

On Blow-Analytic Equivalence of Plane Curves

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Motivation

What is a “good” equivalence choice for the classification of real singularities?

- $\mathcal{C}^0$ -equivalence is very weak,
- $\mathcal{C}^r$ -equivalence ($r > 0$) is too strict:
for example, the Whitney family

$$(\{xy(x-y)(x-\lambda y) = 0\}, 0)$$

generates a continuous $\mathcal{C}^1$ -moduli.

Blow-analytic equivalence: stronger than $\mathcal{C}^0$ -equivalence, more flexible than analytic type classifications.

Definition

$f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is a **blow-analytic function** if there exists a composition of simple blow-ups $\beta = \beta_1 \circ \dots \circ \beta_n: X \rightarrow \mathbb{R}^2$ such that $f \circ \beta$ is analytic.

$h: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a **blow-analytic homeomorphism** if h and h^{-1} have blow-analytic components.

Goal

Classify germs of plane real curves up to blow-analytic homeomorphism.
Two plane curves $(C, 0)$, $(D, 0)$ are **blow-analytically equivalent** if there exists a blow-analytic homeomorphism $h: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that carries $(C, 0)$ to $(D, 0)$.

Blow-Analytic Invariants

$(C, 0)$: a germ of a real plane curve with an isolated singularity at the origin.

- Take $\beta: X \rightarrow \mathbb{R}^2$, a resolution which is
 - simple**: composition of simple blow-ups,
 - good**: $\beta^{-1}(C)$ is simple normal crossing (components are smooth and meet transversally).

- Construct the **weighted dual graph** Γ associated to X :

Γ		X
vertex v_i	$\longleftrightarrow$	except. curve E_i
edge $v_i v_j$	$\longleftrightarrow$	$E_i \cdot E_j = 1$
		$w(v_i) = E_i^2 \pmod 2$

Notation:

- : odd except. curve,
- : even except. curve,
- × : non-compact comp.

- Define:

A : the intersection matrix associated to Γ ,

i.e. $A = (a_{ij})$, where $a_{ii} = w(v_i)$ and $a_{ij} = E_i \cdot E_j \pmod 2$.

Δ : the minimal subtree connecting the non-compact components in Γ .

Γ' : the subgraph of Γ spanned by the vertices not in Δ .

A' : the intersection matrix associated to Γ' .

INVARIANTS:

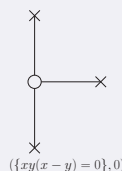
$\mu = \text{corank } A \longleftrightarrow$	surface
$\mu' = \text{corank } A' \longleftrightarrow$	embedded curve

Explicit Results

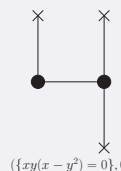
Theorem. [1] All unbranched germs of plane curves are blow-analytically equivalent to a line.

Theorem. [2] Bibranch germs of plane curves are blow-analytically equivalent in the level of the graph Γ if and only if they have the same μ' .

Theorem. [2] A germ of a tribranched plane curve with $\mu' = 0$ is blow-analytically equivalent to one of the following:

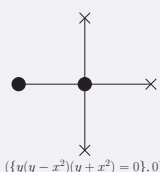


$$(\{xy(x-y) = 0\}, 0)$$

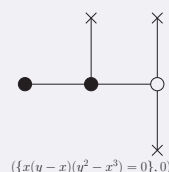


$$(\{x(y-x)(y-x^2) = 0\}, 0)$$

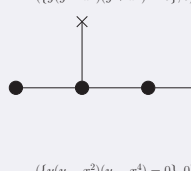
Theorem. [3] A germ of a tribranched plane curve with $\mu' = 1$ is blow-analytically equivalent in the level of the graph Γ to one of the following:



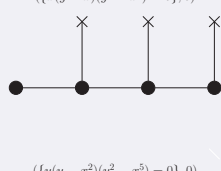
$$(\{y(y-x^2)(y+x^2) = 0\}, 0)$$



$$(\{x(y-x)(y^2-x^3) = 0\}, 0)$$



$$(\{y(y-x^2)(y-x^4) = 0\}, 0)$$



$$(\{y(y-x^2)(y^2-x^5) = 0\}, 0)$$

A Finite Classification

Main Theorem. [3] The number of blow-analytic equivalence classes of plane n -branched germs of curves with $\mu' = k$ is finite, for any n, k in $\mathbb{N}$.

The number of possible non-equivalent Γ can be easily estimated by means of graph theory.

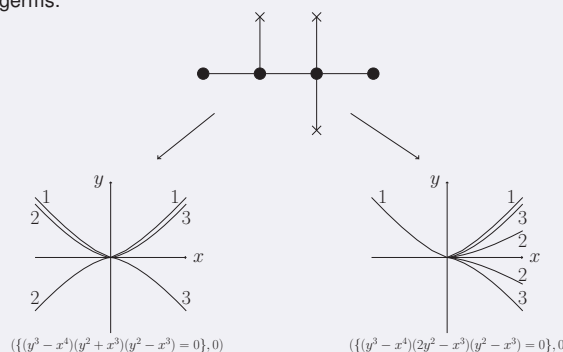
In the tribranched case this is lower or equal than

$$(k^3 - 2k^2 - k + 11)2^{k-2} \quad (\forall k \in \mathbb{N}).$$

From Graphs to Germs

Dual graphs Γ carry less information than resolutions of singularities X .

In some cases, a graph corresponds to two or more blow-analytically distinct curve germs:



INVARIANT:

Order of intersection of the demibranches with the boundary of the exceptional set.

We can obtain information about the blow-analytic type of a germ without having to resolve the singularity.

Proposition. A bibranch germ of plane curve is blow-analytically equivalent to $(\{xy = 0\}, 0)$ if and only if the order of its demibranches is 1212.

Proposition. If an n -branched germ of plane curve has order of demibranches $12 \dots n12 \dots n$, then $\mu' = 0$.

Conjecture. An n -branched germ of plane curve is blow-analytically equivalent to

$$(\{\prod_{i=1}^n (x - \lambda_i y) = 0\}, 0) \quad (\forall i \neq j, \lambda_i \neq \lambda_j)$$

if and only if the order of its demibranches is $12 \dots n12 \dots n$.

References

- [1] M. Kobayashi, T.-C. Kuo, *On Blow-Analytic Equivalence of Embedded Curve Singularities*. Proc. Jpn. Acad. **73**, 97-99 (1997)
- [2] M. Kobayashi, *On Blow-Analytic Equivalence of Branched Curves in $\mathbb{R}^2$* . (preprint)
- [3] C. Valle, *On the Blow-analytic Equivalence of Tribranched Plane Curves*. (preprint)