

MAASS RELATIONS IN HIGHER GENUS

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ABSTRACT. For an arbitrary even genus $2n$ we show that the subspace of Siegel cusp forms of degree $2n$ generated by Ikeda lifts of elliptic cusp forms can be characterized by certain linear relations among Fourier coefficients. This generalizes the classical Maass relations in degree two to higher degrees.

Introduction

The purpose of this paper is to give a characterization of the image of Ikeda's lifting by certain linear relations among Fourier coefficients.

Let us describe our results. We denote by T_r^+ the set of positive definite symmetric half-integral matrices of size r . Put $D_h = \det(2h)$ for $h \in T_{2n}^+$ with a fixed integer n . Let $\mathfrak{d}_h$ be the absolute value of discriminant of $\mathbb{Q}((-1)^n D_h)^{1/2}/\mathbb{Q}$. Put $\mathfrak{f}_h = (\mathfrak{d}_h^{-1} D_h)^{1/2}$. Fix a prime number p . The Siegel series attached to h at p is defined by

$$b_p(h, s) = \sum_{\alpha} \mathbf{e}_p(-\mathrm{tr}(h\alpha)) \nu(\alpha)^{-s},$$

where α extends over all symmetric matrices of rank $2n$ with entries in $\mathbb{Q}_p/\mathbb{Z}_p$. Here, we put $\mathbf{e}_p(x) = e^{-2\pi\sqrt{-1}x'}$, where x' is the fractional part of $x \in \mathbb{Q}_p$, and $\nu(\alpha) = [\alpha\mathbb{Z}^{2n} + \mathbb{Z}^{2n} : \mathbb{Z}^{2n}]$. As is well-known, $b_p(h, s)$ is a product of two polynomials $\gamma_p(h; p^{-s})$ and $F_p(h; p^{-s})$, where

$$\gamma_p(h; X) = (1 - X) \left(1 - \left(\frac{(-1)^n \mathfrak{d}_h}{p} \right) p^n X \right)^{-1} \prod_{j=1}^n (1 - p^{2j} X^2)$$

and the constant term of $F_p(h; X)$ is 1. Put

$$\tilde{F}_p(h; X) = X^{-\mathrm{ord}_p \mathfrak{f}_h} F_p(h; p^{-n-1/2} X).$$

Then, from [6], the following functional equation holds:

$$\tilde{F}_p(h; X) = \tilde{F}_p(h; X^{-1}).$$

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Ikeda constructed a lifting from $S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ to $S_{k+n}(\mathrm{Sp}_{2n}(\mathbb{Z}))$ for each positive integer k such that $k \equiv n \pmod{2}$.

Theorem 0.1 (Ikeda [5]). *Let $f \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ with $k \equiv n \pmod{2}$ be a normalized Hecke eigenform, the L -function of which is given by*

$$\prod_p (1 - \alpha_p p^{k-1/2-s})^{-1} (1 - \alpha_p^{-1} p^{k-1/2-s})^{-1}.$$

Let g be a corresponding cusp form in the Kohnen plus space $S_{k+1/2}^+(4)$ under the Shimura correspondence. We define the function F on the upper half-space $\mathfrak{H}_{2n}$ by

$$F(Z) = \sum_{h \in T_{2n}^+} c_F(h) e^{2\pi\sqrt{-1}\mathrm{tr}(hZ)},$$

where

$$c_F(h) = c_g(\mathfrak{d}_h) \mathfrak{f}_h^{k-1/2} \prod_p \tilde{F}_p(h; \alpha_p).$$

Here, we denote the m -th Fourier coefficient of g by $c_g(m)$. Then F is a cuspidal Hecke eigenform in $S_{k+n}(\mathrm{Sp}_{2n}(\mathbb{Z}))$.

Put $\mathfrak{D}_k = \{m \in \mathbb{N} \mid (-1)^k m \equiv 0, 1 \pmod{4}\}$. In [9] Kohnen defined an integer $\phi(d; h)$ for each $h \in T_{2n}^+$ and each positive divisor d of $\mathfrak{f}_h$, and showed that

$$I_{n,k} : \sum_{m \in \mathfrak{D}_k} c(m) e^{2\pi\sqrt{-1}m\tau} \mapsto \sum_{h \in T_{2n}^+} \sum_{d \mid \mathfrak{f}_h} d^{k-1} \phi(d; h) c(d^{-2} D_h) e^{2\pi\sqrt{-1}\mathrm{tr}(hZ)}$$

is a linear map from $S_{k+1/2}^+(4)$ to $S_{k+n}(\mathrm{Sp}_{2n}(\mathbb{Z}))$, which on Hecke eigenforms coincides with the Ikeda lifting.

He also conjectured that if $F \in S_{k+n}(\mathrm{Sp}_{2n}(\mathbb{Z}))$ has a Fourier expansion of the form

$$F(Z) = \sum_{h \in T_{2n}^+} \sum_{d \mid \mathfrak{f}_h} d^{k-1} \phi(d; h) c(d^{-2} D_h) e^{2\pi\sqrt{-1}\mathrm{tr}(hZ)}$$

for some function $c : \mathfrak{D}_k \rightarrow \mathbb{C}$, then F lies in the image of the map $I_{n,k}$. If $n = 1$, then this conjecture comes down to saying that the Maass space coincides with the image of the Saito-Kurokawa lifting, and hence it is true.

