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Asymptotic behavior of solutions to the drift-diffusion equation of elliptic type

弘前大学大学院理工学研究科　山本 征法 (Masakazu Yamamoto)
Graduate School of Science and Technology
Hirosaki University

東京理科大学理学部数学科　杉山 裕介 (Yuusuke Sugiyama)
Department of Mathematics
Tokyo University of Science

1. Introduction

We consider the following initial-value problem for the drift-diffusion equation:

$$
(1.1) \qquad \begin{cases} \partial_t u + (-\Delta)^{\theta/2} u - \nabla \cdot (u \nabla \psi) = 0, & t > 0,\ x \in \mathbb{R}^n, \\ -\Delta \psi = u, & t > 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases}
$$

where $n \geq 2,\ 1 \leq \theta \leq 2,\ \partial_t = \partial/\partial t,\ (-\Delta)^{\theta/2}\varphi = \mathcal{F}^{-1}[|\xi|^\theta \mathcal{F}[\varphi]],\ \nabla = (\partial_1, \ldots, \partial_n),\ \partial_j = \partial/\partial x_j\ (1 \leq j \leq n),\ \Delta = \partial_1^2 + \cdots + \partial_n^2$, and $u_0 : \mathbb{R}^n \to \mathbb{R}$ is a given initial data. If we put

$$
u_\lambda(t,x) = \lambda^\theta u(\lambda^\theta t, \lambda x), \quad \psi_\lambda(t,x) = \lambda^{\theta-2}\psi(\lambda^\theta t, \lambda x)
$$

for $\lambda > 0$ and solutions (u, ψ) to the drift-diffusion equation, then $(u_\lambda, \psi_\lambda)$ fulfill the equation, and

$$
\sup_{t>0} \|u_\lambda(t)\|_{L^{n/\theta}(\mathbb{R}^n)} = \sup_{t>0} \|u(t)\|_{L^{n/\theta}(\mathbb{R}^n)}
$$

for $\lambda > 0$. Particularly, when $1 < \theta < n$, it follows that

$$
\sup_{t>0} \|\nabla\psi_\lambda(t)\|_{L^{n/(\theta-1)}(\mathbb{R}^n)} = \sup_{t>0} \|\nabla\psi(t)\|_{L^{n/(\theta-1)}(\mathbb{R}^n)}
$$

for $\lambda > 0$, and Hardy-Littlewood-Sobolev's inequality leads that

$$
(1.2) \qquad \|\nabla\psi(t)\|_{L^{n/(\theta-1)}(\mathbb{R}^n)} \leq C \|u(t)\|_{L^{n/\theta}(\mathbb{R}^n)}.
$$

Therefore, we can treat solutions on the scale-invariant class

$$
C\left((0,T), L^{n/\theta}(\mathbb{R}^n)\right)
$$

whenever $1 < \theta < n$. But we call the case $\theta = 1$ the critical since (1.2) does not work. Though well-posedness in several classes, and global in time existence of solutions of (1.1) for $1 \leq \theta \leq 2$ were proved (see [10, 11, 12, 13, 15, 18]). Moreover, the solution satisfies

$$
(1.3) \qquad u \in C^\infty\big((0,\infty), C^\infty(\mathbb{R}^n)\big),
$$

and

$$
(1.4) \qquad \|u(t)\|_{L^p(\mathbb{R}^n)} \leq C(1+t)^{-\frac{n}{\theta}(1-\frac{1}{p})}
$$

for $1 \leq p \leq \infty$. The purpose here is to establish large-time behavior of the solution. When $1 < \theta \leq 2$, the L^p-theory for a parabolic equation yields the asymptotic expansion of the solution as the time variable tends to infinity (cf. [1, 9, 17]). The similar argument as in the above preceding works is effective on the several problems (see for example [2, 3, 4, 5, 7, 8, 16]). However, for (1.1) with $\theta = 1$, the L^p-theory for a parabolic equation does not work since the dissipation balances the nonlinearity. Thus the drift-diffusion equation with $\theta = 1$ is an equation of elliptic type. Throughout this paper, we study (1.1) with $\theta = 1$. Before stating our main theorems, we refer to the following generalized Burgers equation of elliptic type:

$$\begin{cases} \partial_t \omega + (-\partial_x^2)^{1/2}\omega + \omega \partial_x \omega = 0, & t > 0,\ x \in \mathbb{R}, \\ \omega(0, x) = \omega_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.5}$$

