

On the existence of Cohen extensions and $\sum_3^1$ predicates I

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In the present paper, we shall consider only Cohen extensions that do not use notions of forcing which are proper classes in a given model. From now on, according to Takahashi [13], this kind of Cohen extensions we will call Cohenian extensions.

Let $\mathcal{L}$ be the first-order language with the equality symbol "=" and the membership relation symbol " $\in$ ", but without other non-logical symbols. We use ZF for the Zermelo-Fraenkel axiom system (extensionality, regularity, infinity, union, replacement and power set) that is formulated in $\mathcal{L}$, and ZFC for ZF plus the axiom of choice formulated in $\mathcal{L}$.

Suppose that $\mathcal{M}$ is a countable standard transitive model for ZF. For each set m of $\mathcal{M}$, we choose a constant symbol $\underline{m}$ called the name of m . It is understood that different names are chosen for different sets. The language obtained from $\mathcal{L}$ by adding all names of sets in $\mathcal{M}$ is denoted by $\mathcal{L}_{\mathcal{M}}$.

We shall consider the following problem: Let ϕ be a sentence of $\mathcal{L}_{\mathcal{M}}$. Then can we find a Cohenian extension of $\mathcal{M}$ that satisfies ϕ ?

*) The author is in Dr. M. Takahashi's debt for several useful suggestions. Also he gave me that Solovay obtained a simple proof of Takahashi's theorem, but I could not know his proof.

Since there is a sentence of $\mathcal{L}$ that is not true in arbitrary structure for $\mathcal{L}$, some restriction on $\mathcal{G}$ is necessary in order to answer our problem. Now let us consider only $\mathcal{G}$ for which there is a countable standard transitive model $\mathcal{M}$ for $\underline{\text{ZF}}$ that is an extension of $\mathcal{M}$ having the same ordinals as $\mathcal{M}$, and that satisfies $\mathcal{G}$.

Now let us also suppose that $\mathcal{M}$ is one of Easton's model([1]) that satisfies the statement: for every regular cardinal $\aleph_\alpha$, $2^{\aleph_\alpha} < \aleph_{\alpha+1}$. Jensen[3] constructs a countable standard transitive model $\mathcal{N}$ that is an extension of $\mathcal{M}$ having the same ordinals as $\mathcal{M}$, and that satisfies GCH (the generalized continuum hypothesis). Jensen's construction of his model $\mathcal{N}$ uses a notion of forcing that is a proper class of $\mathcal{M}$. We can not construct his model $\mathcal{N}$ using a notion of forcing which is a set of $\mathcal{M}$, for, in arbitrary Cohenian extension, the collapsed cardinals constitute only a set of the Cohenian extension (Cf. Jech[2]). Thus answer to our problem is still negative.

Lévy [5] shows that GCH is a $\prod_2^{\text{ZF}}$ sentence in his hierarchy of set theoretic formulas. This suggests that $\mathcal{G}$ must be restricted to either $\sum_1^{\text{ZF}}$ or $\prod_1^{\text{ZF}}$ sentence of $\mathcal{L}_{\mathcal{M}}$.

Let $\mathcal{G}$ be $\prod_1^{\text{ZF}}$ sentence of $\mathcal{L}_{\mathcal{M}}$ such that there is a standard transitive extension $\mathcal{N}$ of $\mathcal{M}$ that satisfies $\mathcal{G}$. Then we have that $\mathcal{M}$ also satisfies $\mathcal{G}$, for the $\prod_1^{\text{ZF}}$ sentences of $\mathcal{L}_{\mathcal{M}}$ are preserved between $\mathcal{M}$ and $\mathcal{N}$. Thus $\mathcal{G}$ is satisfied in the trivial Cohenian extension $\mathcal{M}$ of $\mathcal{M}$ (use, as a notion of forcing, a linearly ordered structure in $\mathcal{M}$). This give us an affirmative answer to our problem when $\mathcal{G}$ is a $\prod_1^{\text{ZF}}$ sentence of $\mathcal{L}_{\mathcal{M}}$.

Takahashi [13] gives the following answer to our problem for the case of $\sum_1^{\text{ZF}}$ sentences of $\mathcal{L}_m$:

THEOREM (Takahashi [13]). Let m and n be countable standard transitive models for ZFC and assume that n is an extension of m having the same ordinals as m . If ϕ is a $\sum_1^{\text{ZF}}$ sentence of $\mathcal{L}_m$ that is true in n , then there exists a Cohenian extension $m[G]$ of m that satisfies ϕ .

Takahashi's proof of his theorem uses a notion of forcing whose conditions are elements of the Lindenbaum algebra of an infinitary propositional logic.

