

Effective mass and mass renormalization of nonrelativistic QED

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Abstract

The effective mass m_{eff} of the nonrelativistic QED is considered. m_{eff} is defined as the inverse of curvature of the ground state energy with total momentum zero. The effective mass $m_{\text{eff}} = m_{\text{eff}}(e^2, \Lambda, \kappa, m)$ is a function of bear mass $m > 0$, ultraviolet cutoff $\Lambda > 0$, infrared cutoff $\kappa > 0$, and the square of charge e of an electron. Introduce a scaling $m \rightarrow m(\Lambda) = (b\Lambda)^\beta$, $\beta < 0$. Then asymptotics behavior of m_{eff} as $\Lambda \rightarrow \infty$ is studied.

1 Introduction

1.1 The Pauli-Fierz Hamiltonian

This is a joint work with Herbert Spohn.¹ We consider a single, spinless free electron coupled to a quantized radiation field (photons). The Hilbert space of states of photons is the symmetric Fock space:

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \left[\otimes_s^n L^2(\mathbb{R}^3 \times \{1, 2\}) \right],$$

where $\otimes_s^n L^2(\mathbb{R}^3 \times \{1, 2\})$ denotes the n -fold symmetric tensor product of $L^2(\mathbb{R}^3 \times \{1, 2\})$ with $\otimes_s^0 L^2(\mathbb{R}^3 \times \{1, 2\}) = \mathbb{C}$. The inner product in $\mathcal{F}$ is denoted by $(\cdot, \cdot)$ and the Fock vacuum by Ω . On $\mathcal{F}$ we introduce the Bose field

$$a(f) = \sum_{j=1,2} \int f(k, j)^* a(k, j) dk, \quad f \in L^2(\mathbb{R}^3 \times \{1, 2\}), \quad (1.1)$$

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where $a(f)$ and $a^*(f) = a(\bar{f})^*$ are densely defined and satisfy the CCR

$$\begin{aligned} [a(f), a^*(g)] &= (f, g)_{L^2(\mathbb{R}^3 \times \{1,2\})}, \\ [a(f), a(g)] &= 0, \\ [a^*(f), a^*(g)] &= 0. \end{aligned}$$

The free Hamiltonian of $\mathcal{F}$ is read as

$$H_f = \sum_{j=1,2} \int \omega(k) a^*(k, j) a(k, j) dk, \quad (1.2)$$

where the dispersion relation is given by

$$\omega(k) = |k|.$$

The free Hamiltonian H_f acts as

$$\begin{aligned} H_f \Omega &= 0, \\ H_f a^*(f_1) \cdots a^*(f_n) \Omega &= \sum_{j=1}^n a^*(f_1) \cdots a^*(\omega f_j) \cdots a^*(f_n) \Omega. \end{aligned}$$

The Pauli-Fierz Hamiltonian H is defined as a self-adjoint operator acting on

$$\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathcal{F} \cong \int_{\mathbb{R}^3}^{\oplus} \mathcal{F} dx$$

by

$$H = \frac{1}{2m} (p_x \otimes 1 - e A_{\hat{\varphi}})^2 + V \otimes 1 + 1 \otimes H_f,$$

where m and e denote the mass and charge of electron, respectively,

$$p_x = \left(-i \frac{\partial}{\partial x_1}, -i \frac{\partial}{\partial x_2}, -i \frac{\partial}{\partial x_3} \right)$$

and V an external potential. The quantized radiation field $A_{\hat{\varphi}}$ is defined by

$$A_{\hat{\varphi}} = \frac{1}{\sqrt{2}} \int_{\mathbb{R}^3}^{\oplus} (a(f_x) + a^*(\bar{f}_x)) dx, \quad (1.3)$$

where

$$f_x(k, j) = \frac{1}{\sqrt{\omega}} \hat{\varphi}(k) e(k, j) e^{ikx}, \quad (1.4)$$

$e(k, 1), e(k, 2), k/|k|$ form a right-handed dreibain, and $\hat{\varphi}$ is a form factor. $A_{\hat{\varphi}}$ acts for $\Psi \in \mathcal{H}$ as

$$(A_{\hat{\varphi}} \Psi)(x) = (a(f_x) + a^*(\bar{f}_x)) \Psi(x), \quad x \in \mathbb{R}^3.$$

Theorem 1.1 Assume that $\hat{\varphi}/\omega, \hat{\varphi}/\sqrt{\omega}, \sqrt{\omega}\hat{\varphi} \in L^2(\mathbb{R}^3)$ and V is relatively bounded with respect to $-\Delta$ with a relative bound < 1 . Then, for arbitrary values of e , H is self-adjoint on $D(\Delta \otimes 1) \cap D(1 \otimes H_f)$ and bounded from below.

