-1}(P_{e_{l,\infty}})) = 0.$$

Then the assertion follows from $\delta\alpha = 0$.

When l is contained in only one edge, then the edge containing l is $e_{l,\infty}$, and l is not in the orbit $N_{\mathbb{R}}$, which contains rel.int$(e_{l,\infty})$. Hence, in this case, the image of $\operatorname{Span}(e_{l,\infty})$ to $\operatorname{Trop}(O(\sigma'_{P_l}))$ is 0. Hence the equation (9.5.2) holds.

When $L_a(l)$ is the maximal, then l is contained in only one edge, hence the equation (9.5.2) holds.

When l is contained in at least 2 edges, the equation (9.5.2) follows from Lemma 9.5.2 and the assumption of induction. □

LEMMA 9.5.4. *For any $l \in \Xi \cap N_{\mathbb{R}}$, the map*

$$(\operatorname{pr}_1 \circ \iota_{(l,e_{l,0})}, (\operatorname{pr}_2 \circ \iota_{(l,e)})_e) \colon \overline{F}^k(P_l) \to \overline{F}^k(P_{e_{l,0}}) \oplus \bigoplus_e (\mathbb{Q} \cdot g_{l,e} \otimes \overline{F}^{k-1}(P_e))$$

is injective, where $e \in \Xi$ runs through all edges containing l.

PROOF. This follows from Proposition 9.4.2, Lemma 9.4.3, Lemma 9.4.4, and injectivity of

$$K^k_T(\kappa(w)) \xrightarrow{(s^{g_{e_{l,0}}}_{\overline{u_{w,0}}}, (\partial_{\overline{u}})_u)} K^k_T(\kappa(u_{w,0})) \oplus \bigoplus_u K^{k-1}_T(\kappa(u))$$

for w as in Lemma 9.4.2 for l, where $u \in \mathbb{P}^{1,\mathrm{ad}}_{k(\operatorname{supp}(\pi(w)))_{\pi(w)}}$ runs through all non-trivial specializations of w but $u_{w,\infty}$. (Injectivity of this map follows from the fact that tropical K-groups form a cycle module [**?**, Theorem 1.1], Lemma 8.0.6 (1), and [**Ros96**, Proposition 2.2].) □

PROPOSITION 9.5.5. *We have $\alpha = 0$.*

Proof. It suffices to show that $\alpha_l = 0$ for $l \in \Xi$. We show $\alpha_l = 0$ by induction on $L_0(l) \in \mathbb{Z}_{\geq 0}$ in the increasing order.

When $L_0(l) = 0$, i.e., $l = 0$, we have $\alpha_0 = 0$ by definition.

When $L_0(l) \neq 0$ and $l \in N_{\mathbb{R}}$, by Lemma 9.5.3, Lemma 9.5.4, the equation $\delta(\alpha) = 0$, and the assumption of induction, we have $\alpha_l = 0$.

When $L_0(l) \neq 0$ and $l \notin N_{\mathbb{R}}$, by Corollary 9.4.6, the equation $\delta(\alpha) = 0$, and the assumption of induction, we have $\alpha_l = 0$. □

Corollary 9.5.6. *We have*

$$\mathscr{F}^p_{X^\circ} \cong (\pi^\circ)_* \mathscr{F}^p_{(X \times \mathbb{A}^1)^\circ}.$$

9.5.3. A proof of $(R^1 \pi^\circ_* \mathscr{F}^p_{(X\times\mathbb{A}^1)^\circ})_{v_0} = 0$.

In this subsection, we shall show that

$$(R^1 \pi^\circ_* \mathscr{F}^p_{(X\times\mathbb{A}^1)^\circ})_{v_0} = 0.$$

It is enough to show that for

$$\alpha = (\alpha_\gamma)_\gamma \in Z^{p,1}_{\mathrm{Trop}}(\mathrm{Trop}(\varphi'((\pi^\circ)^{-1}(v_0))), \Lambda'),$$

there is another sextuplet $(Y, \phi, \phi', \Phi, \Theta, \Theta')$ consisting of

