Johnson-type homomorphisms, a conjecture by Levine, and the LMO invariant

Anderson Vera[1]
Department of Mathematics, Kyoto University

1 Introduction

This note is based on the talk at the conference Intelligence of Low-dimensional Topology 2020 held at the Research Institute of Mathematical Sciences, Kyoto University. It provides a survey on the results on different filtrations of the mapping class group, their generalizations to homology cobordisms and their relations with the functorial extension of the Le-Murakami-Ohtsuki invariant, called the LMO functor. We refer to [10, 9, 3, 20, 21] for a detailed exposition of these subjects.

The organization of the note is as follows. In Section 2 we introduce the basic definitions about mapping class group and homology cobordisms. Section 3 is devoted to Johnson-type filtrations and Jhonson-type homomorphisms, in particular we state a result which can be considered a weak version of a conjecture stated by Levine about the comparison of two of the filtrations. In Section 4 we explain the explicit relations between the Johnson-type homomorphisms and the tree reduction of the LMO functor.

2 Mapping class group and Homology cobordisms

2.1 Mapping class group and some important subgroups

Let Σ be a compact connected oriented surface of genus g with exactly one boundary component. Denote by $\mathcal{M}$ the mapping class group of Σ, that is, the group of isotopy classes of orientation-preserving homeomorphisms $h : \Sigma \to \Sigma$ which are the identity on the boundary $\partial\Sigma$ of Σ.

From the 3-dimensional point of view it is natural to consider Σ as being part of the boundary of a handlebody V of genus g, that is, we consider an embedding $\iota : \Sigma \to V$ as shown in Figure 1.

Let $* \in \partial\Sigma$ and consider the following notations:

$$\pi = \pi_1(\Sigma, *), \qquad \pi' = \pi_1(V, *), \qquad \mathbb{A} = \ker(\pi \xrightarrow{\iota_\#} \pi'),$$
$$A = \ker(H \xrightarrow{\iota_*} H'),$$
$$H = H_1(\Sigma;\mathbb{Z}), \qquad H' = H_1(V;\mathbb{Z}) \quad \text{and} \qquad K_2 = \ker(\pi \xrightarrow{\iota_\#} \pi' \xrightarrow{\mathrm{ab}'} H') = \mathbb{A}\cdot[\pi,\pi].$$

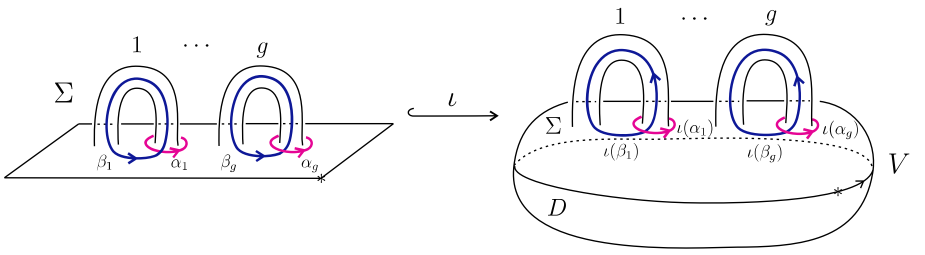


Figure 1: Embedding $\iota : \Sigma \to V$. Here $\partial V = \Sigma \cup D$, where $D \subset \partial V$ is an embedded disk.

The group $\mathcal{M}$ acts naturally on H and π. The *Torelli group*, denoted by $\mathcal{I}$, consists of the elements of $\mathcal{M}$ acting trivially on H, that is,

$$\mathcal{I} = \{h \in \mathcal{M} \mid h_* = \mathrm{Id}_H\} \tag{1}$$

We also consider the following subgroups of $\mathcal{M}$. The *handlebody group*

$$\mathcal{H} = \{h \in \mathcal{M} \mid h_\#(\mathbb{A}) \subset \mathbb{A}\}, \tag{2}$$

the *Lagrangian mapping class group*

$$\mathcal{L} = \{h \in \mathcal{M} \mid h_*(A) \subset A\}, \tag{3}$$

the *Lagrangian Torelli group*

$$\mathcal{I}^L = \{h \in \mathcal{L} \mid h_{*|A} = \mathrm{Id}_A\}, \tag{4}$$

and finally, the *alternative Torelli group*

$$\mathcal{I}^{\mathfrak{a}} = \left\{ h \in \mathcal{L} \;\middle|\; \begin{array}{c} \text{for } x \in \pi : \quad h_\#(x)x^{-1} \in K_2 \\ \text{and for } y \in K_2 : \\ h_\#(y)y^{-1} \in [\pi, [\pi, \pi]] \cdot [\mathbb{A}, \pi] =: K_3 \end{array} \right\}. \tag{5}$$

The group $\mathcal{I}^{\mathfrak{a}}$ can be defined as the subgroup of $\mathcal{M}$, generated by Dehn twists t_γ about curves γ on Σ which are homologically trivial in V. The above subgroups play an important role in the study of homology 3-spheres and the theory of finite-type invariants.