Kohnen and Kojima [10] proved the conjecture for all n with $n \equiv 0, 1 \pmod{4}$. More precisely, they showed that

$$g(\tau) = \sum_{m \in \mathfrak{D}_k} c(m) e^{2\pi\sqrt{-1}m\tau} \in S_{k+1/2}^+(4).$$

Their proof is as follows. There exists $S \in T_{2n-1}^+$ with $\det(2S) = 2$, provided that $n \equiv 0, 1 \pmod{4}$. Writing F_S for the S -th Fourier-Jacobi coefficient of F , and letting $\{F_{S,\mu_0}, F_{S,\mu_1}\}$ be the modular forms of weight $k + \frac{1}{2}$ associated to F_S (cf. §3), they put

$$i(S)F(\tau) = F_{S,\mu_0}(4\tau) + F_{S,\mu_1}(4\tau).$$

Then they showed that $i(S)F$ is an element of $S_{k+1/2}^+(4)$ and is given by

$$i(S)F(\tau) = \sum_{m \in \mathfrak{D}_k} c(m) e^{2\pi\sqrt{-1}m\tau}.$$

This proves the conjecture for all n with $n \equiv 0, 1 \pmod{4}$.

Although there exists no matrix $S \in T_{2n-1}^+$ with $\det(2S) = 2$ if $n \equiv 2, 3 \pmod{4}$ (cf. Lemma 4.2), we shall show the following:

Theorem 0.2 (cf. Theorem 1.5). *Under the notation above, there is $g \in S_{k+1/2}^+(4)$ with the following properties.*

- (1) *If $n \equiv 3 \pmod{4}$, then $c(m) = c_g(m)$.*
- (2) *If $n \equiv 2 \pmod{4}$, then the following assertions holds:*
 - (a) $F = I_{n,k}(g)$;
 - (b) *if m is not a square, then $c(m) = c_g(m)$;*
 - (c) $c(f^2) = c_g(f^2) + (c(1) - c_g(1))f^k$ for every positive integer f .

In order to offer the reader some hint of our idea, we briefly describe the proof of Theorem 0.2 under the simplifying assumption $n = 2$. The first problem is to find auxiliary matrices which supplement the lack of the above matrix S . Fix a rational prime p and let H' be a quaternion algebra with center $\mathbb{Q}$ ramified only at p and the archimedean place. Let ι be the main involution and τ the reduced trace on H' . Put $B' = \{x \in H' \mid x^\iota = -x\}$ and endow B' with a symmetric bilinear form $B(x, y) = p\tau(xy^\iota)/2$. When we identify B with an element of T_3^+ , using a basis of a maximal integral lattice of B' , it is easy to see that $\det(2B) = 2p$. Analogously, we put

$$i(B)F(\tau) = \sum_{\mu} F_{B,\mu}(4p\tau),$$

letting $\{F_{B,\mu}\}$ be a collection of modular forms associated to the B -th Fourier-Jacobi coefficient of F . Then $i(B)F$ is an element of $S_{k+1/2}^+(4p)$. Analyzing the Siegel series closely, we can show that the Fourier expansion of $i(B)F$ is given by

$$i(B)F(\tau) = \sum_{m \in \mathfrak{D}_k} \left(1 - \left(\frac{(-1)^k m}{p}\right)\right) (c(m) - p^k c(p^{-2}m)) e^{2\pi\sqrt{-1}m\tau}.$$

On the other hand, the map

$$P(p) : g \mapsto \sum_{m \in \mathfrak{D}_k} \left(1 - \left(\frac{(-1)^k m}{p} \right) \right) (c_g(m) - p^k c_g(p^{-2}m)) e^{2\pi\sqrt{-1}m\tau}$$

defines an injection from $S_{k+1/2}^+(4)$ into $S_{k+1/2}^+(4p)$. The most difficult part of the proof is to show that $i(B)F$ belongs to the image of $P(p)$.

To solve this problem, we need the newform theory for modular forms of half-integral weight, which we recall in §2. Pursuing this analysis (for detail see §4), we will see that the existence of a function c gives rise to $g \in S_{k+1/2}^+(4)$ satisfying $i(B)F = g|P(p)$ for all primes p and $B \in T_3^+$ subject to $\det(2B) = 2p$. This proves (b) and (c).

If D_h is square, we have $\prod_p \tilde{F}_p(h; p^{1/2}) = 0$ and hence

$$\sum_{d|f_h} d^{-1} \phi(d; h) = 0$$

(see Lemma 1.3). From this, we get (a). Thus one could view the above condition as a natural generalization of Maass relations to higher genus.

Notation

Put $q = \mathbf{e}(\tau) = e^{2\pi\sqrt{-1}\tau}$ for a complex number τ . Let $\mathfrak{H}$ be the upper half-plane. Put

$$\mathfrak{D}_n = \{a \in \mathbb{N} \mid (-1)^n a \equiv 0, 1 \pmod{4}\}$$

with a fixed positive integer n . For a nonzero element $N \in \mathbb{Q}$, we denote the absolute value of the discriminant of $\mathbb{Q}((-1)^n N)^{1/2}/\mathbb{Q}$ by $\mathfrak{d}_N$. Put $\mathfrak{f}_N = (\mathfrak{d}_N^{-1} N)^{1/2}$. For each rational prime p , we put $\underline{\psi}_p(N) = 1, -1, 0$ according as $\mathbb{Q}_p(\sqrt{N})$ is $\mathbb{Q}_p$, an unramified quadratic extension or a ramified quadratic extension.