- an open subvariety $Y \subset X$,
- closed immersions $\phi \colon Y \to \mathbb{G}_m^{L'}$ and $\phi' \colon Y \times \mathbb{A}^1 \to T_{\Sigma'_{\mathrm{new}}}$ to toric varieties,
- a toric morphism $\Phi \colon T_{\Sigma'_{\mathrm{new}}} \to \mathbb{G}_m^{L'}$, and
- a fan structure Θ (resp. Θ') of $\mathrm{Trop}(\phi(Y))$ (resp. $\mathrm{Trop}(\phi'(Y \times \mathbb{A}^1))$)

dominating $(X, \varphi, \varphi', \Psi, \Lambda, \Lambda')$ such that the pull back of α to $Z^{p,1}_{\mathrm{Trop}}(\mathrm{Trop}(\phi'(\pi^{-1}(v_0))), \Theta')$ is equivalent to 0, here we say that a sextuplet $(Y, \phi, \phi', \Phi, \Theta, \Theta')$ *dominates* $(X, \varphi, \varphi', \Psi, \Lambda, \Lambda')$ if

- $(Y, \phi, \phi', \Phi, \Theta, \Theta')$ satisfies Assumptions 9.3.5, 9.3.9, and 9.4.8,
- there exists a commutative diagram

$$\begin{tikzcd}
Y \times \mathbb{A}^1 \arrow[rr, "\pi"] \arrow[d, "\phi'"'] \arrow[ddr, "\varphi'"] & & Y \arrow[d, "\phi"] \arrow[ddr, "\varphi"] & \\
T_{\Sigma'_{\mathrm{new}}} \arrow[rr, "\Phi"] \arrow[dr, "\rho_{\phi'}"'] & & \mathbb{G}_m^{L'} \arrow[dr] & \\
& T_{\Sigma'} \arrow[rr, "\Psi"] & & \mathbb{G}_m^{L},
\end{tikzcd}$$

  where morphisms between troic varieties are toric morphisms, and
- Θ' is sufficiently fine so that the image of each cone in Θ' under the toric morphism $T_{\Sigma'_{\mathrm{new}}} \to T_{\Sigma'}$ are contained in a cone in Λ'.

(When we discuss dominations, we fix toric morphisms.)

We show existance of such a $(Y, \phi, \phi', \Phi, \Theta, \Theta')$ by induction. In each steps of the induction, we exchange $(X, \varphi, \varphi', \Psi, \Lambda, \Lambda')$ and a sextuplet dominating it. We consider $\mathbb{Z} \times \mathbb{Z}[\frac{1}{2}]$ as an ordered set by the lexicographic order.

Notation 9.5.7. *For a sextuplet $(Y, \phi, \phi', \Phi, \Theta, \Theta')$ dominating $(X, \varphi, \varphi', \Psi, \Lambda, \Lambda')$, we put*

$$\Xi_{(\phi',\Theta)} := \Theta' \cap \mathrm{Trop}(\phi'((\pi^\circ)^{-1}(v_0))).$$

For $l \in \Xi_{(\phi',\Theta)} \setminus \{\mathrm{Trop}(\phi'(B))\}$, we put $e_{(\phi',\Theta'),l,\infty} \in \Xi_{(\phi',\Theta')}$ the unique edge which contains l and is contained in $\mathrm{Trop}(\phi'([w_l, \infty)))$*. We define L_b for* $\mathrm{Trop}(\phi'((\pi^\circ)^{-1}(v_0)))$ *in the same way as for* $\mathrm{Trop}(\varphi'((\pi^\circ)^{-1}(v_0)))$.

In this subsection, the character γ always means a singular 1-simplex such that $\gamma((0,1)) \subset \mathrm{rel.int}(e)$ for some $e \in \Xi$ or $\Xi_{(\phi',\Theta')}$. When we write $\alpha_\gamma(\mathrm{Span}(e) \wedge F^p(P_e))$, we assume that $\gamma((0,1)) \subset \mathrm{rel.int}(e)$. It is also the case for other cochains and for e with subscripts.