Example 2.1. *Let t_{α_i} be the (left) Dehn twist about the meridian curve α_i $(1 \leq i \leq g)$ from Figure 1. Then $t_{\alpha_i} \in \mathcal{I}^{\mathfrak{a}} \cap \mathcal{I}^L \cap \mathcal{H}$ but $t_{\alpha_i} \notin \mathcal{I}$. Similarly, let $1 \leq k < l \leq g$ and consider the simple closed curve α_{kl} which turns around the k-th handle and the l-th handle as shown in Figure 4.7 (a). Let $t_{\alpha_{kl}}$ be the (left) Dehn twist about α_{kl}. We have $t_{\alpha_{kl}} \in (\mathcal{I}^{\mathfrak{a}} \cap \mathcal{I}^L \cap \mathcal{H}) \setminus \mathcal{I}$.*

A remarkable result is that (for genus enough large) the Torelli group $\mathcal{I}$ is finitely generated [12]. It can also be shown that $\mathcal{I}^L$ and $\mathcal{I}^{\mathfrak{a}}$ are finitely generated (see [21, Remark 4.15]).

[1]The author is supported by the JSPS Postdoctoral Fellowship for Research in Japan ID No. PE19728

2.2 Homology cobordisms

The notion of homology cobordism was introduced independently by Goussarov [6] and by Habiro [8] in connection with the theory of finite-type invariants. A *homology cobordism* of Σ is the equivalence class of a pair $M = (M, m)$, where M is a compact connected oriented 3-manifold and $m : \partial(\Sigma \times [-1,1]) \to \partial M$ is an orientation-preserving homeomorphism, such that the *bottom* and *top* embeddings $m_\pm(\cdot) := m(\cdot, \pm 1) : \Sigma \to M$ induce isomorphisms in homology. Two pairs (M, m) and (M', m') are *equivalent* if there exists an orientation-preserving homeomorphism $\varphi : M \to M'$ such that $\varphi \circ m = m'$. From now on, we specify a homology cobordism by $M = (M, m_+, m_-)$.

The *composition* $M \circ M'$ of two homology cobordisms of Σ is the equivalence class of $(\widetilde{M}, m'_+, m_-)$, where $\widetilde{M}$ is obtained by gluing the two 3-manifolds M and M' by using the map $m_+ \circ (m'_-)^{-1}$. This composition gives a structure of a monoid. Denote by $\mathcal{C}$ the *monoid of homology cobordisms* of Σ.

Proposition 2.2. [9, Proposition 2.4] The *cylinder map* $c : \mathcal{M} \to \mathcal{C}$ defined by associating to any $h \in \mathcal{M}$ the homology cobordisms $(\Sigma \times [-1,1], h, \mathrm{Id}_\Sigma)$ is injective.

By the above proposition, the monoid $\mathcal{C}$ can be considered as a sort of generalization of $\mathcal{M}$. Let us define the submonoids of $\mathcal{C}$ respectively analogous to the subgroups $\mathcal{I}$, $\mathcal{H}$, $\mathcal{L}$ and $\mathcal{I}^L$ of $\mathcal{M}$.

The monoid of *homology cylinders*

$$\mathcal{IC} = \left\{(M, m_+, m_-) \in \mathcal{C} \mid m_{-,*}^{-1} \circ m_{+,*} = \mathrm{Id}_{H_1(\Sigma)}\right\}, \tag{6}$$

the monoid of *special Lagrangian cobordims*

$$\mathcal{HC} = \{(M, m_+, m_+) \in \mathcal{C} \mid M \cup_{m_-} V = V \text{ as cobordisms } \}, \tag{7}$$

the monoid of *Lagrangian homology cobordims*

$$\mathcal{C}^L = \{(M, m_+, m_-) \in \mathcal{C} \mid m_{+,*}(A) \subset m_{-,*}(A)\}, \tag{8}$$

and the monoid of *Lagrangian homology cylinders*

$$\mathcal{IC}^L = \{(M, m_+, m_-) \in \mathcal{C}^L \mid m_{+,*}|_A = m_{-,*}|_A\}. \tag{9}$$

By definition we have $c(\mathcal{I}) \subset \mathcal{IC}$, $c(\mathcal{H}) \subset \mathcal{HC}$, $c(\mathcal{L}) \subset \mathcal{C}^L$ and $c(\mathcal{I}^L) \subset \mathcal{IC}^L$.