Proof: See Hiroshima [3, 4]. $\square$

1.2 Effective mass

The momentum of the photon field is given by

$$P_f = \sum_{j=1,2} \int k a^*(k, j) a(k, j) dk \quad (1.5)$$

and the total moment by

$$P_{\text{total}} = p_x \otimes 1 + 1 \otimes P_f.$$

Let us assume that

$$V \equiv 0.$$

Then we see that

$$[H, P_{\text{total}\mu}] = 0, \quad \mu = 1, 2, 3.$$

Hence H and $\mathcal{H}$ can be decomposable with respect to $\text{Spec}(P_{\text{total}}) = \mathbb{R}^3$, i.e.,

$$\begin{aligned} \mathcal{H} &= \int_{\mathbb{R}^3}^{\oplus} \mathcal{H}(p) dp, \\ H &= \int_{\mathbb{R}^3}^{\oplus} H(p) dp. \end{aligned}$$

Note that

$$\begin{aligned} e^{-ix \otimes P_f} P_{\text{total}} e^{ix \otimes P_f} &= p_x, \\ e^{-ix \otimes P_f} H e^{ix \otimes P_f} &= \frac{1}{2m} (p_x \otimes 1 - 1 \otimes P_f - e 1 \otimes A_{\hat{\varphi}}(0)) + 1 \otimes H_f, \end{aligned}$$

where

$$A_{\hat{\varphi}}(0) = \frac{1}{\sqrt{2}} (a(f_0) + a(\bar{f}_0)).$$

From this we obtain that for each $p \in \mathbb{R}^3$,

$$\begin{aligned} \mathcal{H}(p) &\cong \mathcal{F}, \\ H(p) &\cong \frac{1}{2m} (p - P_f - e A_{\hat{\varphi}}(0)) + H_f, \end{aligned}$$

Let

$$E_{m,\Lambda}(p) = \inf \text{Spec}(H(p)). \quad (1.6)$$

Let us assume sharp ultraviolet cutoff Λ and infrared cutoff κ , which means

$$\hat{\varphi}(k) = \begin{cases} 0 & \text{for } |k| < \kappa, \\ (2\pi)^{-3/2} & \text{for } \kappa \leq |k| \leq \Lambda, \\ 0 & \text{for } |k| > \Lambda. \end{cases} \quad (1.7)$$

Lemma 1.2 *There exists constants p_* and e_* such that for*

$$(p, e) \in \mathcal{O} = \{(p, e) \in \mathbb{R}^3 \times \mathbb{R} \mid |p| < p_*, |e| < e^*\},$$

$H(p)$ has a ground state $\psi_g(p)$ and it is unique. Moreover $\psi_g(p) = \psi_g(p, e)$ is strongly analytic and $E_{m,\Lambda}(p) = E_{m,\Lambda}(p, e)$ analytic with respect to $(p, e) \in \mathcal{O}$.

Proof: See Hiroshima and Spohn [6, 7]. □

In what follows we assume that $(p, e) \in \mathcal{O}$.

Definition 1.3 *The effective mass $m_{\text{eff}} = m_{\text{eff}}(e^2, \Lambda, \kappa, m)$ is defined by*

$$\frac{1}{m_{\text{eff}}} = \frac{1}{3} \Delta_p E(p, e)|_{p=0}. \quad (1.8)$$

1.3 Mass renormalization

Removal of the ultraviolet cutoff Λ through mass renormalization means to find sequences

$$\Lambda \rightarrow \infty, \quad m \rightarrow 0 \quad (1.9)$$

such that $E_{m,\Lambda}(p) - E_{m,\Lambda}(0)$ has a nondegenerate limit. To achieve this, as a first step we want to find constants

$$\beta < 0, \quad 0 < b$$

such that

$$\lim_{\Lambda \rightarrow \infty} m_{\text{eff}}(e^2, \Lambda, \kappa \Lambda^\beta, (b\Lambda)^\beta) = m_{\text{ph}}, \quad (1.10)$$

where m_{ph} is a given constant. Actually m_{ph} is a physical mass. Namely in the mass renormalization the scaled bare mass goes to zero and the effective mass goes to a physical mass as the ultraviolet cutoff Λ goes to infinity.

We will see later that m_{eff}/m is a function of e^2 , Λ/m and κ/m . Let

$$\frac{m_{\text{eff}}}{m} = f(e^2, \Lambda/m, \kappa/m), \quad (1.11)$$

where $f(0, \Lambda/m, \kappa/m) = 1$ holds. An analysis of (1.10) can be reduced to investigate the asymptotic behavior of f as $\Lambda \rightarrow \infty$. Namely we want to find constants

$$0 \leq \gamma < 1, \quad 0 < b_0$$

such that

$$\lim_{\Lambda \rightarrow \infty} \frac{f(e^2, \Lambda/m, \kappa/m)}{(\Lambda/m)^\gamma} = b_0. \quad (1.12)$$

If we succeed to find constants γ and b_0 such as in (1.12) then by

$$m_{\text{eff}}(e^2, \Lambda, \kappa, m) = m f(e^2, \Lambda/m, \kappa/m),$$

we have

$$m_{\text{eff}}(e^2, \Lambda, \kappa \Lambda^\beta, (b\Lambda)^\beta) = (b\Lambda)^\beta f(e^2, \Lambda/(b\Lambda)^\beta, \kappa/b^\beta) \approx b_0 (b\Lambda)^\beta (\Lambda/(b\Lambda)^\beta)^\gamma. \quad (1.13)$$