For global solutions of (1.5), Iwabuchi [6] established the asymptotic expansion by employing the corresponding Besov spaces (see Section 4). To discuss large-time behavior of the solution of (1.1), we introduce the following integral equation associated with (1.1):

$$u(t) = P(t) * u_0 + \int_0^t P(t-s) * \nabla \cdot (u \nabla (-\Delta)^{-1} u)(s) ds, \tag{1.6}$$

where the Poisson kernel

$$P(t, x) = \pi^{-\frac{n+1}{2}} \Gamma(\tfrac{n+1}{2}) \frac{t}{(t^2 + |x|^2)^{\frac{n+1}{2}}}$$

is the fundamental solution to $\partial_t u + (-\Delta)^{\theta/2} u = 0$, and $*$ denotes the convolution for x. The solution of (1.6) is called the mild solution and solves (1.1).

Theorem 1.1 ([20]). *Let $n \geq 3$, $\theta = 1$, $u_0 \in L^1(\mathbb{R}^n, \sqrt{1 + |x|^2} dx)$, and the solution u of* (1.1) *satisfy* (1.3) *and* (1.4). *Then*

$$\|u(t) - M_u P(t) - m_u \cdot \nabla P(t)\|_{L^p(\mathbb{R}^n)} = o\big(t^{-n(1-\frac{1}{p})-1}\big)$$

as $t \to \infty$ for any $1 < p < \infty$, where $M_u = \int_{\mathbb{R}^n} u_0(y) dy$ and $m_u = \int_{\mathbb{R}^n} (-y) u_0(y) dy$.

In the two-dimensional case, we introduce the following function:

$$J(t, x) = \int_0^t P(t-s) * \nabla \cdot (P \nabla (-\Delta)^{-1} P)(s) ds. \tag{1.7}$$

This function is well-defined, and satisfies

$$J \in C\big((0, \infty), L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)\big),$$

and

$$\|J(t)\|_{L^p(\mathbb{R}^2)} = t^{-2(1-\frac{1}{p})-1} \|J(1)\|_{L^p(\mathbb{R}^2)}$$

for $1 \leq p \leq \infty$ and $t > 0$. We remark that this decay rate is same as one of $\nabla P(t)$. Then we obtain the asymptotic expansion for (1.1) with $n = 2$.

Theorem 1.2 ([20]). *Let $n = 2$, $\theta = 1$, $u_0 \in L^1(\mathbb{R}^2, \sqrt{1 + |x|^2} dx)$, and the solution u of* (1.1) *satisfy* (1.3) *and* (1.4). *Then*

$$\big\|u(t) - M_u P(t) - m_u \cdot \nabla P(t) - M_u^2 J(t)\big\|_{L^p(\mathbb{R}^2)} = o\big(t^{-2(1-\frac{1}{p})-1}\big)$$

as $t \to \infty$ for any $1 < p < \infty$, where $M_u = \int_{\mathbb{R}^2} u_0(y) dy$ and $m_u = \int_{\mathbb{R}^2} (-y) u_0(y) dy$.

Since $J(t)$ corrects the asymptotic expansion, we call this function the correction term. The proofs of Theorems 1.1 and 1.2 are based on the L^p-L^q type estimate for (1.6) with the aid of the energy method.

2. Preliminaries

Hardy-Littlewood-Sobolev's inequality yields the following inequality.

Lemma 2.1. *Let $n \geq 2$, $1 < \sigma < n$, $1 < p < \frac{n}{\sigma}$ and $\frac{1}{p_*} = \frac{1}{p} - \frac{\sigma}{n}$. Then there exists a positive constant C such that*

$$\|(-\Delta)^{-\sigma/2}\varphi\|_{L^{p_*}(\mathbb{R}^n)} \leq C\|\varphi\|_{L^p(\mathbb{R}^n)}$$

for any $\varphi \in L^p(\mathbb{R}^n)$.

