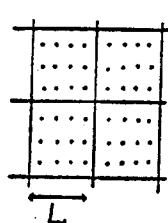


SCALING, RENORMALIZATION & CROSSOVER FOR STATIC & DYNAMICAL CRITICAL PHENOMENA IN CLASSICAL & QUANTUM SYSTEMS (M. SUZUKI)

1. DYNAMIC CELL ANALYSIS $\rightarrow$ SCALING LAW
2. LINEAR & NONLINEAR DYNAMICAL SCALING RELATIONS IN R.G. THEORY
3. DERIVATION OF FISHER'S FINITE-SIZE SCALING LAW USING R.G. THEORY
4. DYNAMIC FINITE-SIZE SCALING LAW-CROSSOVER EFFECT IN DYNAMICS
5. RELATION BETWEEN QUANTUM SPIN SYSTEM & ISING SYSTEM
— QUANTUM-CLASSICAL CROSSOVER —

(Ref. M. Suzuki, Prog. Theor. Phys. 51 (1974) 1257.)

Dynamic cell analysis $\rightarrow$ scaling law



TDGL model fluctuation

$$\frac{\partial S}{\partial t} = -\left(\frac{\Gamma}{k_B T}\right) \frac{\delta \mathcal{H}}{\delta S} + \eta(r, t)$$

$$\langle \eta(r, t) \eta(r', t') \rangle = 2 \Gamma \delta(r - r') \delta(t - t')$$

scale trans. $\epsilon = (T - T_c)/T_c$, $h = \mu_B H / k_B T$

$$R \rightarrow R' = R/L, \quad \epsilon \rightarrow \epsilon' = \epsilon L^{\gamma}, \quad h \rightarrow h' = h L^x$$

$$S \rightarrow S' = L^{d-x} S \quad ; \quad x = d - \beta/\nu$$

Coarse graining of time $t \rightarrow t' = L^{-z} t$

$$\text{(Markoffian appr.)} \quad \Gamma \rightarrow \Gamma' = L^{\phi} \Gamma$$

$\Downarrow$ new eq. of motion is invariant

$$\frac{\partial S'}{\partial t'} = -\left(\frac{\Gamma'}{k_B T'}\right) \frac{\delta \mathcal{H}'}{\delta S'} + \eta' ; \quad \eta' = \eta L^{x+\phi}$$

$$\text{if } z = 2x - d + \phi = \frac{\gamma}{\nu} + \phi = 2 - \eta + \phi$$

$$\text{Note that } (\mathcal{H}/k_B T) = L^{-d} (\mathcal{H}'/k_B T')$$

$$\text{and } [l.h.] = L^{d-x} L^z, \quad [r.h.] = L^{\phi} L^{-d} L^{-(d-x)}$$

invariance of eq. of motion for scale trans.

$$\Downarrow \text{time corr. func. } S(R, \epsilon, h; \omega) = \int_0^{\infty} \langle S(0, 0) S(R, t) \rangle e^{-i\omega t} dt$$

$$S(R, \epsilon, h; \omega) = L^{2(x-d)+z} S(R/L, L^{\gamma} \epsilon, h L^x, L^z \omega)$$

$$\uparrow \text{static matching } \int S(R, \epsilon, h) d\omega \propto R^{-(d-2+\gamma)}$$

Solution $\rightarrow$ dynamic scaling

$$S(R, \epsilon, h; \omega) = R^{2(x-d)+z} S(R, \epsilon^{1/\gamma}, \epsilon h^{-\gamma/x}; \omega R^z)$$

$$\text{where } \gamma = \frac{1}{\nu} \quad \text{and} \quad z = 2 - \eta + \phi$$

Linear and Nonlinear Dynamic Scaling Relations in the Renormalization Group Theory

→ Proof of Existence → Prog. Theor. Phys. 53 (1975) 1657, 55 (1976) 383, 58A (1976) 435
Generating function: (M.S. Phys. Letters 58A)

$$\Phi(\lambda, h, \varepsilon, t) = \frac{1}{\Omega} \log \int \prod_{k < 1} d\sigma_k e^{\lambda M} e^{t\Gamma} e^{hM - \mathcal{H}}$$

$$\mathcal{H}_2 = \sum_{n=1}^{\infty} \Omega^{1-n} \sum_{k_1, k_2, \dots, k_n < 1} u_{2n, 2}(k_1, \dots, k_n) \sigma_{2k_1} \dots \sigma_{2k_n}$$

$$\Gamma_2 = \int \prod_{i < 1} d\sigma_{2i} \frac{\partial}{\partial \sigma_{2i}} \left(\frac{\partial \mathcal{H}_2}{\partial \sigma_{2i}} + \frac{\partial}{\partial \sigma_{2i}} \right) + \dots$$

to rescale the variables as

$$k \rightarrow bk, \quad \Omega \rightarrow b^d \Omega, \quad \sigma_{2k} \rightarrow b^{-\eta/2} \sigma_{2k}, \quad t \rightarrow b^{-z} t$$

$$t_2 \Gamma_2 = t_{2+1} \Gamma_{2+1} = \text{invariant}$$

(confirmed perturbationally c.f. M.S. & F.T.)
(Prog. Theor. Phys. 52 (1974) 722)