LEMMA 9.5.8. *Let $g \in \mathcal{O}(X)[T]$ be a polynomial irreducible over $K(X)_{v_0}$. We consider g as a morphism $X \times \mathbb{A}^1 \to \mathbb{A}^1$. Then there exists $(Y_g, \phi_g, \Phi_g, \Theta_g, \Theta'_g)$ such that a sextuplet $(Y_g, \phi_g, \phi'_g := (\varphi', \phi_g, g), \Phi_g, \Theta_g, \Theta'_g)$ dominates $(X, \varphi, \varphi', \Psi, \Lambda, \Lambda')$.*

PROOF. This follows from Lemma 9.3.12, Lemma 9.3.14, and Lemma 9.4.7. $\square$

LEMMA 9.5.9. *Let $(A, r) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 0}$. We assume that*

$$\alpha_\gamma(\mathrm{Span}(e) \wedge F_{p-1}(P_e)) = \{0\}$$

for any γ in $((\mathrm{res.alg.deg}, L_b) > (A, r))$-locus $\subset \mathrm{Trop}(\varphi'((\pi^\circ)^{-1}(v_0)))$. Let $l \in \Xi$ contained in at least 3 edges such that $(\mathrm{res.alg.deg}(l), L_b(l)) = (A, r)$.

Then there exist a sextuplet $(Y_l, \phi_l, \phi'_l, \Phi_l, \Theta_l, \Theta'_l)$ dominating $(X, \varphi, \varphi', \Psi, \Lambda, \Lambda')$ and $\beta_l \in C^{p,0}(\mathrm{Trop}(\phi'_l(w_l)), \Theta'_l)$ such that

- *the map*

$$\mathrm{Trop}(\rho_{\phi'_l})\colon \mathrm{Trop}(\phi'_l((\pi^\circ)^{-1}(v_0))) \to \mathrm{Trop}(\varphi'((\pi^\circ)^{-1}(v_0)))$$

  is injective on the inverse image of

$$(((A, 0) \leq (\mathrm{res.alg.deg}, L_b) \leq (A, r)) - locus) \setminus \{l\},$$

  and
- *we have*

$$(\mathrm{Trop}(\rho_{\phi'_l})^*\alpha + \delta\beta_l)_\gamma(\mathrm{Span}(e) \wedge F_{p-1}(P_e)) = \{0\}$$

  for any $e \in \Xi_{(\phi'_l,\Theta'_l)} \setminus \{e_{(\phi'_l,\Theta'_l),l,\infty}\}$ containing $\mathrm{Trop}(\phi'_l(w_l))$ and any γ, (when $l = \mathrm{Trop}(\varphi'(B))$, we put $\{e_{(\phi',\Theta'),\mathrm{Trop}(\phi'(B)),\infty}\} := \emptyset$).

PROOF. This follows from Remark 8.0.5, Remark 8.0.8, Lemma 8.0.14 Assumption 9.3.5 (4), Corollary 9.4.10, Lemma 9.5.8, and surjectivity of

$$\wedge^k(\kappa(w_0)^\times)_\mathbb{Q} \xrightarrow{(\partial_{\overline{u_0}})_{u \neq u_{w_0,\infty}}} \bigoplus_{u_0 \neq u_{w_0,\infty}} \wedge^{k-1}(\kappa(u_0)^\times)_\mathbb{Q}$$

for w_0 as in Corollary 9.4.10, where $u_0 \in \mathbb{P}^{1,\mathrm{ad}}_{K(X)_{v_0}}$ runs through all non-trivial specializations of w_0 except for $u_{w_0,\infty}$. (Surjectivity follows from Lemma 8.0.6 (1). See Remark 6.2.10.) More presicely, we can take ϕ'_l of the form $(\varphi', \phi_l, (\tilde{g}_j)_j)$, where $g_j \in \mathcal{O}(X)[T]$ is a polynomial irrducible over $K(X)_{v_0}$ (which is considered as a morphism $Y_l \times \mathbb{A}^1 \to \mathbb{A}^1$,) such that

- $[w_l, \infty] \subset [(g_j = 0), \infty]$, or
- $[\infty, b_j] \cap \mathrm{Sk}_{\varphi'}\, \pi^{-1}(v_0) = [\infty, w_j]$ for some type 2 point $w_j \in \pi^{-1}(v_0)$ such that $\mathrm{res.alg.deg}(w_j) < A$, where $b_j \in \pi^{-1}(v_0)$ is the zero of g_j.