We denote by $S_\kappa(\mathrm{Sp}_r(\mathbb{Z}))$ the space of Siegel cusp forms of weight κ with respect to the Siegel modular group $\mathrm{Sp}_r(\mathbb{Z})$. Let T_r be the set of symmetric half-integral matrices of size r and T_r^+ the set consisting of positive definite elements of T_r . We denote the h -th Fourier coefficient by $c_F(h)$ for $F \in S_\kappa(\mathrm{Sp}_r(\mathbb{Z}))$ and $h \in T_r^+$. Put $D_h = 4^{\lceil r/2 \rceil} \det h$, where $\lceil \cdot \rceil$ is the Gauss bracket. To simplify notation, we abbreviate $\mathfrak{d}_h = \mathfrak{d}_{D_h}$, $\mathfrak{f}_h = \mathfrak{f}_{D_h}$ for $h \in T_{2n}^+$. Put $h[A] = {}^t A h A$ and $h(A, B) = {}^t A h B$ for matrices A and B if they are well-defined.

1. Main Theorem

We recall first of all a formula for the Siegel series. Let p be a rational prime and $\mathbb{F}_p$ a finite field with p elements. Put $T_{2n,p} = T_{2n} \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

Fix a nondegenerate matrix $h \in T_{2n,p}$ and let q_p be a quadratic form on $V = \mathbb{F}_p^{2n}$ obtained from the quadratic form $x \mapsto h[x]$ by reducing modulo p . Let $\text{Rad}(V)$ be the radical of V , i.e.,

$$\text{Rad}(V) = \{x \in V \mid h[x] \equiv 0 \text{ and } h[x+y] \equiv h[x] + h[y] \pmod{p} \text{ for all } y \in V\}.$$

Writing $V = V_1 \oplus \text{Rad}(V)$, we put

$$\begin{aligned} s_p &= s_p(h) = \dim_{\mathbb{F}_p} \text{Rad}(V), \\ \lambda_p(h) &= \begin{cases} 1 & \text{if } V_1 \text{ is hyperbolic or } s_p = 2n, \\ -1 & \text{otherwise,} \end{cases} \\ H_{n,p}(h; X) &= \begin{cases} 1 & \text{if } s_p = 0. \\ \prod_{j=1}^{(s_p-1)/2} (1 - p^{2j-1} X^2) & \text{if } 2 \nmid s_p. \\ (1 + \lambda_p(h) p^{(s_p-1)/2} X) \prod_{j=1}^{s_p/2-1} (1 - p^{2j-1} X^2) & \text{otherwise.} \end{cases} \end{aligned}$$

Set

$$\mathcal{D}_p(h) = \text{GL}_{2n}(\mathbb{Z}_p) \backslash \{G \in \text{M}_{2n}(\mathbb{Z}_p) \cap \text{GL}_{2n}(\mathbb{Q}_p) \mid h[G^{-1}] \in T_{2n,p}\},$$

where $\text{GL}_{2n}(\mathbb{Z}_p)$ operates by left multiplication. Putting $D_h = \det(2h)$ and letting $\mathfrak{d}_h$ be the discriminant ideal of $\mathbb{Q}_p(((-1)^n D_h)^{1/2})/\mathbb{Q}_p$, we put

$$\mathfrak{f}_p(h) = (\text{ord}_p D_h - \text{ord}_p \mathfrak{d}_h)/2.$$

Now the formula is

$$\begin{aligned} (1.1) \quad \tilde{F}_p(h; X) &= \sum_{G \in \mathcal{D}_p(h)} X^{-\mathfrak{f}_p(h) + 2 \text{ord}_p \det G} H_{n,p}(h[G^{-1}]; X) \\ &\quad \times \begin{cases} 1 & \text{if } \mathfrak{f}_p(h[G^{-1}]) = 0. \\ 1 - \left(\frac{\mathfrak{d}_h}{p}\right) p^{-1/2} X & \text{if } \mathfrak{f}_p(h[G^{-1}]) > 0. \end{cases} \end{aligned}$$

This formula is due to Kitaoka [7] at least when $p \neq 2$. A complete and general proof including the case $p = 2$ was obtained by Feit [4]. Böcherer [1] obtained the formula globally in connection with the Fourier coefficients of Siegel Eisenstein series (cf. [9]).

Let $h \in T_{2n}^+$. Following [9], we define the map $\rho_h : \mathbb{N} \rightarrow \mathbb{C}$ via the relation

$$\sum_{a \in \mathbb{N}} \rho_h(a) a^{-s} = \prod_{p \mid \mathfrak{f}_h} (1 - p^{-2s}) H_{n,p}(h; p^{-s}).$$

For each positive divisor a of $\mathfrak{f}_h$, an integer $\phi(a; h)$ is defined by

$$\phi(a; h) = \sqrt{a} \sum_{d \in \mathbb{N}, d^2 \mid a} \sum_{G \in \mathcal{D}(h), \det G = d} \rho_{h[G^{-1}]}(d^{-2}a),$$

where

$$\mathcal{D}(h) = \mathrm{GL}_{2n}(\mathbb{Z}) \setminus \{G \in \mathrm{M}_{2n}(\mathbb{Z}) \cap \mathrm{GL}_{2n}(\mathbb{Q}) \mid h[G^{-1}] \in T_{2n}\}.$$

Note that there is a natural bijection between $\mathcal{D}(h)$ and $\prod_p \mathcal{D}_p(h)$.