3 Johnston-type homomorphisms

3.1 Johnson homomorphisms

From the works of Johnson [11] and Morita [18] we can consider a stepwise approximation of the action of $\mathcal{M}$ on π by considering the action of $\mathcal{M}$ on the nilpotent quotients $\pi/\Gamma_{n+1}\pi$ for $n \geq 1$, where $\{\Gamma_n\pi\}_{n\geq 1}$ is the *lower central series of* π (i.e. $\Gamma_1\pi = \pi$ and $\Gamma_{n+1}\pi = [\pi, \Gamma_n\pi]$ for $n \geq 1$). This gives rise to the so-called *Johnson filtration* of $\mathcal{M}$:

$$\mathcal{I} = J_1\mathcal{M} \supset J_2\mathcal{M} \supset J_3\mathcal{M} \cdots \tag{10}$$

where $J_n\mathcal{M}$ consists of the elements in $\mathcal{M}$ acting trivially on $\pi/\Gamma_{n+1}\pi$.

Let $\mathfrak{Lie}(H) = \oplus_{n\geq 1}\mathfrak{Lie}_n(H)$ be the free graded Lie algebra on H. Denote by $D_n(H)$ the kernel of the Lie bracket map $[\ ,\ ] : H \otimes \mathfrak{Lie}_{n+1}(H) \to \mathfrak{Lie}_{n+2}(H)$. Each one of the terms $J_n\mathcal{M}$ of the Johnson filtration comes equipped with a group homomorphism, the *n-th Johnson homomorphism*,

$$\tau_n : J_n\mathcal{M} \longrightarrow D_n(H) \tag{11}$$

such that $\ker(\tau_n) = J_{n+1}\mathcal{M}$.

The Johnson filtration and the Johnson homomorphisms of $\mathcal{M}$ extend in a natural way to $\mathcal{C}$, see [4]. Given $M = (M, m_+, m_-) \in \mathcal{C}$, since m_+ and m_- induce isomorphisms in homology in all degrees, by Stallings' theorem [19, Theorem 3.4], the maps $m_{\pm,*}$: $\pi/\Gamma_n\pi \to \pi_1(M,*)/\Gamma_n\pi_1(M,*)$ are isomorphisms for all $n \geq 2$. The *Johnson filtration* of $\mathcal{C}$ is the decreasing sequence of submonoids

$$\mathcal{IC} = J_1\mathcal{C} \supset J_2\mathcal{C} \supset J_3\mathcal{C} \cdots \tag{12}$$

where $J_n\mathcal{C}$ consists of the elements $(M, m_+, m_)) \in \mathcal{C}$ such that $m_{-,*}^{-1} \circ m_{+,*}$ acts trivially on $\pi/\Gamma_{n+1}\pi$. Notice that under the cylinder map we have $c(J_n\mathcal{M}) \subset J_n\mathcal{C}$.

The homomorphism (11) extends to a monoid homomorphism,

$$\tau_n : J_n\mathcal{C} \longrightarrow D_n(H) \tag{13}$$

such that $\ker(\tau_n^L) = J_{n+1}^L\mathcal{C}$ and called the n-th *Johnson homomorphism* for $\mathcal{C}$.

3.2 Johnson-Levine homomorphisms

Levine [13, 15] introduced a filtration for $\mathcal{I}^L$, which we call *Johnson-Levine filtration*,

$$\mathcal{I}^L = J_1^L\mathcal{M} \supset J_2^L\mathcal{M} \supset J_3^L\mathcal{M} \supset \cdots \tag{14}$$

where

$$J_n^L\mathcal{M} = \{h \in \mathcal{I}^L \mid \iota_\# h_\#(\mathbb{A}) \subset \Gamma_{n+1}\pi'\}$$

for $n \geq 1$. Notice that $J_n\mathcal{M} \subset J_n^L\mathcal{M}$. Levine also introduced a family of group homomorphisms

$$\tau_n^L : J_n^L\mathcal{M} \longrightarrow D_n(H') \tag{15}$$

such that $\ker(\tau_n^L) = J_{n+1}^L\mathcal{M}$. Here $\mathfrak{Lie}(H') = \oplus_{n\geq 1}\mathfrak{Lie}_n(H')$ is the free graded Lie algebra on H' and $D_n(H')$ the kernel of the Lie bracket map $[\ ,\ ] : H' \otimes \mathfrak{Lie}_{n+1}(H') \to \mathfrak{Lie}_{n+2}(H')$.