It follows that

$$\|\nabla(-\Delta)^{-1}\varphi\|_{L^\infty(\mathbb{R}^n)} \leq C\left((1+t)\|\varphi\|_{L^\infty(\mathbb{R}^n)} + (1+t)^{-n+1}\|\varphi\|_{L^1(\mathbb{R}^n)}\right) \tag{2.1}$$

for any $\varphi \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $t > 0$. Indeed

$$|\nabla(-\Delta)^{-1}\varphi(x)| \leq C\left(\int_{|x-y|\leq 1+t} + \int_{|x-y|>1+t}\right)\frac{|\varphi(y)|}{|x-y|^{n-1}}dy$$

immediately gives (2.1). Since the Poisson kernel fulfills

$$\|\partial_t^k(-\Delta)^{\sigma/2}P(t)\|_{L^p(\mathbb{R}^n)} = t^{-n(1-\frac{1}{p})-k-\sigma}\|\partial_t^k\nabla^\alpha P(1)\|_{L^p(\mathbb{R}^n)}$$

for $k \in \mathbb{Z}_+$, $\sigma \geq 0$, $1 \leq p \leq \infty$ and $t > 0$, we obtain the following lemmas.

Lemma 2.2. *Let $n \geq 1$, $1 \leq p \leq q \leq \infty$, $k \in \mathbb{Z}_+$ and $\sigma \geq 0$. Then there exists a positive constant C such that*

$$\|\partial_t^k(-\Delta)^{\sigma/2}P(t) * \varphi\|_{L^q(\mathbb{R}^n)} \leq Ct^{-n(\frac{1}{p}-\frac{1}{q})-k-\sigma}\|\varphi\|_{L^p(\mathbb{R}^n)}$$

for any $\varphi \in L^p(\mathbb{R}^n)$ and $t > 0$.

Lemma 2.3. *Let $n \geq 1$, $k \in \mathbb{Z}_+$ and $\varphi \in L^1(\mathbb{R}^n, (1+|x|^2)^{k/2}\,dx)$. Then*

$$\left\|P(t) * \varphi - \sum_{|\alpha|\leq k}\frac{\nabla^\alpha P(t)}{\alpha!}\int_{\mathbb{R}^n}(-y)^\alpha\varphi(y)dy\right\|_{L^p(\mathbb{R}^n)} = o\left(t^{-n(1-\frac{1}{p})-k}\right)$$

as $t \to \infty$ for any $1 \leq p \leq \infty$. In addition, if $|x|^{k+1}\varphi \in L^1(\mathbb{R}^n)$, then

$$\left\|P(t) * \varphi - \sum_{|\alpha|\leq k}\frac{\nabla^\alpha P(t)}{\alpha!}\int_{\mathbb{R}^n}(-y)^\alpha\varphi(y)dy\right\|_{L^p(\mathbb{R}^n)} \leq Ct^{-n(1-\frac{1}{p})-k}(1+t)^{-1}$$

for any $1 \leq p \leq \infty$ and $t > 0$.

Proposition 2.4. *Let $n \geq 2$, $\theta = 1$, and the solution u of (1.1) satisfy (1.3) and (1.4). Then there exist positive constants C and T such that*

$$\|(-\Delta)^{1/4}u(t)\|_{L^2(\mathbb{R}^n)} \leq Ct^{-1/2}(1+t)^{-n/2} \tag{2.2}$$

for any $t \geq T$.

Proof. We multiply (1.1) by $t^q(-\Delta)^{1/2}u$ for sufficiently large $q > 0$, and have

$$\begin{aligned}
&\frac{d}{dt}\left(t^q\|(-\Delta)^{1/4}u\|_{L^2(\mathbb{R}^n)}^2\right) + 2t^q\|(-\Delta)^{1/2}u\|_{L^2(\mathbb{R}^n)}^2 \\
&= -t^q\int_{\mathbb{R}^n}\nabla u\cdot\nabla(-\Delta)^{-1}u(-\Delta)^{1/2}udx + t^q\int_{\mathbb{R}^n}u^2(-\Delta)^{1/2}udx \\
&\quad + qt^{q-1}\|(-\Delta)^{1/4}u\|_{L^2(\mathbb{R}^n)}^2.
\end{aligned} \tag{2.3}$$