Taking

$$\beta = \frac{-\gamma}{1-\gamma} < 0, \quad b = 1/b_1^{1/\gamma},$$

we see that by (1.13)

$$\lim_{\Lambda \rightarrow \infty} m_{\text{eff}}(e^2, \Lambda, \kappa \Lambda^\beta, (b\Lambda)^\beta) = \lim_{\Lambda \rightarrow \infty} b_0 \left(\frac{\Lambda}{b_1^{1/\gamma}} \right)^\beta \left(\frac{\Lambda}{(\Lambda/(b_1)^{1/\gamma})^\beta} \right)^\gamma = b_0 b_1,$$

where b_1 is a parameter, which is adjusted such as

$$b_0 b_1 = m_{\text{ph}}.$$

Hence we will be able to establish (1.10). It is easily seen that

$$f(e^2, \Lambda/m, \kappa/m) = 1 + \alpha \frac{8}{3\pi} \log\left(\frac{\Lambda/m + 2}{\kappa/m + 2}\right) + O(\alpha^2),$$

where $\alpha = e^2/4\pi$, which suggests

$$f(e^2, \Lambda/m, \kappa/m) \approx (\Lambda/m)^{8\alpha/3\pi},$$

for sufficiently small α and large Λ , and therefore

$$\gamma = 8\alpha/3\pi.$$

One may assume that

$$f(e^2, \Lambda/m, \kappa/m) \approx (\Lambda/m)^{\alpha(8/3\pi) + \alpha^2 b}$$

for sufficiently small α with some constant b . Then by expanding m_{eff}/m to order α^2 one may expect that

$$f(e^2, \Lambda/m, \kappa/m) \approx 1 + \alpha \frac{8}{3\pi} \log\left(\frac{\Lambda}{m}\right) + \frac{1}{2} \alpha^2 \left(\frac{8}{3\pi} \log\left(\frac{\Lambda}{m}\right) \right)^2 + b \alpha^2 \log\left(\frac{\Lambda}{m}\right) + O(\alpha^3) \quad (1.14)$$

for sufficiently small α and large Λ . It is, however, that (1.14) is not confirmed. Instead of (1.14) we prove that there exists a constant $C > 0$ such that

$$f(e^2, \Lambda/m, \kappa/m) = 1 + \alpha \frac{8}{3\pi} \log\left(\frac{\Lambda/m + 2}{\kappa/m + 2}\right) + \alpha^2 C \sqrt{\Lambda/m} + O(\alpha^3).$$

The effective mass and its renormalization have been studied from a mathematical point of view by many authors. Spohn [10] investigates the effective mass of the Nelson model [9] from a functional integral point of view. Lieb and Loss [8] studied mass renormalization and binding energies of models of matter coupled to radiation fields including the Pauli-Fierz model. Hainzl and Seiringer [2] computed exactly the leading order in α of the effective mass of the Pauli-Fierz Hamiltonian with spin.

2 Perturbative expansions

The effective masses for $H(p)$ and

$$\frac{1}{2m} : (p - P_f - eA_{\hat{\varphi}}(0))^2 : + H_f$$

are identical. Then in what follows we redefine $H(p)$ as

$$H(p) = \frac{1}{2m} : (p - P_f - eA_{\hat{\varphi}}(0))^2 : + H_f.$$

Furthermore for notational convenience we write A and $E(p)$ for $A_{\hat{\varphi}}(0)$ and $E_{m,\Lambda}(p)$, respectively.

2.1 Formulae

Lemma 2.1 *We have*

$$\frac{m}{m_{\text{eff}}} = 1 - \frac{2}{3} \sum_{\mu=1,2,3} \frac{(\psi_g(0), (P_f + eA)_{\mu} (H(0) - E(0))^{-1} (P_f + eA)_{\mu} \psi_g(0))}{(\psi_g(0), \psi_g(0))}.$$

Proof: It is seen that $E(p, e) = E(p, -e) = E(-p, e)$. Then

$$\left. \frac{\partial}{\partial p_{\mu}} E(p, e) \right|_{p_{\mu}=0} = 0, \quad \mu = 1, 2, 3, \quad (2.1)$$

follows. Moreover it is seen that $E(p, e)$ is a function of e^2 and

$$\left. \frac{d^{2m-1}}{de^{2m-1}} E(p, e) \right|_{e=0} = 0. \quad (2.2)$$

In this proof, $f'(p)_{\mu}$ means the strong derivative of $f(p)$ with respect to p_{μ} . Since

$$H(p)\psi_g(p) = E(p)\psi_g(p),$$

we have

$$H'(p)_{\mu}\psi_g(p) + H(p)\psi_g'(p)_{\mu} = E'(p)_{\mu}\psi_g(p) + E(p)\psi_g'(p)_{\mu} \quad (2.3)$$

and

$$\begin{aligned} & H''(p)_{\mu}\psi_g(p) + 2H'(p)_{\mu}\psi_g'(p)_{\mu} + H(p)\psi_g''(p)_{\mu} \\ &= E''(p)_{\mu}\psi_g(p) + 2E'(p)_{\mu}\psi_g'(p)_{\mu} + E(p)\psi_g''(p)_{\mu}. \end{aligned} \quad (2.4)$$

By (2.1) it follows that $E'(0)_{\mu} = 0$, and by (2.3) with $p = 0$,

$$\begin{aligned} & (P_f + eA)_{\mu}\psi_g(0) \in D((H(0) - E(0))^{-1}), \\ & \psi_g'(0)_{\mu} = (H(0) - E(0))^{-1} (P_f + eA)_{\mu}\psi_g(0). \end{aligned}$$

Then we have by (2.3) and (2.4),

$$\begin{aligned} \frac{m}{m_{\text{eff}}} &= \frac{1}{3} \sum_{\mu=1,2,3} \frac{(\psi_g(0), E''(0)_\mu \psi_g(0))}{(\psi_g(0), \psi_g(0))} \\ &= 1 - \frac{2}{3} \sum_{\mu=1,2,3} \frac{((P_f + eA)_\mu \psi_g(0), (H(0) - E(0))^{-1} (P_f + eA)_\mu \psi_g(0))}{(\psi_g(0), \psi_g(0))} \end{aligned}$$