Note that $\lambda M = \lambda \Omega^{1/2} \sigma_{0,0}$ and $hM = h \Omega^{1/2} \sigma_0$

renormalization proc. ($b^{-1} < k < 1$)

→ does not change results

with $\lambda_{2+1} = b^x \lambda_2$, $h_{2+1} = b^y h_2$,
we have

$$\Phi_2(\lambda, h, \varepsilon, t) = b^{-d} \Phi_{2+1}(\lambda, h, \varepsilon, t) + f_2(\varepsilon, t)$$

$$\Phi_{2+1}(\lambda, h, \varepsilon, t) = \Phi_2(\lambda b^x, h b^y, \varepsilon b^z, t b^{-z}) \quad \{u_{2n, 2}\}$$

$$\Phi_2(\lambda, h, \varepsilon, t) = b^{-d} \Phi_2(\lambda b^x, h b^y, \varepsilon b^z, t b^{-z}) + f_2(\varepsilon, t)$$

Differentiate w.r. to λ and define $m_2 \equiv \left(\frac{\partial \Phi_2}{\partial \lambda} \right)_{\lambda=0}$

$$\therefore m_2(h, \varepsilon, t) = b^{x-d} m_2(h b^y, \varepsilon b^z, t b^{-z})$$

General sol.

$$m(t) = h^{1/\delta} F_1(h \varepsilon^{-\beta \delta}, t \varepsilon^{\nu z}) \text{ or } m(t) = \varepsilon^\beta F_2(h \varepsilon^{-\beta \delta}, t \varepsilon^{\nu z})$$

where

$$\nu = \frac{1}{y}, \quad \delta = \frac{x}{d-x} = \frac{d+2-\eta}{d-2+\eta}, \quad \beta = \frac{x}{\delta y} = \frac{\gamma}{\delta-1}$$

Note that $m(0) = h^{1/\delta} F_1(0, 0)$

$$m(t) = m(0) f_1\left(\frac{m(0)}{\varepsilon^\beta}, t \varepsilon^{\nu z}\right)$$

$$\text{or } m(t) = \varepsilon^\beta f_2\left(\frac{\varepsilon^\beta}{m(0)}, t \varepsilon^{\nu z}\right)$$

(i) linear relaxation $m(0) \ll \varepsilon^\beta$

$$\tau^{(2)} = \int_0^\infty f_1(0, t \varepsilon^{\nu z}) dt \propto \varepsilon^{-\nu z} \propto \varepsilon^{-\Delta^{(2)}}$$

(ii) nonlinear relax. $\varepsilon^\beta \ll m(0)$

$$\tau^{(n, 2)} = \varepsilon^\beta \int_0^\infty f_2(0, t \varepsilon^{\nu z}) dt \propto \varepsilon^{\beta - \nu z} \propto \varepsilon^{-\Delta^{(n, 2)}}$$

$$\therefore \Delta^{(n, 2)} = \Delta^{(2)} - \beta = \frac{1}{2} + \frac{1}{3}(4-d) + \dots \quad \begin{matrix} \text{Fisher} \\ \text{Racz} \end{matrix}$$

Crossover from lin. to nonlin. at $m(0) = m^* = \varepsilon^\beta$

Derivation of Fisher's Finite-Size Scaling Law

on the basis of the RG theory

a) type A $L \times L \times \dots \times L$; $\varepsilon = (T - T_c(\infty)) / T_c(\infty)$

Consider $\Phi_L(\varepsilon, k_0, h) = \frac{1}{\Omega} \log \int \prod_{k_0 < k < 1} d\sigma_{2,k} e^{hM - \mathcal{H}_L}$

$$\boxed{k_0 = L^{-1}} \quad (\text{no constraint})$$

to rescale the variables as

$$k \rightarrow bk, \quad k_0 \rightarrow bk_0, \quad h \rightarrow b^x h, \quad \varepsilon \rightarrow b^y \varepsilon$$