By using (2.1) and (1.4), we see that

$$
\begin{aligned}
t^q\left|\int_{\mathbb{R}^3} \nabla u\cdot\nabla(-\Delta)^{-1}u(-\Delta)^{1/2}u dx\right| &\le Ct^q\left\|\nabla(-\Delta)^{-1}u(t)\right\|_{L^\infty(\mathbb{R}^n)}\left\|(-\Delta)^{1/2}u\right\|^2_{L^2(\mathbb{R}^n)} \\
&\le \frac{1}{3}t^q\left\|(-\Delta)^{1/2}u\right\|^2_{L^2(\mathbb{R}^n)}
\end{aligned} \tag{2.4}
$$

and

$$
\begin{aligned}
t^q\left|\int_{\mathbb{R}^n} u^2(-\Delta)^{1/2}u dx\right| &\le t^q\left\|u\right\|^2_{L^4(\mathbb{R}^n)}\left\|(-\Delta)^{1/2}u\right\|_{L^2(\mathbb{R}^n)} \\
&\le Ct^q(1+t)^{-3n}+\frac{1}{3}t^q\left\|(-\Delta)^{1/2}u\right\|^2_{L^2(\mathbb{R}^n)}
\end{aligned} \tag{2.5}
$$

for large $t>0$. Gagliardo-Nirenberg's inequality and (1.4) lead that

$$
\begin{aligned}
qt^{q-1}\left\|(-\Delta)^{1/4}u\right\|^2_{L^2(\mathbb{R}^n)} &\le Ct^{q-1}\left\|u\right\|_{L^2(\mathbb{R}^n)}\left\|(-\Delta)^{1/2}u\right\|_{L^2(\mathbb{R}^n)} \\
&\le Ct^{q-1}(1+t)^{-n/2}\left\|(-\Delta)^{1/2}u\right\|_{L^2(\mathbb{R}^n)} \\
&\le Ct^{q-2}(1+t)^{-n}+\frac{1}{3}t^q\left\|(-\Delta)^{1/2}u\right\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned} \tag{2.6}
$$

By applying (2.4), (2.5) and (2.6) to (2.3), we complete the proof. □

3. Outline of the Proof of Main Results

In this section, we outline the proof of our main theorem. The detailed proofs will appear in [20]. Before proving Theorem 1.2, we prepare the following two propositions.

Proposition 3.1. *Let $n=2$, $\theta=1$, $u_0\in L^1(\mathbb{R}^2,\sqrt{1+|x|^2}\,dx)$, and the solution u of* (1.1) *satisfy* (1.3) *and* (1.4). *Assume that $1<p<\infty$. Then*

$$
\|u(t)-M_uP(t)\|_{L^p(\mathbb{R}^2)}\le Ct^{-2(1-\frac{1}{p})}(1+t)^{-1}\log(2+t) \tag{3.1}
$$

for any $t>0$.

Proposition 3.2. *Let $n=2$, $\theta=1$, $u_0\in L^1(\mathbb{R}^2,\sqrt{1+|x|^2}\,dx)$, and the solution u of* (1.1) *satisfy* (1.3) *and* (1.4). *Assume that $1<p<\infty$ and $0<\sigma<\frac{1}{4p}$. Then there exist positive constants C and T such that*

$$
\left\|(-\Delta)^{\sigma/2}\left(u(t)-M_uP(t)\right)\right\|_{L^p(\mathbb{R}^2)}\le Ct^{-2(1-\frac{1}{p})-\sigma}(1+t)^{-1}\log(2+t)
$$

for any $t\ge T$.

The above propositions are proved by the L^p-L^q estimate for (1.6) together with (1.4) and (2.2).

Outline of Theorem 1.2. From (1.6) and (1.7), we see

$$
\begin{aligned}
&u(t) - M_u P(t) - m_u \cdot \nabla P(t) - M_u^2 J(t) \\
=&P(t) * u_0 - M_u P(t) - m_u \cdot \nabla P(t) \\
&+ \int_0^{t/2} \nabla P(t-s) * \big((u\nabla(-\Delta)^{-1}u)(s) - M_u^2 (P\nabla(-\Delta)^{-1}P)(1+s)\big)\, ds \\
&+ M_u^2 \int_0^{t/2} \nabla P(t-s) * \big((P\nabla(-\Delta)^{-1}P)(1+s) - (P\nabla(-\Delta)^{-1}P)(s)\big)\, ds \\
&+ \int_{t/2}^{t} P(t-s) * \nabla \cdot \big((u\nabla(-\Delta)^{-1}u)(s) - M_u^2 (P\nabla(-\Delta)^{-1}P)(s)\big)\, ds.
\end{aligned} \tag{3.2}
$$