Note that by Lemma 8.0.6 (4), the morphism ϕ'_l of this form always satisfies the first property of the assertion. $\square$

REMARK 9.5.10. *In Lemma 9.5.9, for any $l' \in \Xi$ such that*

$$(A, 0) \leq (\mathrm{res.alg.deg}(l'), L_b(l')) \leq (A, r),$$

by the first assertion of Lemma 9.5.9, we have

$$(\mathrm{res.alg.deg}(\mathrm{Trop}(\phi'_l(w_{l'}))), L_b(\mathrm{Trop}(\phi'_l(w_{l'})))) = (\mathrm{res.alg.deg}(l'), L_b(l')).$$

REMARK 9.5.11. *For $A \in \mathbb{Z}_{\geq 1}$ and $r \in \mathbb{Z}_{\geq 0} + \frac{1}{2}$,*

$$((A, r - \frac{1}{2}) \leq (\mathrm{res}\,.\,\mathrm{alg}\,.\,\mathrm{deg}, L_b) \leq (A, r + \frac{1}{2})) - \textit{locus}$$

is homeomorphic to a direct sum of spaces which are gluings of finitely many copies of $[0,1]$ at $\{0\}$. (The image of $((A, r - \frac{1}{2}) = (\mathrm{res}\,.\,\mathrm{alg}\,.\,\mathrm{deg}, L_b)) -$ locus is the copies of $\{0\}$.)

PROPOSITION 9.5.12. *There exists a sextuplet $(Y_\dagger, \phi_\dagger, \phi'_\dagger, \Phi_\dagger, \Theta_\dagger, \Theta'_\dagger)$ dominating $(X, \varphi, \varphi', \Psi, \Lambda, \Lambda')$ such that the pull-back $\mathrm{Trop}(\rho_{\phi'_\dagger})^*\alpha$, is equivalent to an element*

$$\alpha' \in C^{p,1}(\mathrm{Trop}(\phi'_\dagger(\pi^{-1}(v_0))), \Theta'_\dagger)$$

satisfying

$$\alpha'_\gamma(\mathrm{Span}(e) \wedge F_{p-1}(P_e)) = \{0\}$$

for any $e \in \Xi_{(\phi'_\dagger, \Theta'_\dagger)}$ and any γ. (Here

$$\mathrm{Trop}(\rho_{\phi'_\dagger})\colon \mathrm{Trop}(\phi'_\dagger((\pi^\circ)^{-1}(v_0))) \to \mathrm{Trop}(\varphi'((\pi^\circ)^{-1}(v_0)))$$

is the induced map.)

PROOF. This follows from Lemma 9.5.9, Remark 9.5.10, and Remark 9.5.11 by induction on $(\mathrm{res}\,.\,\mathrm{alg}\,.\,\mathrm{deg}, L_b) \in \mathbb{Z} \times \mathbb{Z}[\frac{1}{2}]$ in the descending order. □

By Proposition 9.5.12, without loss of generality, we may assume that

$$\alpha_\gamma(\mathrm{Span}(e) \wedge F_{p-1}(P_e)) = \{0\}$$

for any γ.