For an integer e , we define $l_e \in \mathbb{C}[X, X^{-1}]$ by

$$l_e(X) = \begin{cases} \frac{X^{e+1} - X^{-e-1}}{X - X^{-1}} & \text{if } e \geq 0. \\ 0 & \text{if } e < 0. \end{cases}$$

For each prime number p , we put

$$\ell_{p,N} = l_{\mathrm{ord}_p \mathfrak{f}_N} - \underline{\psi}_p((-1)^n N) p^{-1/2} l_{\mathrm{ord}_p \mathfrak{f}_N - 1}$$

(see Notation for the definitions of $\mathfrak{f}_N$ and $\underline{\psi}_p$). The arguments in [9] assert that

$$(1.2) \quad \mathfrak{f}_h^{k-1/2} \prod_p \tilde{F}_p(h; X_p) = \sum_{d|\mathfrak{f}_h} d^{k-1} \phi(d; h) (d^{-1} \mathfrak{f}_h)^{k-1/2} \prod_p \ell_{p, d^{-2} D_h}(X_p).$$

Note that $c_g(m) = c_g(\mathfrak{d}_m) \mathfrak{f}_m^{k-1/2} \prod_p \ell_{p,m}(\alpha_p)$ if $g|\tilde{T}(p^2) = p^{k-1/2}(\alpha_p + \alpha_p^{-1})g$ for all primes p . Hence the following theorem follows from Theorem 0.1 and (1.2).

Theorem 1.1 (Kohnen [9]). *The notation being as in Theorem 0.1, we have*

$$c_F(h) = \sum_{d|\mathfrak{f}_h} d^{k-1} \phi(d; h) c_g(d^{-2} D_h).$$

Thus a linear map $I_{n,k} : S_{k+1/2}^+(4) \rightarrow S_{k+n}(\mathrm{Sp}_{2n}(\mathbb{Z}))$ arises as in Introduction.

Definition 1.2. If $k \equiv n \pmod{2}$, then the space $S_{k+n}^M(\mathrm{Sp}_{2n}(\mathbb{Z}))$ is defined in the following way: $F \in S_{k+n}(\mathrm{Sp}_{2n}(\mathbb{Z}))$ is an element of $S_{k+n}^M(\mathrm{Sp}_{2n}(\mathbb{Z}))$ if there is a function $c : \mathfrak{D}_k \rightarrow \mathbb{C}$ such that all $h \in T_{2n}^+$ satisfy

$$(1.3) \quad c_F(h) = \sum_{d|\mathfrak{f}_h} d^{k-1} \phi(d; h) c(d^{-2} D_h).$$

Lemma 1.3. *If $n \equiv 2 \pmod{4}$ and h is an element of T_{2n}^+ with $\mathfrak{d}_h = 1$, then we have*

$$\sum_{d|\mathfrak{f}_h} d^{k-1} \phi(d; h) (d^{-1} \mathfrak{f}_h)^k = 0.$$

Remark 1.4. Lemma 1.3 implies that the parameter c is not uniquely determined by $F \in S_{k+n}^M(\mathrm{Sp}_{2n}(\mathbb{Z}))$ if $n \equiv 2 \pmod{4}$. More precisely, a function $c' : \mathfrak{D}_k \rightarrow \mathbb{C}$ defined by

$$c'(m) = \begin{cases} c(m) & \text{if } m \notin \mathbb{Q}^{\times 2} \\ c(m) + a \cdot \mathfrak{f}_m^k & \text{if } m \in \mathbb{Q}^{\times 2} \end{cases}$$

is also a parameter of F for any complex number a . We shall show that these are all parameters of F (cf. Corollary 4.5).

Proof. Whereas $\mathfrak{D}_k$ contains 1, there are no matrices $B \in T_{2n}^+$ such that $D_B = 1$ under our hypothesis on n . It follows from (1.1) that $\tilde{F}_p(h; p^{1/2}) = 0$ for some p (cf. [5, Lemma 15.2]). This, together with (1.2) and $\prod_p \ell_{p, d^{-2}D_h}(p^{1/2}) = (d^{-1}\mathfrak{f}_h)^{1/2}$, proves the relation we want. $\square$

Our main result is the following:

Theorem 1.5. *Suppose that $k \equiv n \pmod{2}$ and $F \in S_{k+n}(\mathrm{Sp}_{2n}(\mathbb{Z}))$ admits Fourier coefficients of the form (1.3). Then there is $g \in S_{k+1/2}^+(4)$ with the following properties:*

- (1) *If $n \equiv 0, 1, 3 \pmod{4}$, then $c(m) = c_g(m)$ for every $m \in \mathfrak{D}_k$.*
- (2) *If $n \equiv 2 \pmod{4}$, then the following assertions holds:*
 - (a) $F = I_{n,k}(g)$;
 - (b) *if m is not a square, then $c(m) = c_g(m)$;*
 - (c) $c(f^2) = c_g(f^2) + (c(1) - c_g(1))f^k$ *for every positive integer f .*

Corollary 1.6. The space $S_{k+n}^M(\mathrm{Sp}_{2n}(\mathbb{Z}))$ coincides with the image of $I_{n,k}$.

2. Modular forms of half-integral weight

We recall certain properties of newforms for the Kohnen plus space (see [11] for detail). Fix a multiple N of 4. The space of cusp forms of weight $k + \frac{1}{2}$ with respect to $\Gamma_0(N)$ is denoted by $S_{k+1/2}(N)$. We denote the m -th Fourier coefficient of $g \in S_{k+1/2}(N)$ by $c_g(m)$. The Kohnen plus space $S_{k+1/2}^+(N)$ consists of all functions $g \in S_{k+1/2}(N)$ such that $c_g(m) = 0$ unless $m \in \mathfrak{D}_k$.