Lemma 2.3 gives that

$$
\left\| P(t) * u_0 - M_u P(t) - m_u \cdot \nabla P(t) \right\|_{L^p(\mathbb{R}^2)} = o\left(t^{-2(1-\frac{1}{p})-1}\right) \tag{3.3}
$$

as $t \to \infty$. The second term on the right-hand side is rewritten by

$$
\begin{aligned}
&\int_0^{t/2} \nabla P(t-s) * \big((u\nabla(-\Delta)^{-1}u)(s) - M_u^2 (P\nabla(-\Delta)^{-1}P)(1+s)\big)\, ds \\
=&\int_0^{t/2} \int_{\mathbb{R}^2} (\nabla P(t-s, x-y) - \nabla P(t-s, x)) \\
&\qquad \cdot \big((u\nabla(-\Delta)^{-1}u)(s,y) - M_u^2 (P\nabla(-\Delta)^{-1}P)(1+s,y)\big)\, dyds
\end{aligned}
$$

since $\int_{\mathbb{R}^2}((u\nabla(-\Delta)^{-1}u)(s,y) - M_u^2(P\nabla(-\Delta)^{-1}P)(1+s,y))dy = 0$ for $s > 0$. We introduce $R(t) = o(t)$ as $t \to \infty$, then, by Taylor's theorem, we see that

$$
\begin{aligned}
&\int_0^{t/2} \int_{\mathbb{R}^2} (\nabla P(t-s, x-y) - \nabla P(t-s, x)) \\
&\qquad \cdot \big((u\nabla(-\Delta)^{-1}u)(s,y) - M_u^2 (P\nabla(-\Delta)^{-1}P)(1+s,y)\big)\, dyds \\
=&\int_0^{t/2} \int_{|y|\le R(t)} \int_0^1 (-y\cdot\nabla)\nabla P(t-s, x-y+\lambda y) \\
&\qquad \cdot \big((u\nabla(-\Delta)^{-1}u)(s,y) - M_u^2 (P\nabla(-\Delta)^{-1}P)(1+s,y)\big)\, d\lambda dyds \\
&+ \int_0^{t/2} \int_{|y|> R(t)} (\nabla P(t-s, x-y) - \nabla P(t-s, x)) \\
&\qquad \cdot \big((u\nabla(-\Delta)^{-1}u)(s,y) - M_u^2 (P\nabla(-\Delta)^{-1}P)(1+s,y)\big)\, dyds.
\end{aligned}
$$

By employing Lemma 2.2 together with (1.4), Lemma 2.1 and Proposition 3.1, we have that

$$
\begin{aligned}
&\Bigg\| \int_0^{t/2} \int_{|y|\le R(t)} \int_0^1 (-y\cdot\nabla)\nabla P(t-s, x-y+\lambda y) \\
&\qquad \cdot \big((u\nabla(-\Delta)^{-1}u)(s,y) - M_u^2 (P\nabla(-\Delta)^{-1}P)(1+s,y)\big)\, d\lambda dyds \Bigg\|_{L^p(\mathbb{R}^2)} \\
\le& CR(t) \int_0^{t/2} (t-s)^{-(1-\frac{1}{p})-2}(1+s)^{-2} ds = o\big(t^{-2(1-\frac{1}{p})-1}\big)
\end{aligned}
$$

as $t \to \infty$. Similarly, we obtain that

$$\begin{aligned}
&\Big\| \int_0^{t/2} \int_{\mathbb{R}^2} (\nabla P(t-s, x-y) - \nabla P(t-s, x)) \\
&\qquad \cdot \big((u\nabla(-\Delta)^{-1}u)(s,y) - M_u^2 (P\nabla(-\Delta)^{-1}P)(1+s,y) \big)\, dyds \Big\|_{L^p(\mathbb{R}^2)} \\
=& O\big(t^{-2(1-\frac{1}{p})-1}\big)
\end{aligned}$$

as $t \to \infty$. Hence Lebesgue's monotone theorem yields that

$$\begin{aligned}
&\Big\| \int_0^{t/2} \int_{|y|>R(t)} (\nabla P(t-s, x-y) - \nabla P(t-s, x)) \\
&\qquad \cdot \big((u\nabla(-\Delta)^{-1}u)(s,y) - M_u^2 (P\nabla(-\Delta)^{-1}P)(1+s,y) \big)\, dyds \Big\|_{L^p(\mathbb{R}^2)} \\
=& o\big(t^{-2(1-\frac{1}{p})-1}\big)
\end{aligned}$$