Thus the lemma follows. $\square$

Let

$$\psi_g(0) = \sum_{n=0}^{\infty} \frac{e^n}{n!} \varphi_n, \quad E(0) = \sum_{n=0}^{\infty} \frac{e^{2n}}{(2n)!} E_{2n}.$$

Note that

$$\varphi_{2m} \in \bigoplus_{m=0}^{\infty} \mathcal{F}^{(2m)}, \quad \varphi_{2m+1} \in \bigoplus_{m=0}^{\infty} \mathcal{F}^{(2m+1)}.$$

We want to get the explicit form of φ_n . Let

$$\begin{aligned} \mathcal{F}_{\text{fin}} &= \{ \{ \Psi^{(n)} \}_{n=0}^{\infty} \in \mathcal{F} \mid \Psi^{(m)} = 0 \text{ for } m \geq \ell \text{ with some } \ell \}, \\ \mathcal{F}_0 &= \left\{ \{ \Psi^{(n)} \}_{n=0}^{\infty} \in \mathcal{F}_{\text{fin}} \mid \begin{aligned} &\text{(i) } \Psi^{(0)} = 0, \\ &\text{(ii) } \text{supp}_{(k_1, \dots, k_n) \in \mathbb{R}^{3n}} \Psi^{(n)}(k_1, \dots, k_n, j_1, \dots, j_n) \not\supset \{(0, \dots, 0)\} \end{aligned} \right\}. \end{aligned}$$

Lemma 2.2 *We see that $\mathcal{F}_0 \subset D(H_0^{-1})$.*

Proof: Let $\Psi = \{ \Psi^{(n)} \}_{n=0}^{\infty} \in \mathcal{F}_0$. Since

$$\begin{aligned} &(H_0 \Psi)^{(n)}(k_1, \dots, k_n, j_1, \dots, j_n) \\ &= \left[\frac{1}{2} (k_1 + \dots + k_n)^2 + \sum_{j=1}^n \omega(k_j) \right] \Psi^{(n)}(k_1, \dots, k_n, j_1, \dots, j_n), \end{aligned}$$

we see that

$$\begin{aligned} &(H_0^{-1} \Psi)^{(n)}(k_1, \dots, k_n, j_1, \dots, j_n) \\ &= \left[\frac{1}{2} (k_1 + \dots + k_n)^2 + \sum_{j=1}^n \omega(k_j) \right]^{-1} \Psi^{(n)}(k_1, \dots, k_n, j_1, \dots, j_n). \end{aligned}$$

Since $\text{supp}_{(k_1, \dots, k_n) \in \mathbb{R}^{3n}} \Psi^{(n)}(k_1, \dots, k_n, j_1, \dots, j_n) \not\supset \{(0, \dots, 0)\}$, we obtain that

$$\|H_0^{-1} \Psi\|_{\mathcal{F}}^2 = \sum_{n=1}^{\text{finite}} \|(H_0^{-1} \Psi)^{(n)}\|_{\mathcal{F}^{(n)}}^2 < \infty.$$

Then the lemma follows. $\square$

We split $H(0)$ as

$$H(0) = H_0 + eH_1 + \frac{e^2}{2} H_2,$$

where

$$\begin{aligned} H_0 &= \frac{1}{2}P_f^2 + H_f, \\ H_1 &= \frac{1}{2}(P_f \cdot A + A \cdot P_f) = P_f \cdot A = A \cdot P_f, \\ H_2 &=: A^2: . \end{aligned}$$

Lemma 2.3 *We have $E_0 = E_1 = E_2 = E_3 = 0$ and*

$$\varphi_0 = \Omega, \quad \varphi_1 = 0, \quad \varphi_2 = -H_0^{-1}H_2\Omega, \quad \varphi_3 = 3H_0^{-1}H_1H_0^{-1}H_2\Omega.$$

In particular $\varphi_2 \in \mathcal{F}^{(2)}$ and $\varphi_3 \in \mathcal{F}^{(1)} \cap \mathcal{F}^{(3)}$.

Proof: Let us set $H(0)$, $E(0)$ and $\psi_g(0)$ as H , E and ψ_g , respectively. It is obvious that $E_0 = 0$ and $\varphi_0 = a\Omega$ with arbitrary $a \in \mathbb{C}$, and by (2.2), $E_1 = E_3 = 0$. Set $a = 1$. We denote the strong derivative of $f = f(e)$ with respect to e by f' . We have

$$H'\psi_g + H\psi_g' = E'\psi_g + E\psi_g' \quad (2.5)$$

and

$$H''\psi_g + 2H'\psi_g' + H\psi_g'' = E''\psi_g + 2E'\psi_g' + E\psi_g'' \quad (2.6)$$

From (2.6) it follows that

$$(\psi_g, H''\psi_g) + (\psi_g, 2H'\psi_g') + (\psi_g, H\psi_g'') = E''(\psi_g, \psi_g) + (\psi_g, 2E'\psi_g') + (\psi_g, E\psi_g''). \quad (2.7)$$

Put $e = 0$ in (2.7). Then

$$(\Omega, H_2\Omega) + (\Omega, 2H_1\Omega) + (\Omega, H_0\varphi_2) = E_2(\Omega, \Omega). \quad (2.8)$$