An operator $U(a)$ on formal power series is defined by

$$\sum_{m \in \mathbb{N}} c(m) q^m |U(a) = \sum_{m \in \mathbb{N}} c(am) q^m$$

for a positive integer a . An operator $U_k(a^2)$ defined by

$$\sum_{m \in \mathfrak{D}_k} c(m)q^m |U_k(a^2) = \sum_{m \in \mathfrak{D}_k} c(a^2 m)q^m$$

maps $S_{k+1/2}^+(N)$ into itself if each prime factor of a is that of $4^{-1}N$.

In what follows, we assume that $k > 0$ and $N/4$ is square-free. The space of newforms $S_{k+1/2}^{\text{new},+}(N)$ for $S_{k+1/2}^+(N)$ is by definition the orthogonal complement of

$$\sum_{p|4^{-1}N} \left(S_{k+1/2}^+(p^{-1}N) + S_{k+1/2}^+(p^{-1}N) |U_k(p^2) \right),$$

where p extends over all prime divisors of $4^{-1}N$, in $S_{k+1/2}^+(N)$ with respect to the Petersson inner product. The space of newforms for $S_{2k}(\Gamma_0(4^{-1}N))$ is denoted by $S_{2k}^{\text{new}}(4^{-1}N)$. Let $\tilde{T}(p^2)$ (resp. $T(p)$) be the usual Hecke operator on the space of modular forms of half-integral (resp. integral) weight.

Theorem 2.1. *Suppose that $N/4$ is square-free and k is positive.*

- (1) $S_{k+1/2}^+(N) = \oplus_{a,d \geq 1, ad|4^{-1}N} S_{k+1/2}^{\text{new},+}(4a) |U_k(d^2)$.
- (2) *The operators $\tilde{T}(p^2)$ and $U_k(q^2)$, where $(p, 4^{-1}N) = 1$ and $q|4^{-1}N$, fix $S_{k+1/2}^{\text{new},+}(N)$. Moreover, $S_{k+1/2}^{\text{new},+}(N)$ has an orthogonal $\mathbb{C}$ -basis which consists of common Hecke eigenforms of these operators.*
- (3) *There is a bijective correspondence, up to scalar multiple, between Hecke eigenforms in $S_{2k}^{\text{new}}(4^{-1}N)$ and those in $S_{k+1/2}^{\text{new},+}(N)$ in the following way. If $f \in S_{2k}^{\text{new}}(4^{-1}N)$ is a primitive form, i.e.,*

$$f|T(p) = \omega_p f, \quad f|U(q) = \omega_q f$$

for every prime number $p \nmid 4^{-1}N$ and prime divisor q of $4^{-1}N$, then there is a nonzero Hecke eigenform $g \in S_{k+1/2}^{\text{new},+}(N)$ such that

$$g|\tilde{T}(p^2) = \omega_p g, \quad g|U_k(q^2) = \omega_q g$$

for every prime number $p \nmid 4^{-1}N$ and prime divisor q of $4^{-1}N$.

Proposition 2.2. *Let $g \in S_{k+1/2}^{\text{new},+}(N)$, p a prime divisor of $4^{-1}N$, and ϵ either 1 or -1 . Then the following conditions are equivalent:*

- (i) *if $\psi_p((-1)^k m) = \epsilon$, then $c_g(m) = 0$;*
- (ii) *$g|U_k(p^2) = \epsilon p^{k-1} g$.*

3. Jacobi forms and the Fourier-Jacobi coefficients

It is useful to recall certain properties of Jacobi forms before turning to the proof of Theorem 1.5. Let $B \in T_{2n-1}^+$. Put $L = \mathbb{Z}^{2n-1}$ and $L^* = (2B)^{-1}L$. Let $\mathcal{T}_B^+$ be the set of all pairs $(a, \alpha) \in \mathbb{Z} \times L^*$ such that $a > B[\alpha]$. Choose a complete representative $\Xi(B)$ for L^*/L . For $\mu \in \Xi(B)$ we define the theta function $\vartheta_\mu^B(\tau, w)$ by

$$\vartheta_\mu^B(\tau, w) = \sum_{\alpha \in L} \mathbf{e}(\tau B[\alpha + \mu] + 2B(\alpha + \mu, w)).$$

A Jacobi cusp form Φ of weight κ with index B is a holomorphic function on $\mathfrak{H} \times \mathbb{C}^{2n-1}$ subject to the following conditions:

(i) for every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$\Phi\left(\frac{a\tau + b}{c\tau + d}, \frac{w}{c\tau + d}\right) = (c\tau + d)^\kappa \mathbf{e}\left(\frac{cB[w]}{c\tau + d}\right) \Phi(\tau, w);$$

(ii) for every $\xi, \eta \in L$, we have

$$\Phi(\tau, w + \xi\tau + \eta) = \mathbf{e}(-B(\xi, w)/2 - B[\xi]\tau) \Phi(\tau, w);$$

(iii) Φ has a Fourier expansion of the form

$$\Phi(\tau, w) = \sum_{(a, \alpha) \in \mathcal{T}_B^+} c_\Phi(a, \alpha) q^a \mathbf{e}(2B(\alpha, w)).$$

As a consequence of (ii), a Jacobi form Φ is written as a sum

$$\Phi(\tau, w) = \sum_{\mu \in \Xi(B)} \Phi_\mu(\tau) \vartheta_\mu^B(\tau, w),$$

where

$$\Phi_\mu(\tau) = \sum_{a \in \mathbb{N}, a > B[\mu]} c_\Phi(a, \mu) \mathbf{e}((a - B[\mu])\tau).$$

The B -th Fourier-Jacobi coefficient F_B of $F \in S_\kappa(\mathrm{Sp}_{2n}(\mathbb{Z}))$ is defined by

$$F_B(\tau, w) = \sum_{(a, \alpha) \in \mathcal{T}_B^+} c_F(B_{a, \alpha}) q^a \mathbf{e}(2B(\alpha, w)), \quad B_{a, \alpha} = \begin{pmatrix} B & B\alpha \\ {}_t\alpha B & a \end{pmatrix}.$$

Then F_B is a Jacobi cusp form of weight κ with index B . The proof of the following proposition proceeds like the argument given for the relevant part of the proof of [3, Theorem 5.6].