as $t \to \infty$. Therefore, it follows that

$$\begin{aligned}
&\Big\| \int_0^{t/2} \nabla P(t-s) * \big((u\nabla(-\Delta)^{-1}u)(s) - M_u^2 (P\nabla(-\Delta)^{-1}P)(1+s) \big)\, ds \Big\|_{L^p(\mathbb{R}^2)} \\
=& o\big(t^{-2(1-\frac{1}{p})-1}\big)
\end{aligned} \tag{3.4}$$

as $t \to \infty$. We see at once that

$$\begin{aligned}
&\Big\| \int_0^{t/2} \nabla P(t-s) * \big((P\nabla(-\Delta)^{-1}P)(1+s) - (P\nabla(-\Delta)^{-1}P)(s) \big)\, ds \Big\|_{L^p(\mathbb{R}^2)} \\
=& o\big(t^{-2(1-\frac{1}{p})-1}\big)
\end{aligned} \tag{3.5}$$

as $t \to \infty$. We represent the last term on the right-hand side of (3.2) by

$$\begin{aligned}
&\int_{t/2}^t P(t-s) * \nabla \cdot \big((u\nabla(-\Delta)^{-1}u)(s) - M_u^2 (P\nabla(-\Delta)^{-1}P)(s) \big)\, ds \\
=& \int_{t/2}^t \nabla(-\Delta)^{-\sigma/2} P(t-s) * (-\Delta)^{\sigma/2} \big((u\nabla(-\Delta)^{-1}u)(s) - M_u^2 (P\nabla(-\Delta)^{-1}P)(s) \big)\, ds
\end{aligned}$$

for some small $\sigma > 0$. Thus, by employing Lemma 2.2 together with Lemma 2.1, (2.2) and Proposition 3.2, we conclude that

$$\begin{aligned}
&\Big\| \int_{t/2}^t P(t-s) * \nabla \cdot \big((u\nabla(-\Delta)^{-1}u)(s) - M_u^2 (P\nabla(-\Delta)^{-1}P)(s) \big)\, ds \Big\|_{L^p(\mathbb{R}^2)} \\
\le& C \int_{t/2}^t (t-s)^{-(1-\sigma)} s^{-2(1-\frac{1}{p})-2-\sigma} ds = o\big(t^{-2(1-\frac{1}{p})-1}\big)
\end{aligned} \tag{3.6}$$

as $t \to \infty$. By applying (3.3), (3.4), (3.5) and (3.6) to (3.2), we complete the outline. □

Theorem 1.1 is proved in the similar way.

4. The Drift-Diffusion Equation and the Burgers Equation

We expect that the solution of the two dimensional drift-diffusion equation and one of the Burgers equation have a similar decay structure since those nonlinear terms decay with same order. Namely

$$\|\omega\partial_x\omega(t)\|_{L^1(\mathbb{R})} = O(t^{-1})$$

and

$$\|u\nabla(-\Delta)^{-1}u(t)\|_{L^1(\mathbb{R}^2)} = O(t^{-1})$$

as $t \to \infty$. To discuss an asymptotic expansion for (1.5) we make the following definition:

$$\begin{aligned} J_\omega(t,x) =& -\frac{1}{2}\int_0^{t/2}\int_{\mathbb{R}} \left(\partial_x P(t-s,x-y) - \partial_x P(t,x)\right) P(s,y)^2 dyds \\ & - \int_{t/2}^t P(t-s) * (P\partial_x P)(s)ds. \end{aligned} \tag{4.1}$$

This function is well-defined in $C((0,\infty), L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$, and satisfies

$$\|J_\omega(t)\|_{L^p(\mathbb{R})} = t^{-(1-\frac{1}{p})-1}\|J_\omega(1)\|_{L^p(\mathbb{R})}$$

for any $1 \le p \le \infty$ and $t > 0$. Then, for $1 \le p \le \infty$, the decaying solution $\omega(t)$ of (1.5) fulfills