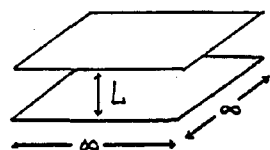
$$\begin{cases} \Phi_L(h, \varepsilon, k_0) = b^{-d} \Phi_{L+1}(h, \varepsilon, k_0) + f_L(\varepsilon), \\ \Phi_{L+1}(h, \varepsilon, k_0) = \Phi_L(hb^x, \varepsilon b^y, k_0 b) \end{cases}$$

$$M_L \equiv \frac{\partial \Phi_L}{\partial h} \rightarrow M_L(h, \varepsilon, k_0) = b^{x-d} M_L(hb^x, \varepsilon b^y, k_0 b)$$

$$\text{General sol. } M(h, \varepsilon, L) = L^{-\beta/\nu} m(h\varepsilon^{-\beta\delta}, \varepsilon L^{1/\nu})$$

(Fisher's Finite-size scaling law)

b) type B $\infty \times \infty \times L$;



$$\begin{cases} \varepsilon = (T - T_c(\infty)) / T_c(\infty) \\ \dot{\varepsilon} = (T - T_c(L)) / T_c(\infty) \end{cases}$$

③

$$\text{Consider } \Phi_L(h, \varepsilon, k_0) = \frac{1}{\Omega} \log \int \prod_{\substack{0 \leq k_x, k_y < 1 \\ k_0 < k_z < 1}} d\sigma_{2,k} e^{-\mathcal{H}_L + hM}$$

we have

$$\Phi_L(h, \dot{\varepsilon}, k_0) = b^{-d} \Phi_L(h, \dot{\varepsilon}, k_0) + f_L(\dot{\varepsilon})$$

Similarly to Type A, we get

$$M_L(h, \dot{\varepsilon}, k_0) = b^{x-d} M_L(hb^x, \dot{\varepsilon} b^y, k_0 b)$$

$$\therefore M(h, \dot{\varepsilon}, k_0) = L^{-\beta/\nu} m_2(h\dot{\varepsilon}^{-\beta\delta}, \dot{\varepsilon} L^{1/\nu})$$

(for no constraint)

Dynamic Finite-Size Scaling Law

— Crossover Effect in Dynamics

For simplicity, consider Type B

$$\Phi_L(\lambda, h, \dot{\epsilon}, k_0, t) = \frac{1}{\Omega} \log \int \prod_{0 < k_x, k_y < 1} d\sigma_{x,k} e^{\lambda M} e^{t \Gamma_L} e^{hM - \mathcal{H}_L}$$

$k_0 < k_z < 1$

to rescale the variables as

$$k \rightarrow bk, \quad k_0 \rightarrow bk_0, \quad t \rightarrow b^{-z}t$$

$$\Phi_L(\lambda, h, \dot{\epsilon}, k_0, t) = b^{-d} \Phi_L(\lambda b^x, h b^x, \dot{\epsilon} b^y, k_0 b, t b^{-z}) + f_L(\dot{\epsilon} t)$$

$$\therefore M_L(h, \dot{\epsilon}, k_0, t) = b^{x-d} M_L(h b^x, \dot{\epsilon} b^y, k_0 b, t b^{-z})$$

General sol.

$$M(h, \dot{\epsilon}, L, t) = L^{-\beta/\nu} f_1(h \dot{\epsilon}^{-\beta\delta}, \dot{\epsilon} L^{1/\nu}, t L^{-z})$$

Dynamic Finite-Size Scaling Law

$$\odot M(t) = t^{-\psi} m(L t^{-\frac{1}{z}}) \quad \text{at } \dot{\epsilon} = 0; \quad \boxed{\psi = \frac{\beta}{z\nu}}$$

$$\odot M(t) \propto t^{-\dot{\psi}} \quad \text{for } t \rightarrow \infty, \quad L < \infty; \quad \boxed{\dot{\psi} = \frac{\beta}{z\nu}}$$

$$\text{Then, } m(\xi) \approx M_0 \xi^{-z(\psi - \dot{\psi})} \quad \text{as } \xi \rightarrow 0$$

$$(\quad m(\xi) = \text{constant as } \xi \rightarrow \infty \quad)$$

$$\text{Thus, } M(t) \approx M_0 L^{-z(\psi - \dot{\psi})} t^{-\dot{\psi}}$$

$$M(t) \approx M_0 L^{-z\psi} (t/L^z)^{-\dot{\psi}}$$

$$(i) \quad t > t^x \Rightarrow M(t) \sim t^{-\dot{\psi}} \quad (d=2)$$

$$(ii) \quad t < t^x \Rightarrow M(t) \sim t^{-\psi} \quad (d=3)$$

$$\text{where } t^x = L^z \quad (\text{Crossover time})$$

$$\text{or } \omega \geq \omega^x; \quad \omega^x = L^{-z} = k_0^z$$

Crossover effect in Critical Dynamics

Relationship between Quantal Spin Systems

and Ising Systems — Quantum-Classical Crossover

(M. Suzuki, Prog. Theor. Phys. 56 (1976) No.5)

$$\text{Ex. } \mathcal{H}_Q^{(d)} = - \sum \sigma_i^z \sigma_j^z - \Gamma \sum \sigma_j^x$$

$$\boxed{\text{Th. 1}} \quad \mathcal{H}_Q^{(d)} \text{ at } T=0 \overset{\text{equivalent}}{\longleftrightarrow} \mathcal{H}_{\text{Ising}}^{(d+1)}$$