We shall show that Takahashi's theorem may be proved with a very simple notion of forcing. Since in order to do forcing over m we need only to be able to code the forcing language and to define the forcing relation in m , and these do not need the axiom of choice (Cf. Jensen [4]), our proof will improve Takahashi's theorem such, that the theorem applies to models m and n that do not satisfy the axiom of choice. Also, we shall apply Takahashi's theorem to some $\sum_3^1$ predicates. We will present more applications in a following paper "II".

We assume that the readers are familiar with the notions of first-order languages, formal system of Zermelo-Fraenkel set theory in such a language, models for such a system and the analytical hierarchy, and the theory of forcing. The book of Shoenfield [8] provides one of the best accounts of these notions and their theories. For the theory of forcing, the readers should consult the excellent papers of Shenfield [9] and Solovay [12, §1].

Our notations and terminologies are those of Shoenfield[9,10] and Solovay[12,§1], but with the following differences: We use the symbol " $\equiv$ " for logical equivalence, and small Greek letters " α ", " β " and " γ " denote reals which are total functions from ω into ω , but letter " σ " is a special variable for ordinals.

1. Shoenfield Absoluteness Theorem. Let us begin with a theorem which is a model theoretic version of well known Shoenfield absoluteness theorem([8]). This is considerably important in our further work.

Let $\mathcal{M}$ and $\mathcal{N}$ be standard transitive models for $\underline{ZF}$ and assume that $\mathcal{N}$ is an extension of $\mathcal{M}$ having the same countable ordinals as $\mathcal{M}$. Then we have

THEOREM 1. The Σ_2^1 and Π_2^1 sentences of $\mathcal{L}_{\mathcal{M}}$ are absolute between $\mathcal{M}$ and $\mathcal{N}$.

Proof. Shoenfield[8] shows that if φ is a Σ_2^1 sentence of $\mathcal{L}_{\mathcal{M}}$ then there is a $\Delta_1^{\underline{ZF}}$ formula $\lambda(\sigma)$ having only one free variable σ and the same names as φ such that

$$(*) \quad \varphi \equiv \exists \sigma < \omega_1 \lambda(\sigma)$$

By (*), the absoluteness of the $\Delta_1^{\underline{ZF}}$ sentences (Cf. Karp[5]) and the hypothesis of the theorem,

$$\begin{aligned} \mathcal{M} \models \varphi &\equiv \mathcal{M} \models \exists \sigma < \omega_1 \lambda(\sigma) \\ &\equiv \exists \sigma < \omega_1^{\mathcal{M}} [\mathcal{M} \models \lambda(\sigma)] \\ &\equiv \exists \sigma < \omega_1^{\mathcal{N}} [\mathcal{N} \models \lambda(\sigma)] \\ &\equiv \mathcal{N} \models \exists \sigma < \omega_1 \lambda(\sigma) \end{aligned}$$

$$\equiv \mathcal{M} \models \varphi.$$

For a Π_2^1 sentence φ , consider the negation of φ which is Σ_2^1 .

C.Q.F.D.

2. Main Theorem. Now we turn to our main theorem which is a slight improvement of Takahashi's theorem.

Let $\mathcal{M}$ and $\mathcal{N}$ be countable standard transitive models for ZF and assume that $\mathcal{N}$ is an extension of $\mathcal{M}$ having the same ordinals as $\mathcal{M}$.

THEOREM 2. If φ is a Σ_1^{ZF} sentence of $\mathcal{L}$ having only names $c_1, \dots, c_n$, then there exists a Cohenian extension $\mathcal{M}[G]$ of $\mathcal{M}$ that satisfies φ .

Proof. Without loss of generality, we may assume that φ has only one name c .

Let $\lambda(x, y)$ be a Δ_0 formula of $\mathcal{L}$ having only two free variables x and y such that

$$\varphi \equiv \exists x \lambda(x, c)$$

is provable in ZF. Since φ is true in $\mathcal{N}$, there is a set s of $\mathcal{N}$ such that $\lambda(x, y)$ is satisfied in $\mathcal{N}$ when x and y are interpreted by s and c respectively. If s is already in $\mathcal{M}$, then our theorem is trivial. Therefore we may assume that s is not in $\mathcal{M}$.

Now let us consider two partially ordered structures

$$\mathcal{C}_c = (\mathbb{H}_{\aleph_0}(\omega, \text{TC}(c \cup \omega)), \subseteq)$$

and

$$\mathcal{C}_s = (\mathbb{H}_{\aleph_0}(\omega, \text{TC}(s)), \subseteq).$$

Then C_c and C_s are notions of forcing which are sets in $\mathcal{M}$ and $\mathcal{N}$ respectively. Now let G_c and G_s be a $\mathcal{N}$ -generic filter on C_c and a $\mathcal{N}[G_c]$ -generic filter on C_s respectively. Notice that G_c is also $\mathcal{M}$ -generic filter on C_c . Thus there is a bijection g_c from ω onto $TC(c \cup \omega)$ in $\mathcal{M}[G_c]$ and $\mathcal{N}[G_c]$. Let g be a bijection from ω onto $TC(c \cup \omega) \cup TC(s)$ in $\mathcal{N}[G_c, G_s]$ such that for every natural number i ,