Proposition 3.1. *Suppose $\kappa = k + n$, κ is even and $B \in T_{2n-1}^+$. Let $F \in S_\kappa(\mathrm{Sp}_{2n}(\mathbb{Z}))$. Put $D_B = \det(2B)/2$ and*

$$i(B)F(\tau) = \sum_{\mu \in \Xi(B)} F_{B, \mu}(4D_B\tau).$$

Then $i(B)F$ is an element of $S_{k+1/2}^+(4D_B)$.

4. Proof of Theorem 1.5

Let v be a place of $\mathbb{Q}$ and B a nondegenerate symmetric matrix over $\mathbb{Q}_v$. We define the Hasse invariant $h_v(B)$ for B by

$$h_v(B) = \prod_{i \leq j} (a_i, a_j)_v,$$

where $(\ , \)_v$ is the Hilbert symbol over $\mathbb{Q}_v$, if we take any orthogonal basis $\{x_i\}$ such that $B[x_i] = a_i$. We modify $h_v(B)$ by putting

$$\eta_v(B) = h_v(B)(\det B, (-1)^{n-1} \det B)_v (-1, -1)_v^{n(n-1)/2}$$

so that an anisotropic kernel of B has dimension $2 - \eta_p(B)$ when v is a finite place p .

Remark 4.1. It should be remarked that the Hasse invariant is sometimes defined by replacing $\leq$ by $<$.

We start with an easy lemma.

Lemma 4.2. *Let $B \in T_{2n-1}^+$. Assume that D_B is square-free. Then*

$$\prod_{p|D_B} \eta_p(B) = \begin{cases} 1 & \text{if } n \equiv 0, 1 \pmod{4}. \\ -1 & \text{if } n \equiv 2, 3 \pmod{4}. \end{cases}$$

Proof. It follows from the definition of $\eta_\infty(B)$ that

$$(4.1) \quad \eta_\infty(B) = \begin{cases} 1 & \text{if } n \equiv 0, 1 \pmod{4}. \\ -1 & \text{if } n \equiv 2, 3 \pmod{4}. \end{cases}$$

Note that $\mathbb{Z}_p^{2n-1}$ is a maximal $\mathbb{Z}_p$ -integral lattice with respect to B for all primes p by assumption. Taking the classification of maximal $\mathbb{Z}_p$ -integral lattices (cf. [2, §9]) into account, we can observe that p divides D_B whenever $\eta_p(B) = -1$. Our assertion now follows from the product formula of the Hilbert symbol. $\square$

Lemma 4.3. *For an arbitrary rational prime p , there exists $B \in T_{2n-1}^+$ with $D_B = p$.*

Proof. It suffices to prove that there exists a positive definite even integral lattice L of rank $2n - 1$ with discriminant $2p$. We endow a lattice $M = \mathbb{Z}$ with the quadratic form $x \mapsto 2(-1)^{n-1}px^2$. Let H' be a definite quaternion algebra over $\mathbb{Q}$ which is ramified only at p and fix a maximal order R' . Take a lattice L' such that $L' \otimes_{\mathbb{Z}} \mathbb{Z}_p = \mathfrak{P}_p'^{-1}$, where let $\mathfrak{P}_p'$ be the maximal ideal of $R' \otimes_{\mathbb{Z}} \mathbb{Z}_p$, and $L' \otimes_{\mathbb{Z}} \mathbb{Z}_q = R' \otimes_{\mathbb{Z}} \mathbb{Z}_q$

for every rational prime q distinct from p . We endow the lattice $M' = \{x \in L' \mid x^t = -x\}$ with the quadratic form $x \mapsto 2(-1)^n p x x^t$, where $x \mapsto x^t$ is the main involution of H' .

For each prime q , we define a lattice L_q over $\mathbb{Z}_q$ as follows:

$$L_q = \begin{cases} (M \otimes_{\mathbb{Z}} \mathbb{Z}_q) \oplus H_q^{n-1} & \text{if } n \equiv 0, 1 \pmod{4} \\ (M' \otimes_{\mathbb{Z}} \mathbb{Z}_q) \oplus H_q^{n-2} & \text{if } n \equiv 2, 3 \pmod{4} \end{cases}$$

(cf. (4.1)). Here $H_q \simeq \mathbb{Z}_q^2$ is a hyperbolic plane over $\mathbb{Z}_q$. Note that L_q has discriminant $2p$. The Minkowski-Hasse theorem (see [8, Theorem 4.1.2]) shows that there exists a positive definite quadratic space V over $\mathbb{Q}$ such that $V \otimes_{\mathbb{Q}} \mathbb{Q}_q \simeq L_q \otimes_{\mathbb{Z}_q} \mathbb{Q}_q$ for every rational prime q . We can choose a lattice $L \subset V$ such that $L \otimes_{\mathbb{Z}} \mathbb{Z}_q \simeq L_q$ for every rational prime q . Then L has the required properties. $\square$

Lemma 4.4. *Let $B \in T_{2n-1}^+$ and $m \in \mathbb{N}$. Assume that D_B equals a rational prime p . Then the following conditions are equivalent:*