Proof Put $n = \Gamma/kT$ and note that

$$\mathcal{F} = -kT \log \text{Tr} \exp(-\mathcal{H}_Q^{(d)}/kT)$$

$$= -\frac{\Gamma}{n} \lim_{m \rightarrow \infty} \text{Tr} \left[\exp\left(\sum \frac{J_{ij}}{m\Gamma} \sigma_i^z \sigma_j^z\right) \exp\left(\frac{1}{m} \sum \sigma_j^x\right) \right]^{mn}$$

Trotter's formula

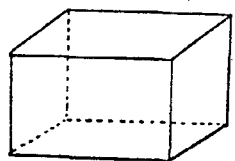
$$e^{A+B} = \lim_{m \rightarrow \infty} \left(e^{\frac{A}{m}} e^{\frac{B}{m}} \right)^m$$

Proof →

Generalized T.F.
M. Suzuki:
Commun.
Math. Phys.
51 (1976) 183

$$\mathcal{F} = -\frac{\Gamma}{n} \lim_{m \rightarrow \infty} \log \left[\left(\frac{1}{2} \sinh \frac{2}{m} \right)^{\frac{Nm n}{2}} \sum_{\sigma_{ij} = \pm 1} \exp \mathcal{H}_{\text{eff}}^{(n,m)} \right]$$

$$\mathcal{H}_{\text{eff}}^{(n,m)} = \frac{1}{m\Gamma} \sum_{i,j \in R^d} \sum_{k=1}^{mn} J_{ij} \sigma_{i,k} \sigma_{j,k} + \frac{1}{2} (\log \coth \frac{1}{m}) \sum \sigma_{i,k} \sigma_{i,k+1}$$



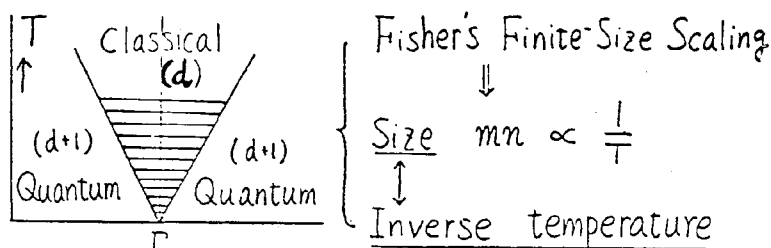
$$\left. \right\} mn \propto \frac{1}{T}$$

Corollary

Monte-Carlo Calculations
of Quantum Spin Systems.

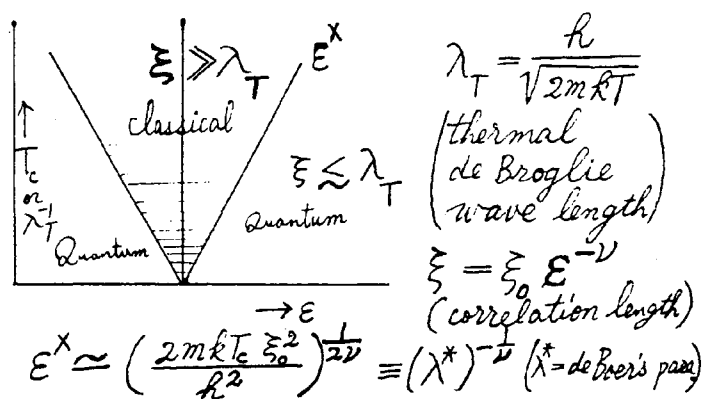
Real space (d dimensions) (Suzuki, Miyashita, Kuroda)

Th.2 Quantum - Classical Crossover



Quantum Crossover Effect in the Gas-Liquid P.T.

(M. Suzuki, Prog. Theor. Phys. 56(1976) No.3)
 crossover at $\xi \approx \lambda_T = h(2mkT)^{-1/2}$
 $\text{He}^4, \text{He}^3 \dots \gamma = 1.13$ ($d+1=4$ dimensions)
 $\longleftrightarrow \text{Xe}, \text{CO}_2 \dots \gamma = 1.25$



数值計算 (R.J. Elliott et al. J. Phys. C: Solid St. Phys. 4 ('71) 2359.
 A. Yanase et al. J. Phys. Soc. Japan (in press)
 J. Oitmaa et al. J. Phys. C: Solid St. Phys. 9 ('76) 2093.)

Applications

Z. Friedman: Phys. Rev. Lett. 36 ('76) 1326.
 K. Subbarao: two preprints.