Since the left-hand side of (2.8) vanishes, we have $E_2 = 0$. From (2.5) with $e = 0$ and the fact $E_0 = E_1 = 0$, it follows that

$$H_1\Omega + H_0\varphi_1 = 0,$$

from which it holds that $H_0\varphi_1 = 0$. Since H_0 has the unique eigenvector Ω (the ground state) with eigenvalue zero, it follows that $\varphi_1 = b\Omega$ with some constant b . $\varphi_1 \in \bigoplus_{m=0}^{\infty} \mathcal{F}^{(2m+1)}$ which implies $b = 0$. Hence $\varphi_1 = 0$ follows. By (2.6) with $e = 0$, we have

$$H_2\Omega + 2H_1\varphi_1 + H_0\varphi_2 = 0.$$

Since $H_2\Omega \in \mathcal{F}_0$, we see that by Lemma 2.2, $H_2\Omega \in D(H_0^{-1})$. Thus we have $\varphi_2 = -H_0^{-1}H_2\Omega$. From the identity

$$H'''\psi_g + 3H''\psi_g' + 3H'\psi_g'' + H\psi_g''' = E'''\psi_g + 3E''\psi_g' + 3E'\psi_g'' + E\psi_g''' \quad (2.9)$$

it follows that at $e = 0$,

$$3H_1\varphi_2 + H_0\varphi_3 = 0.$$

Since $H_1\varphi_2 = -H_1H_0^{-1}H_2\Omega \in \mathcal{F}_0$, Lemma 2.2 ensures that $H_1\varphi_2 \in D(H_0^{-1})$. Hence $\varphi_3 = -3H_0^{-1}H_1\varphi_2 = 3H_0^{-1}H_1H_0^{-1}H_2\Omega$. Then the lemma is proven. $\square$

2.2 Order e^4

In this subsection we expand m/m_{eff} up to order e^4 . We define A^- and A^+ by

$$A^- = \frac{1}{\sqrt{2}}a(f), \quad A^+ = \frac{1}{\sqrt{2}}a^*(f).$$

Then $A = A^+ + A^-$.

Lemma 2.4 *We have*

$$\begin{aligned} \frac{m}{m_{\text{eff}}} &= 1 - e^2 \frac{2}{3} \sum_{\mu=1}^3 \left(\Omega, A_\mu H_0^{-1} A_\mu \Omega \right) \\ &\quad - e^4 \frac{2}{3} \sum_{\mu=1}^3 \left\{ 2 \left(\Psi_3^\mu, H_0^{-1} \Psi_1^\mu \right) + \left(\Psi_2^\mu, H_0^{-1} \Psi_2^\mu \right) - 2 \left(\Psi_2^\mu, H_0^{-1} H_1 H_0^{-1} \Psi_1^\mu \right) \right. \\ &\quad \left. - \frac{1}{2} \left(\Psi_1^\mu, H_0^{-1} H_2 H_0^{-1} \Psi_1^\mu \right) + \left(\Psi_1^\mu, H_0^{-1} H_1 H_0^{-1} H_1 H_0^{-1} \Psi_1^\mu \right) \right\} + O(e^6), \end{aligned} \quad (2.10)$$

where

$$\begin{aligned} \Psi_1^\mu &= A_\mu \Omega, \\ \Psi_2^\mu &= -\frac{1}{2} P_{\text{f}\mu} H_0^{-1} (A^+ \cdot A^+) \Omega, \\ \Psi_3^\mu &= \frac{1}{2} \left\{ -A_\mu H_0^{-1} (A^+ \cdot A^+) \Omega + \frac{1}{2} P_{\text{f}\mu} H_0^{-1} (P_{\text{f}} \cdot A + A \cdot P_{\text{f}}) H_0^{-1} (A^+ \cdot A^+) \Omega \right\}. \end{aligned}$$

Proof: In Lemma 2.1 we have seen that

$$\frac{m}{m_{\text{eff}}} = 1 - \frac{2}{3} \sum_{\mu=1,2,3} \frac{((P_{\text{f}} + eA)_\mu \psi_{\text{g}}(0), (H(0) - E(0))^{-1} (P_{\text{f}} + eA)_\mu \psi_{\text{g}}(0))}{(\psi_{\text{g}}(0), \psi_{\text{g}}(0))}. \quad (2.11)$$

We can strongly expand $(H(0) - E(0))^{-1}$ as

$$\begin{aligned} (H(0) - E(0))^{-1} &= H_0^{-1} - e H_0^{-1} H_1 H_0^{-1} \\ &\quad + e^2 \left(-\frac{1}{2} H_0^{-1} H_2 H_0^{-1} + H_0^{-1} H_1 H_0^{-1} H_1 H_0^{-1} \right) + O(e^3). \end{aligned} \quad (2.12)$$

Here we set

$$H_j = \begin{cases} H_j, & j = 1, 2, \\ -E_j, & j \geq 3. \end{cases}$$

Note that

$$\varphi_0 \in \mathcal{F}^{(0)}, \varphi_2 \in \mathcal{F}^{(2)}, \varphi_3 \in \mathcal{F}^{(3)} \cap \mathcal{F}^{(1)}, \varphi_4 \in \mathcal{F}^{(4)} \cap \mathcal{F}^{(2)}.$$

In particular

$$\frac{1}{(\psi_g, \psi_g)} = 1 - e^4 \left(\frac{1}{2} \varphi_2, \frac{1}{2} \varphi_2 \right) - e^4 \left(\Omega, \frac{1}{24} \varphi_4 \right) + O(e^6) = 1 - e^4 \frac{1}{4} (\varphi_2, \varphi_2) + O(e^6). \quad (2.13)$$