- (i) *there exists $(a, \alpha) \in \mathcal{T}_B^+$ such that $D_{B_{a,\alpha}} = m$;*
- (ii) *$(-1)^n m \equiv 0, 1 \pmod{4}$ and*

$$\left(\frac{(-1)^n m}{p} \right) \neq \begin{cases} -1 & \text{if } n \equiv 0, 1 \pmod{4}. \\ 1 & \text{if } n \equiv 2, 3 \pmod{4}. \end{cases}$$

Proof. We will consider the case when $n \equiv 2, 3 \pmod{4}$. The other case follows similarly. For m to be of the form D_h for some $h \in T_{2n}^+$, we must have $m \in \mathfrak{D}_n$. Choose an element $c \in \mathbb{Z}_p^\times$ in such a way that $\underline{\psi}_p(c) = -1$. Since an anisotropic kernel of the quadratic form associated to B over $\mathbb{Q}_p$ has dimension 3 by Lemma 4.2, B over $\mathbb{Z}_p$ is equivalent to

$$H_p^{n-2} \oplus K_p \oplus (-1)^{n-1} p c$$

by the classification of maximal lattices (cf. [2, §9]), where K_p is the norm form of the unramified quadratic extension $\mathbb{Q}_p(\sqrt{c})$. Since

$$(-1)^n D_{B_{a,\alpha}} = (-1)^n 4p(a - B[\alpha]) \equiv c\mu^2 \pmod{4p}$$

for some $\mu \in \mathbb{Z}_p$, (i) implies that $(-1)^n m$ is not a square modulo p or is divisible by p . We can easily check the other implication of our statement. $\square$

For there to be $B \in T_r^+$ with $\det(2B) = 1$, it is necessary and sufficient that r is a multiple of 8. Thus the following corollary is a consequence of Lemmas 4.3 and 4.4.

Corollary 4.5. We have

$$\{D_h \mid h \in T_{2n}^+\} = \begin{cases} \mathfrak{D}_n & \text{if } n \equiv 0, 1, 3 \pmod{4}. \\ \mathfrak{D}_n - \{1\} & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

Lemma 4.6. *Suppose that $n \equiv 2, 3 \pmod{4}$. Choose a rational prime p and $B \in T_{2n-1}^+$ in such a way that $D_B = p$. Letting $(a, \alpha) \in \mathcal{T}_B^+$, we have*

$$\tilde{F}_q(B^\sim; X) = \begin{cases} \ell_{q, D_{B^\sim}}(X) & \text{if } q \neq p. \\ \ell_{p, D_{B^\sim}}(X) - p^{-1/2} \ell_{p, p^{-2} D_{B^\sim}}(X) & \text{if } q = p. \end{cases}$$

Here, we abbreviate $B^\sim = B_{a, \alpha} = \begin{pmatrix} B & B\alpha \\ \alpha B & a \end{pmatrix}$ to simplify notation.

Proof. As explained in the proof of Lemma 4.4, there is $A \in \mathrm{GL}_{2n}(\mathbb{Z}_p)$ such that

$$B^\sim[A] = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}^{\oplus(n-2)} \oplus \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1-c}{4} \end{pmatrix} \oplus B',$$

where $B' \in T_{2,p}$ is of the form

$$B' = \begin{pmatrix} (-1)^{n-1}pc & * \\ * & * \end{pmatrix}.$$

Note that $\lambda_p(B^\sim) = -1$, $s_p(B^\sim) = s_p(B')$ and

$$\left(\frac{\mathfrak{D}_{B^\sim}}{p} \right) = \left(\frac{c\mathfrak{D}_{B'}}{p} \right) = - \left(\frac{\mathfrak{D}_{B'}}{p} \right).$$

It is easy to see that $\underline{\psi}_p((-1)^n D_{B^\sim}) = -\underline{\psi}_p(-D_{B'})$ and $\mathfrak{f}_p(B^\sim) = \mathfrak{f}_p(B')$. Moreover, the map

$$G \mapsto \begin{pmatrix} \mathbf{1}_{2n-2} & \\ & G \end{pmatrix} A$$

induces a bijection between $\mathcal{D}_p(B')$ and $\mathcal{D}_p(B^\sim)$. Combining these facts with (1.1), we have

$$\tilde{F}_p(B^\sim; X) = (-1)^{\mathfrak{f}_p(B')} \tilde{F}_p(B'; -X).$$

For any nondegenerate matrix $h \in T_{2,p}$, it is well-known that

$$\tilde{F}_p(h; X) = \sum_{a=0}^{\epsilon_p(h)} p^{-a/2} \ell_{p, D_{p^{-a}h}}(X),$$

where

$$\epsilon_p(h) = \max\{a \in \mathbb{N} \cup \{0\} \mid p^{-a}h \in T_{2,p}\}$$

(for example, see [6]). We conclude that

$$\tilde{F}_p(B^\sim; X) = (-1)^{\mathfrak{f}_p(B')} \tilde{F}_p(B'; -X)$$

$$\begin{aligned}
&= (-1)^{f_p(B')} (\ell_{p, D_{B'}}(-X) + p^{-1/2} \ell_{p, D_{p^{-1}B'}}(-X)) \\
&= \ell_{p, D_{B \sim}}(X) - p^{-1/2} \ell_{p, p^{-2}D_{B \sim}}(X),
\end{aligned}$$

observing that $\epsilon_p(B') = 1$ if and only if $f_p(B') > 0$. We can prove the formula for primes q distinct from p by a simpler way. $\square$

The following corollary immediately follows from Lemmas 4.4 and 4.6.