$$\begin{aligned} &\Bigg\| \omega(t) - M_\omega P(t) + \frac{1}{4\pi}M_\omega^2 \partial_x P(t)\log(2+t) - M_\omega^2 J_\omega(t) \\ &- \left(m_\omega - \frac{1}{2}\int_0^\infty \int_{\mathbb{R}} \left(\omega(s,y)^2 - M_\omega^2 P(1+s,y)^2\right) dyds + \frac{1}{4\pi}M_\omega^2 \log 2\right)\partial_x P(t)\Bigg\|_{L^p(\mathbb{R})} \\ =& o\left(t^{-(1-\frac{1}{p})-1}\right) \end{aligned}$$

as $t \to \infty$, where $M_\omega = \int_{\mathbb{R}} \omega_0(y)dy$ and $m_\omega = \int_{\mathbb{R}}(-y)\omega_0(y)dy$ (cf. [6, 20]). The logarithmic term on this is derived from the following procedure: The mild solution of (1.5) is given by

$$\omega(t) = P(t) * \omega_0 - \frac{1}{2}\int_0^{t/2} \partial_x P(t-s) * (\omega^2)(s)ds - \int_{t/2}^t P(t-s) * (\omega\partial_x\omega)(s)ds. \tag{4.2}$$

We rewrite the nonlinear term by

$$\begin{aligned} &\int_0^{t/2} \partial_x P(t-s) * (\omega^2)(s)ds \\ =& \int_0^{t/2} \partial_x P(t-s) * \left(\omega(s)^2 - M_\omega^2 P(1+s)^2\right) ds + M_\omega^2 \int_0^{t/2} \partial_x P(t-s) * (P^2)(1+s)ds \\ =& \partial_x P(t) \int_0^\infty \int_{\mathbb{R}} \left(\omega(s)^2 - M_\omega^2 P(1+s,y)^2\right) dyds + M_\omega^2 \partial_x P(t) \int_0^{t/2}\int_{\mathbb{R}} P(1+s,y)^2 dyds \\ &+ M_\omega^2 \int_0^{t/2}\int_{\mathbb{R}} \left(\partial_x P(t-s,x-y) - \partial_x P(t,x)\right) P(s,y)^2 dyds + \rho_1(t) + \rho_2(t) + \rho_3(t), \end{aligned}$$

where

$$\rho_1(t) = \int_0^{t/2}\int_{\mathbb{R}} (\partial_x P(t-s, x-y) - \partial_x P(t,x)) \left(\omega(s,y)^2 - M_\omega^2 P(1+s,y)^2\right) dyds,$$

$$\rho_2(t) = -\, \partial_x P(t,x) \int_{t/2}^{\infty}\int_{\mathbb{R}} \left(\omega(s,y)^2 - M_\omega^2 P(1+s,y)^2\right) dyds,$$

$$\rho_3(t) = M_\omega^2 \int_0^{t/2}\int_{\mathbb{R}} (\partial_x P(t-s, x-y) - \partial_x P(t,x)) \left(P(1+s,y)^2 - P(s,y)^2\right) dyds.$$

The third term on the right-hand side is a part of $J_\omega(t)$, and the second term will lead the logarithmic term. Indeed we see that $\int_0^{t/2}\int_{\mathbb{R}} P(1+s,y)^2 dyds = \int_0^{t/2}(1+s)^{-1}ds \int_{\mathbb{R}} P(1,y)^2 dy = \frac{1}{2\pi}(\log(2+t) - \log 2)$. Since $\omega(t)$ converges to $M_\omega P(t)$, we can confirm that

$$\|\rho_1(t)\|_{L^p(\mathbb{R})}, \|\rho_2(t)\|_{L^p(\mathbb{R})}, \|\rho_3(t)\|_{L^p(\mathbb{R})} = o(t^{-(1-\frac{1}{p})-1})$$

as $t \to \infty$. In the study for (1.1), the similar logarithmic term as above appears seemingly. Namely, in the same manner as above, the nonlinear term on (1.6) provides

$$\begin{aligned}
&M_u^2 \nabla P(t) \cdot \int_0^{t/2}\int_{\mathbb{R}^2} (P\nabla(-\Delta)^{-1}P)(1+s,y) dyds \\
=&M_u^2 \nabla P(t) \cdot \int_0^{t/2} (1+s)^{-1} ds \int_{\mathbb{R}^2} (P\nabla(-\Delta)^{-1}P)(1,y) dy
\end{aligned}$$

into the asymptotic expansion. However, since $\int_{\mathbb{R}^2}(P\nabla(-\Delta)^{-1}P)(1,y)dy = 0$, this term is vanishing.

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