Moreover we have

$$\begin{aligned} (P_f + eA)_\mu \psi_g(0) &= eA_\mu \Omega + e^2 \left(\frac{1}{2} P_{f\mu} \varphi_2 \right) + e^3 \left(\frac{1}{2} A_\mu \varphi_2 + \frac{1}{6} P_{f\mu} \varphi_3 \right) + O(e^4) \\ &= e\Psi_1^\mu + e^2\Psi_2^\mu + e^3\Psi_3^\mu + O(e^4). \end{aligned} \quad (2.14)$$

Substitute (2.12), (2.13) and (2.14) into (2.11). Then the lemma follows. $\square$

For each $k \in \mathbb{R}^3$ let us define the projection $Q(k)$ on $\mathbb{R}^3$ by

$$Q(k) = \sum_{j=1,2} |e_j(k)\rangle \langle e_j(k)|.$$

We set

$$\hat{\varphi}_j = \hat{\varphi}(k_j), \quad \omega_j = \omega(k_j), \quad Q(k_j) = Q_j, \quad j = 1, 2.$$

Let

$$\begin{aligned} \frac{1}{F_j} &= \frac{1}{r_j^2/2 + r_j}, \quad j = 1, 2, \\ \frac{1}{F_{12}} &= \frac{1}{(r_1^2 + 2r_1r_2X + r_2^2)/2 + r_1 + r_2}, \quad r_1, r_2 \geq 0, \quad -1 \leq X \leq 1. \end{aligned}$$

Lemma 2.5 *We have*

$$\frac{m}{m_{\text{eff}}} = 1 - \alpha a_1(\Lambda/m, \kappa/m) - \alpha^2 a_2(\Lambda/m, \kappa/m) + O(\alpha^3),$$

where

$$a_1(\Lambda/m, \kappa/m) = \frac{8}{3\pi} \log \left(\frac{\Lambda/m + 2}{\kappa/m + 2} \right) \quad (2.15)$$

and

$$\begin{aligned} &a_2(\Lambda/m, \kappa/m) \\ &= \frac{(4\pi)^2}{(2\pi)^6} \frac{2}{3} \int_{-1}^1 dX \int_{\kappa/m}^{\Lambda/m} dr_1 \int_{\kappa/m}^{\Lambda/m} dr_2 \pi r_1 r_2 \times \\ &\times \left\{ - \left(\frac{1}{F_1} + \frac{1}{F_2} \right) \frac{1}{F_{12}} (1 + X^2) + \left(\frac{1}{F_{12}} \right)^3 \frac{r_1^2 + 2r_1r_2X + r_2^2}{2} (1 + X^2) \right. \\ &+ \left(\frac{1}{F_1} + \frac{1}{F_2} \right) \left(\frac{1}{F_{12}} \right)^2 r_1 r_2 X (-1 + X^2) - \frac{1}{F_1} \frac{1}{F_2} (1 + X^2) \\ &\left. + \left(\frac{r_1^2}{F_1^2} + \frac{r_2^2}{F_2^2} \right) \frac{1}{F_{12}} (1 - X^2) + \frac{1}{F_1} \frac{1}{F_2} \frac{1}{F_{12}} r_1 r_2 X (-1 + X^2) \right\}. \end{aligned} \quad (2.16)$$

Proof: Note that

$$\begin{aligned} a_1(\Lambda, \kappa) &= \frac{2}{3}(\sqrt{4\pi})^2(A_\mu^+\Omega, H_0^{-1}A_\mu^+\Omega) \\ &= \frac{8}{3\pi} \log\left(\frac{\Lambda/m+2}{\kappa/m+2}\right). \end{aligned}$$

Thus (2.15) follows. To see $a_2(\Lambda, \kappa)$ we exactly compute the five terms on the right-hand side of (2.10) separately. Let

$$\begin{aligned} \frac{1}{E_j} &= \frac{1}{|k_j|^2/2 + \omega_j}, \quad j = 1, 2, \\ \frac{1}{E_{12}} &= \frac{1}{|k_1 + k_2|^2/2 + \omega_1 + \omega_2}. \end{aligned}$$

(1) We have

$$\begin{aligned} 2(\Psi_3^\mu, H_0^{-1}\Psi_1^\mu) &= (\Omega, -(A^- \cdot A^-)H_0^{-1}A_\mu H_0^{-1}A_\mu^+\Omega) \\ &+ \frac{1}{2}(\Omega, (A^- \cdot A^-)H_0^{-1}(P_f \cdot A + A \cdot P_f)H_0^{-1}P_{f\mu}H_0^{-1}A_\mu^+\Omega). \\ &= - \iint dk_1^3 dk_2^3 \frac{|\hat{\varphi}_1|^2}{2\omega_1} \frac{|\hat{\varphi}_2|^2}{2\omega_2} \frac{1}{E_{12}} \left(\frac{1}{E_1} + \frac{1}{E_2}\right) \text{tr}(Q_1 Q_2). \end{aligned} \quad (2.17)$$

(2) We have

$$\begin{aligned} &(\Psi_2^\mu, H_0^{-1}\Psi_2^\mu) \\ &= \left(\frac{1}{2}\right)^2 (P_{f\mu}H_0^{-1}(A^+ \cdot A^+)\Omega, H_0^{-1}P_{f\mu}H_0^{-1}(A^+ \cdot A^+)\Omega) \\ &= \left(\frac{1}{2}\right)^2 \iint dk_1^3 dk_2^3 \frac{|\hat{\varphi}_1|^2}{2\omega_1} \frac{|\hat{\varphi}_2|^2}{2\omega_2} \left(\frac{1}{E_{12}}\right)^3 |k_1 + k_2|^2 2\text{tr}(Q_1 Q_2). \end{aligned} \quad (2.18)$$