Corollary 4.7. Suppose that $F \in S_{k+n}(\mathrm{Sp}_{2n}(\mathbb{Z}))$ has Fourier coefficients of the form (1.3). If D_B equals a prime number p , then the Fourier expansion of $i(B)F$ is as follows:

$$i(B)F(\tau) = \sum_{m \in \mathfrak{D}_k} \left(1 - \left(\frac{(-1)^k m}{p} \right) \right) (c(m) - p^k c(p^{-2}m)) q^m.$$

Put

$$\begin{aligned}
S_{k+1/2}^{++}(4p) &= \left\{ g \in S_{k+1/2}^+(4p) \mid c_g(m) = 0 \text{ whenever } \left(\frac{(-1)^k m}{p} \right) = 1 \right\}, \\
S_{k+1/2}^{\text{new}, ++}(4p) &= \left\{ g \in S_{k+1/2}^{\text{new}, +}(4p) \mid c_g(m) = 0 \text{ whenever } \left(\frac{(-1)^k m}{p} \right) = 1 \right\}.
\end{aligned}$$

We define the map $P(p) : S_{k+1/2}^+(N) \rightarrow S_{k+1/2}^+(Np)$ by

$$\begin{aligned}
g|P(p)(\tau) &= g(\tau) - p^{1-k} g(\tilde{T}(p^2) - U_k(p^2))(\tau) \\
&= \sum_{m \in \mathfrak{D}_k} \left(1 - \left(\frac{(-1)^k m}{p} \right) \right) (c_g(m) - p^k c_g(p^{-2}m)) q^m,
\end{aligned}$$

provided that N and p are coprime. The following lemma is an easy consequence of Theorem 2.1 and [11, Propositions 2, 4 and Corollary 2 (2)].

Lemma 4.8. *The direct sum $S_{k+1/2}^+(4) \oplus S_{k+1/2}^{\text{new}, ++}(4p)$ is isomorphic to $S_{k+1/2}^{++}(4p)$ via the $\mathbb{C}$ -linear map $(g, h) \mapsto g|P(p) + h$.*

Lemma 4.9. *The notation being as in Corollary 4.7, there is $g_B \in S_{k+1/2}^+(4)$ such that*

$$i(B)F = g_B|P(p).$$

Proof. Proposition 3.1, Corollary 4.7 and Lemma 4.8 give $g \in S_{k+1/2}^+(4)$ and $h \in S_{k+1/2}^{\text{new}, ++}(4p)$ such that

$$i(B)F = g|P(p) + h.$$

Seeking a contradiction, we suppose $h \neq 0$. Note that $c_h(mp^{2f}) = p^{(k-1)f}c_h(m)$ by Proposition 2.2. Choose $m \in \mathbb{N}$ with properties:

$$c_h(m) \neq 0, \quad p^{-2}m \notin \mathfrak{D}_k.$$

Of course, we have

$$\left(\frac{(-1)^k m}{p}\right) \neq 1.$$

Comparing the coefficients of $q^{mp^{2f}}$, we have

$$c(m) = c_g(m) + \frac{c_h(m)}{1 - \left(\frac{(-1)^k m}{p}\right)}$$

and

$$c(mp^{2f}) - p^k c(mp^{2f-2}) = c_g(mp^{2f}) - p^k c_g(mp^{2f-2}) + c_h(mp^{2f})$$

for $f \in \mathbb{N}$. Therefore we have

$$c(mp^{2f}) = c_g(mp^{2f}) + c_h(m)p^{(k-1)f} \left(1 + p + p^2 + \cdots + p^{f-1} + \frac{p^f}{1 - \left(\frac{(-1)^k m}{p}\right)}\right).$$

It follows that the limit $\lim_{f \rightarrow \infty} p^{-kf}c(mp^{2f})$ exists. On the other hand, for every prime q distinct from p , we have $c(mq^{2f}) = O(q^{(k-1/2+\epsilon)f})$ for any $\epsilon > 0$ by the Ramanujan-Petersson conjecture combined with Theorem 2.1. This is a contradiction since p is arbitrary. Hence h must be zero, and $g_B = g$ works. $\square$

Lemma 4.10. *Suppose that p , B and g_B are related as in Lemma 4.9, and similarly for p' , B' and $g_{B'}$. Then we have $g_B = g_{B'}$.*

Proof. Corollary 4.7 and Lemma 4.9 show that $g_B|P(p)P(p') = g_{B'}|P(p)P(p')$. Since $P(p)$ and $P(p')$ are injective in view of Theorem 2.1 (1), we have $g_B = g_{B'}$ as claimed. $\square$

Now we are ready to prove Theorem 1.5. We may assume that $n \equiv 2, 3 \pmod{4}$ by [10]. We define g to be the function g_B in Lemma 4.9, which is independent of the choice of B on account of Lemma 4.10.

Fix a positive integer m such that $(-1)^k m \equiv 0, 1 \pmod{4}$. If there exists a rational prime p such that

$$\left(\frac{(-1)^k m}{p}\right) \neq 1,$$

then Corollary 4.7, Lemmas 4.3 and 4.9 show that

$$c(m) - p^k c(p^{-2}m) = c_g(m) - p^k c_g(p^{-2}m).$$

If $(-1)^k m$ is not a square, then we can find such a prime and deduce that $c(m) = c_g(m)$, which proves the case $n \equiv 3 \pmod{4}$.

Assume that $n \equiv 2 \pmod{4}$. We then have

$$c(f^2) = c_g(f^2) + (c(1) - c_g(1))f^k.$$

In view of Lemma 1.3 and Remark 1.4, the proof of Theorem 1.5 is now complete.

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