$$g(2i) = g_c(i).$$

Consider the binary relation R_g on ω defined as follows

$$R_g = \{(i, j) \in \omega \times \omega : g(i) \in g(j)\}.$$

Then g is an isomorphism between two structures (ω, R_g) and $(TC(c \cup \omega) \cup TC(s), \in)$. Let c^* and s^* be two natural numbers such that

$$g(c^*) = c$$

and

$$g(s^*) = s.$$

Since $\chi(x, y)$ is a Δ_0 formula of $\mathcal{L}$, and $(TC(c \cup \omega) \cup TC(s), \in)$ is a transitive substructure of $\mathcal{N}$,

$$(TC(c \cup \omega) \cup TC(s), \in) \models \chi(x, y)[s, c],$$

and thus

$$(\omega, R_g) \models \chi(x, y)[s^*, c^*].$$

Let $\psi_0(x, y)$ be an arithmetical predicate having only two free variables x and y , without names, which says that x and y are reals

such that x and y are the codes of binary relations on ω , and for all natural numbers i and j

$$(*) \quad x(\langle 2i, 2j \rangle) = 0 \equiv y(\langle 2i, 2j \rangle) = 0$$

and

$$y(\langle i, 2j+1 \rangle) = 1 \& y(\langle 2i+1, j \rangle) = 1.$$

Let $\psi_1(x)$ be a $\prod_1^1$ predicate having only one free variable x , without names, which says that x is a real that is the code of a well-founded binary relation on ω , and for all natural numbers j and k

$$\forall i (x(\langle i, j \rangle) = 0 \equiv x(\langle i, k \rangle) = 0) \rightarrow j = k.$$

Finally let $\psi_2(x)$ be an arithmetical predicate having only one free variable x , without names, that is the logical conjunction of a predicate which says that x is a real and the predicate obtained from the formula $\chi(x, y)$ by replacing x with $\underline{s}^*$, y with $\underline{c}^*$, $u \in v$ with $x(\langle i, j \rangle) = 0$, $\forall u$ with $\forall i$ and $\exists u$ with $\exists i$. Then

$$\exists x (\psi_0(x, y) \& \psi_1(x) \& \psi_2(x))$$

is a $\sum_2^1$ predicate of $\mathcal{L}_m$ having only one free real variable y and two names $\underline{c}^*$ and $\underline{s}^*$, and we express this predicate as $\psi(y)$ for simplicity.

Consider the binary relation S_{g_c} defined as follows

$$S_{g_c} = \{ (2i, 2j) \in \omega \times \omega : g_c(i) \in g_c(j) \}$$

which is in $\mathcal{R}[G_c]$ and $\mathcal{R}[G_c]$.

Let α and β be the codes of R_g and S_{g_c} respectively. Notice β is in $\mathcal{M}[G_c]$, so in $\mathcal{N}[G_c]$ and $\mathcal{N}[G_c, G_s]$. Since R_g is a well-founded binary relation on ω such that for all natural numbers i, j and k

$$(2i, 2j) \in R_g \equiv (2i, 2j) \in S_{g_c},$$

and

$$(i, 2j+1) \notin S_{g_c} \equiv (2i+1, j) \notin S_{g_c}$$

$$\forall n((n, j) \in R_g \equiv (n, k) \in R_g) \rightarrow j = k,$$

and $\psi_2(\alpha)$ says that $\chi(x, y)$ is satisfied in (ω, R_g) when x and y interpreted by s^* and c^* respectively, we have

$$\mathcal{N}[G_c, G_s] \models \psi_0(x, y) \ \& \ \psi_1(x) \ \& \ \psi_2(x) [\alpha, \beta].$$

thus

$$\mathcal{N}[G_c, G_s] \models \psi(y) [\beta].$$

Now observe that $\mathcal{M}[G_c]$ is a submodel of $\mathcal{N}[G_c, G_s]$ having the same countable ordinals as $\mathcal{N}[G_c, G_s]$, for, since the notions of forcing $(H_{\aleph_0}(\omega, TC(c \cup \omega)), \subseteq)$ and $(H_{\aleph_0}(\omega, TC(s)), \subseteq)$ satisfy the $\aleph_0$ -chain condition in $\mathcal{M}$ and $\mathcal{N}$ respectively, the cardinals, so the countable ordinals, are preserved between the two models $\mathcal{M}[G_c]$ and $\mathcal{N}[G_c, G_s]$. By theorem 1, the Σ_2^1 predicate $\psi(y)$ is also satisfied in $\mathcal{M}[G_c]$ when y is interpreted by β . Let γ be a real in $\mathcal{M}[G_c]$ such that

$$\mathcal{M}[G_c] \models \psi_0(x, y) \ \& \ \psi_1(x) \ \& \ \psi_2(x) [\gamma, \beta].$$