(3) We have

$$\begin{aligned} &-2(\Psi_2^\mu, H_0^{-1}H_1H_0^{-1}\Psi_1^\mu) \\ &= \frac{1}{2}(P_{f\mu}H_0^{-1}(A^+ \cdot A^+)\Omega, H_0^{-1}(P_f \cdot A + A \cdot P_f)H_0^{-1}A_\mu^+\Omega) \\ &= \iint dk_1^3 dk_2^3 \frac{|\hat{\varphi}_1|^2}{2\omega_1} \frac{|\hat{\varphi}_2|^2}{2\omega_2} \left(\frac{1}{E_{12}}\right)^2 \left(\frac{1}{E_1} + \frac{1}{E_2}\right) (k_2, Q_1 Q_2 k_1). \end{aligned} \quad (2.19)$$

(4) We have

$$\begin{aligned} &-\frac{1}{2}(\Psi_1^\mu, H_0^{-1}H_2H_0^{-1}\Psi_1^\mu) \\ &= -\frac{1}{2}(A_\mu^+\Omega, H_0^{-1}((A^+ \cdot A^+) + 2(A^+ \cdot A^-) + (A^- \cdot A^-))H_0^{-1}A_\mu^+\Omega) \\ &= - \iint dk_1^3 dk_2^3 \frac{|\hat{\varphi}_1|^2}{2\omega_1} \frac{|\hat{\varphi}_2|^2}{2\omega_2} \frac{1}{E_1} \frac{1}{E_2} \text{tr}(Q_1 Q_2). \end{aligned} \quad (2.20)$$

(5) We have

$$\begin{aligned}
& \left(\Psi_1^\mu, H_0^{-1} H_1 H_0^{-1} H_1 H_0^{-1} \Psi_1^\mu \right) \\
&= \left(\frac{1}{2} \right)^2 \left(A_\mu^+ \Omega, H_0^{-1} (P_f \cdot A + A \cdot P_f) H_0^{-1} (P_f \cdot A + A \cdot P_f) H_0^{-1} A_\mu^+ \Omega \right) \\
&= \iint dk_1^3 dk_2^3 \frac{|\hat{\varphi}_1|^2}{2\omega_1} \frac{|\hat{\varphi}_2|^2}{2\omega_2} \frac{1}{E_{12}} \left\{ \left(\frac{1}{E_1} \right)^2 (k_1, Q_2 k_1) + \left(\frac{1}{E_2} \right)^2 (k_2, Q_1 k_2) \right\} \\
&+ \iint dk_1^3 dk_2^3 \frac{|\hat{\varphi}_1|^2}{2\omega_1} \frac{|\hat{\varphi}_2|^2}{2\omega_2} \frac{1}{E_{12}} \frac{1}{E_1} \frac{1}{E_2} (k_2, Q_1 Q_2 k_1). \tag{2.21}
\end{aligned}$$

Changing variables to the polar coordinate, we obtain (2.16) from Lemma 2.4, (2.17), (2.18), (2.19), (2.20), (2.21) and the facts

$$\begin{aligned}
\text{tr}[Q_1 Q_2] &= 1 + (\hat{k}_1, \hat{k}_2)^2, \\
(k_1, Q_2 Q_1 k_2) &= (k_1, k_2) ((\hat{k}_1, \hat{k}_2)^2 - 1), \\
(k_1, Q_2 k_1) &= |k_1|^2 (1 - (\hat{k}_1, \hat{k}_2)^2).
\end{aligned}$$

Thus the proof is complete. $\square$

3 Main theorem

The main theorem is as follows.

Theorem 3.1 *There exist strictly positive constants $C_{\min}$ and $C_{\max}$ such that*

$$C_{\min} \leq \lim_{\Lambda \rightarrow \infty} \frac{a_2(\Lambda/m, \kappa/m)}{\sqrt{\Lambda/m}} \leq C_{\max}.$$

Proof: We show an outline of a proof. See Hiroshima and Spohn [7] for details. By (2.16) we can see that

$$a_2(\Lambda, \kappa) = \frac{(4\pi)^2}{(2\pi)^6} \frac{2}{3} \sum_{j=1}^6 b_j(\Lambda/m), \tag{3.1}$$

where

$$\begin{aligned}
b_1(\Lambda/m) &= - \int (1 + X^2) \left(\frac{1}{F_1} + \frac{1}{F_2} \right) \frac{1}{F_{12}}, \\
b_2(\Lambda/m) &= \int (1 + X^2) \left(\frac{1}{F_{12}} \right)^3 \frac{r_1^2 + 2r_1 r_2 X + r_2^2}{2}, \\
b_3(\Lambda/m) &= \int X(-1 + X^2) r_1 r_2 \left(\frac{1}{F_1} + \frac{1}{F_2} \right) \left(\frac{1}{F_{12}} \right)^2, \\
b_4(\Lambda/m) &= - \int (1 + X^2) \frac{1}{F_1} \frac{1}{F_2}, \\
b_5(\Lambda/m) &= \int (1 - X^2) \left(\frac{r_1^2}{F_1^2} + \frac{r_2^2}{F_2^2} \right) \frac{1}{F_{12}}, \\
b_6(\Lambda/m) &= \int X(-1 + X^2) r_1 r_2 \frac{1}{F_1} \frac{1}{F_2} \frac{1}{F_{12}},
\end{aligned}$$

where

$$\int = \int_{-1}^1 dX \int_{\kappa/m}^{\Lambda/m} dr_1 \int_{\kappa/m}^{\Lambda/m} dr_2 \pi r_1 r_2.$$