LEMMA 9.5.13. *Let $(A, r) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 0}$. Assume $\alpha_\gamma = 0$ for any γ in $((\mathrm{res}\,.\,\mathrm{alg}\,.\,\mathrm{deg}, L_b) < (A, r))$-locus. We fix $l \in \Xi \setminus \{\mathrm{Trop}(\varphi'(B))\}$ such that $(\mathrm{res}\,.\,\mathrm{alg}\,.\,\mathrm{deg}(l), L_b(l)) = (A, r)$.*

Then there exist a sextuplet $(Y_l, \phi_l, \phi'_l, \Phi_l, \Theta_l, \Theta'_l)$ dominating $(X, \varphi, \varphi', \Psi, \Lambda, \Lambda')$ and $\beta_l \in C^{p,0}(\mathrm{Trop}(\phi'_l(w_l)), \Theta'_l)$ such that

- *the map*

  $$\mathrm{Trop}(\rho_{\phi'_l})\colon \mathrm{Trop}(\phi'_l((\pi^\circ)^{-1}(v_0))) \to \mathrm{Trop}(\varphi'((\pi^\circ)^{-1}(v_0)))$$

  is injective on the inverse image of $(\mathrm{res}\,.\,\mathrm{alg}\,.\,\mathrm{deg} \geq A) -$ locus,
- *$\beta_l(\mathrm{Span}(e) \wedge F_{p-1}(e)) = \{0\}$ for any $e \in \Xi_{(\phi'_l, \Theta'_l)}$ containing $\mathrm{Trop}(\phi'_l(w_l))$, and*
- *we have*

  $$(\mathrm{Trop}(\rho_{\phi'_l})^*\alpha + \delta\beta_l)_\gamma = 0$$

  for any $\gamma\colon [0,1] \to e_{(\phi'_l, \Theta'_l), l, \infty}$.

PROOF. This follows from, when l is contained in at least 3 edges (resp. only 1 edge), Lemma 8.0.6 (2), Lemma 8.0.14, Corollary 8.0.11, Assumption 9.3.5 (4), Corollary 9.4.9, Corollary 9.4.10, and Lemma 9.5.8 (resp. Lemma 8.0.6 (3), Corollary 9.4.11, and Lemma 9.5.8). More presicely, we can take ϕ'_l of the form $(\varphi', \phi_l, (\tilde{g}_j)_j)$ where $g_j \in \mathcal{O}(X)[T]$ is a polynomial irreducible over $K(X)_{v_0}$ such that

$$[\infty, b_j] \cap \mathrm{Sk}_{\varphi'}(\pi^{-1}(v_0)) = [\infty, w'_j]$$

for some type 2 point $w'_j \in \pi^{-1}(v_0)$ such that $\mathrm{res}\,.\,\mathrm{alg}\,.\,\mathrm{deg}(w'_j) < A$, where $b_j \in \mathrm{Sk}\,\varphi'(\pi^{-1}(v_0))$ is the zero of g_j. (We consider g_j as a morphism $Y_l \times \mathbb{A}^1 \to \mathbb{A}^1$). Note that by Lemma 8.0.6 (4), the morphism ϕ'_l of this form always satisfies the first property of the assertion. □

REMARK 9.5.14. *In Lemma 9.5.13, for any $l' \in \Xi$ such that*

$$A \leq \operatorname{res}.\operatorname{alg}.\deg(l'),$$

by the first assertion of Lemma 9.5.9, we have

$$(\operatorname{res}.\operatorname{alg}.\deg(\operatorname{Trop}(\phi'_l(w_{l'}))), L_b(\operatorname{Trop}(\phi'_l(w_{l'})))) = (\operatorname{res}.\operatorname{alg}.\deg(l'), L_b(l')).$$

PROPOSITION 9.5.15. *There exist a sextuplet $(\phi, \phi', \Phi, \Theta, \Theta')$ dominating $(\varphi, \varphi', \Psi, \Lambda, \Lambda')$ such that the pull back of α is equivalent to 0.*

PROOF. This follows from Remark 9.5.11, 9.5.14, and Lemma 9.5.13, by induction on $(\operatorname{res}.\operatorname{alg}.\deg, L_b) \in \mathbb{Z} \times \mathbb{Z}[\frac{1}{2}]$ in the increasing order. $\square$

Consequently, we have the following.

COROLLARY 9.5.16.

$$(R^1\pi^{\circ}_{*}\mathscr{F}^{p}_{(X\times\mathbb{A}^1)^{\circ}})_{v_0} = 0.$$

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