Consider the binary relation R_γ defined as follows

$$R_\gamma = \{(i,j) \in \omega \times \omega : \gamma(\langle i,j \rangle) = 0\}.$$

Then (ω, R) is a well-founded and extensional structure such that

$$(\omega, R_\gamma) \models \chi(x,y)[s^*, c^*].$$

By Mostowski Collapsing Theorem ([7]), there are unique transitive set u and unique isomorphism π from (ω, R_γ) onto $(u, \in)$ in $\mathcal{M}[G_c]$. Thus

$$(u, \in) \models \chi(x,y)[\pi(s^*), \pi(c^*)].$$

By (*) S_{g_c} is a subset of R_γ , and hence the inverse function of g_c is the restriction of π to the set of even natural numbers.

Since c^* is a even natural number,

$$\pi(c^*) = g_c^{-1}(c^*) = g^{-1}(c^*) = c.$$

Notice that $(u, \in)$ is a substructure of $\mathcal{M}[G_c]$, and $\chi(x,y)$ is a Δ_0 formula of $\mathcal{L}$. Therefore we have

$$\mathcal{M}[G_c] \models \chi(x,y)[\pi(s^*), c],$$

so

$$\mathcal{M}[G_c] \models \phi.$$

C.Q.F.D.

3. Application. Now we shall give some applications of our theorem 2 that are concerned with the analytical hierarchy.

Let $\mathcal{M}$ and $\mathcal{N}$ be countable transitive models for $\underline{ZF}$ + "there exists an inaccessible cardinal" and assume that $\mathcal{N}$ is an extension of $\mathcal{M}$ having the same ordinals. Let $\exists \beta \exists (\alpha, \beta)$ be a Σ_3^1 sentence with one name $\underline{\alpha}$ for a real in $\mathcal{N}$.

THEOREM 3. If the predicate $\exists \beta \varphi(\underline{\alpha}, \beta)$ is satisfied in $\mathcal{M}$, then there exists a Cohenian extension $\mathcal{M}[G]$ which has a standard transitive submodel for $\underline{ZF} + \exists \beta \varphi(\underline{\alpha}, \beta)$.

Proof. Let σ be an inaccessible cardinal in $\mathcal{M}$ and $R^{\mathcal{M}}(\sigma)$ the set of sets in $\mathcal{M}$ with ranks less than σ . Let $\psi(x, y, \underline{\alpha})$ be the $\Delta_1^{\underline{ZF}}$ formula " $(x, \in)$ is a transitive model for $\underline{ZF} + \varphi(\underline{\alpha}, y)$ ". Then $\psi(x, y, \underline{\alpha})$ is satisfied in $\mathcal{M}$ when x and y are interpreted by $R^{\mathcal{M}}(\sigma)$ and some real in $R^{\mathcal{M}}(\sigma)$ respectively. Since $\exists x \exists y \psi(x, y, \underline{\alpha})$ is a $\Sigma_1^{\underline{ZF}}$ formula, by our theorem 2 there exists a Cohenian extension $\mathcal{M}[G]$ of $\mathcal{M}$ in which this formula is true. This means that there is a transitive standard submodel of $\mathcal{M}[G]$ in which $\exists \beta \varphi(\underline{\alpha}, \beta)$ is true. C.Q.F.D.

The technique in the proof of theorem 3 has some interest and many applications, and we present here one more application.

Let $\mathcal{M}$ and $\mathcal{N}$ be standard transitive models for $\underline{ZF}$ having the same ordinals and assume that $\mathcal{N}$ satisfies $\underline{MC}$ (there exists a measurable cardinal). Let $P(\underline{\alpha}, \beta)$ be a Π_2^1 predicate which says that $\beta = \alpha^\#$ (Cf. Solovay [11]). Since $\exists \beta P(\underline{\alpha}, \beta)$ is provable in $\underline{ZF} + \underline{MC}$, for each real α in $\mathcal{M}$, $\exists \beta P(\underline{\alpha}, \beta)$ is true in $\mathcal{N}$. Let $\psi(x, y, \underline{\alpha})$ be a $\Delta_1^{\underline{ZF}}$ formula which says that $(x, \in)$ is a transitive model of $\underline{ZF} + P(\underline{\alpha}, y)$. Then applying a similar argument in the proof of theorem 3 to this formula, we have

THEOREM 4. There exists a Cohenian extension $\mathcal{M}[G]$ of $\mathcal{M}$ which has a standard transitive submodel for $\underline{ZF} + "\alpha^\# \text{ exists}"$.

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