Let $\rho_\Lambda(\cdot, \cdot) : [0, \infty) \times [-1, 1] \rightarrow \mathbb{R}$ be defined by

$$\rho_\Lambda = \rho_\Lambda(r, X) = r^2 + 2\Lambda rX + \Lambda^2 + 2r + 2\Lambda = (r + \Lambda X + 1)^2 + \Delta,$$

where

$$\Delta = \Lambda^2(1 - X^2) + 2\Lambda(1 - X) - 1. \quad (3.2)$$

Then we can show that there exist constants C_1, C_2, C_3 and C_4 such that for sufficiently large $\Lambda > 0$,

$$\begin{aligned} (1) \quad & \int_{-1}^1 dX \int_0^\Lambda dr \frac{1}{\rho_\Lambda(r, X)} \leq C_1 \frac{1}{\Lambda}, \\ (2) \quad & \int_{-1}^1 dX \int_0^\Lambda dr \left(\frac{1}{\rho_\Lambda(r, X)} \right)^2 \leq C_2 \frac{1}{\Lambda^{5/2}}, \\ (3) \quad & \int_{-1}^1 dX \int_0^\Lambda dr \frac{1}{\rho_\Lambda(r, X)} \frac{1}{r+2} \leq C_3 \frac{\log \Lambda}{\Lambda^2}, \\ (4) \quad & \int_{-1}^1 dX \int_0^\Lambda dr \left(\frac{1}{\rho_\Lambda(r, X)} \right)^2 (1 - X^2) \leq C_4 \frac{1}{\Lambda^3}. \end{aligned}$$

Using (1)–(4) we can prove that there exists a constant $C > 0$ such that

$$\begin{aligned} |b_j(\Lambda/m)| &\leq C[\log(\Lambda/m)]^2, \quad j = 1, 4, \\ |b_2(\Lambda/m)| &\leq C(\Lambda/m)^{1/2}, \\ |b_j(\Lambda/m)| &\leq C \log(\Lambda/m), \quad j = 3, 5, 6. \end{aligned}$$

Hence there exists a constant $C_{\max}$ such that

$$\lim_{\Lambda \rightarrow \infty} \frac{a_2(\Lambda/m, \kappa/m)}{\sqrt{\Lambda/m}} \leq C_{\max}.$$

Next we can show that there exists a positive constant $\xi > 0$ such that

$$\lim_{\Lambda \rightarrow \infty} \sqrt{\Lambda/m} \frac{d}{d(\Lambda/m)} b_2(\Lambda/m) > \xi,$$

which implies that there exists a constant ξ' such that

$$\xi' \leq \lim_{\Lambda \rightarrow \infty} \frac{b_2(\Lambda/m)}{\sqrt{\Lambda/m}}.$$

Thus we have

$$C_{\min} \leq \lim_{\Lambda \rightarrow \infty} \frac{a_2(\Lambda/m, \kappa/m)}{\sqrt{\Lambda/m}} \leq C_{\max}.$$

□

Remark 3.2 Theorem 3.1 may suggests $\gamma \geq 1/2$ uniformly in e but $e \neq 0$.

Remark 3.3 (1) $a_2(\Lambda/m, \kappa/m)/\sqrt{\Lambda/m}$ converges to a nonnegative constant as $\Lambda \rightarrow \infty$. (2) By (3.1), we can define $a_2(\Lambda/m, 0)$ since $b_j(\Lambda/m)$ with $\kappa = 0$ are finite. Moreover $a_2(\Lambda/m, 0)$ also satisfies Theorem 3.1. (3) In the case of $\kappa = 0$, Chen [1] established that $H(0)$ has a ground state $\psi_g(0)$ but does not for $H(p)$ with $p \neq 0$.

4 Concluding remarks

The Pauli-Fierz Hamiltonian with the dipole approximation, H_{dip} , is defined by H with $A_{\hat{\varphi}}$ replaced by $1 \otimes A_{\hat{\varphi}}(0)$, i.e.,

$$H_{\text{dip}} = \frac{1}{2m}(p \otimes 1 - e1 \otimes A_{\hat{\varphi}}(0))^2 + V \otimes 1 + 1 \otimes H_f.$$

Set $V \equiv 0$. Note that

$$[H_{\text{dip}}, P_{\text{total}}] \neq 0.$$

It is established in [5] that there exists a unitary operator $U : \mathcal{H} \rightarrow \mathcal{H}$ such that

$$UH_{\text{dip}}U^{-1} = -\frac{1}{2(m + \delta m)}\Delta \otimes 1 + 1 \otimes H_f + e^2 G,$$

where

$$\begin{aligned} \delta m &= m + e^2 \frac{2}{3} \|\hat{\varphi}/\omega\|^2, \\ G &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{t^2 \|\hat{\varphi}/(t^2 + \omega^2)\|^2}{m + (2e^2/3) \|\hat{\varphi}/\sqrt{t^2 + \omega^2}\|^2} dt. \end{aligned}$$

Hence

$$[UH_{\text{dip}}U^{-1}, P_{\text{total}}] = 0.$$

Then we can define the effective mass m_{eff} for $UH_{\text{dip}}U^{-1}$, and which is

$$m_{\text{eff}}/m = 1 + \alpha \frac{4}{3\pi} (\Lambda/m - \kappa/m).$$

Hence $\gamma = 1$, then the mass renormalization for H_{dip} is not available.

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