The Johnson-Levine filtration extends naturally to homology cobordisms as the descending chain of submonoids

$$\mathcal{IC}^L = J_1^L\mathcal{C} \supset J_2^L\mathcal{C} \supset J_3^L\mathcal{C} \supset \cdots \tag{16}$$

where

$$J_n^L\mathcal{C} = \{M \in \mathcal{IC}^L \mid \iota_\# m_{-,*}^{-1} m_{+,*}(\mathbb{A}) \subset \Gamma_{n+1}\pi'\}.$$

Clearly we have $J_n\mathcal{C} \subset J_n^L\mathcal{C}$ and $c(J_n^L\mathcal{M}) \subset J_n^L\mathcal{C}$. The n-th Johnson-Levine homomorphism extends to a monoid homomorphism

$$\tau_n^L : J_n^L\mathcal{C} \longrightarrow D_n(H') \tag{17}$$

such that $\ker(\tau_n^L) = J_{n+1}^L\mathcal{C}$.

3.3 A conjecture by Levine

It is natural to ask about the relationship between the Johnson filtration and the Johnson-filtration. Levine proposed the following.

Conjecture 3.1 (Levine [15]). *For every $n \geq 1$, we have $J_n^L\mathcal{M} = J_n\mathcal{M} \cdot (\mathcal{H} \cap \mathcal{I}^L)$.*

Levine showed this conjecture for $n \in \{1, 2\}$. In [20] we obtained a comparison of the Johnson and Johnson-Levine filtrations for homology cobordisms but up to some surgery equivalence relations called Y_r-*equivalence* (defined for any $r \geq 1$). The notion of Y_r-equivalence was introduced independently by Goussarov [6, 5] and Habiro [8] in their study of finite-type invariants. Let us explain these relations, we follow [8].

Let G be a graph that can be decomposed into two subgraphs, say $G = G' \cup G^o$, such that G' is a unitrivalent graph and G^o is a union of looped edges of G. The subgraph G' is called the *shape* of G. Let us consider a compact oriented 3-manifold M (possibly with boundary) . A *graph clasper* in M is an embedding $\mathbb{G} \hookrightarrow \mathrm{int}(M)$ of a thickening $\mathbb{G}$ of G, see Figure 2. We still denote the image of the embedding by G. The *degree* of a graph clasper is the number of trivalent vertices of its shape. From now on, we assume that the degree of graph claspers is greater than or equal to 1.

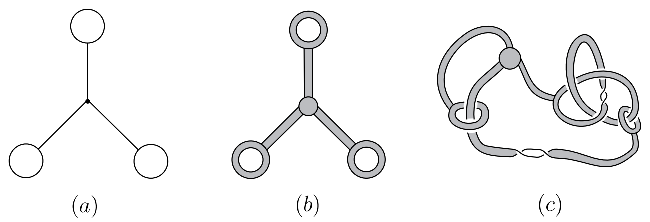


Figure 2: (a) Graph G. (b) Thickening $\mathbb{G}$. (c) Embedding $\mathbb{G} \hookrightarrow M$.

A graph clasper G in M carries surgery instructions for modifying M as follows. Suppose that G has degree 1. Consider a regular neighbourhood $N(G)$ of G in $\mathrm{int}(M)$. Perform surgery in $N(G)$ along the framed six-component link L illustrated in Figure 3.

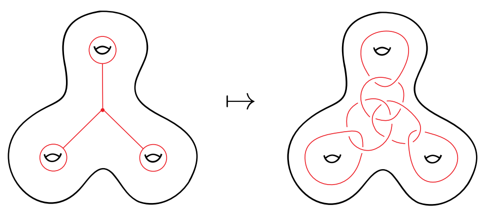


Figure 3: Framed link associated to a degree 1 clasper.

Denote the result by $N(G)_L$. We obtain a new 3-manifold M_G by setting

$$M_G := (M \setminus \mathrm{int}(N(G))) \cup N(G)_L.$$

If G is of degree > 1 we apply the *fission rule*, illustrated in Figure 4, until obtaining a disjoint union of degree 1 claspers. Then M_G is defined by performing surgery as before along each degree 1-clasper. If the degree of G is r, we say that M_G is obtained from M by a Y_r-*surgery*.

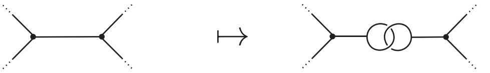


Figure 4: Fission rule.

The Y_r-*equivalence* is the equivalence relation among 3-manifolds generated by Y_r-surgeries and orientation-preserving homeomorphisms.

Habiro proved that $\mathcal{IC}/Y_r$ is a group [8, Theorem 5.4], see also [6, Theorem 9.2]. Consider the short exact sequence

$$1 \longrightarrow \mathcal{IC}/Y_r \xrightarrow{\subset} \mathcal{C}/Y_r \xrightarrow{\rho_1} \mathrm{Sp}(H) \longrightarrow 1, \tag{18}$$

where $\rho_1(M) = m_{-,*}^{-1} m_{+,*} : H \to H$ for $M = (M, m_+, m_-) \in \mathcal{C}$. This map is well-defined on $\mathcal{C}/Y_r$ by [17, Lemma 6.1]. It follows from (18) that $\mathcal{C}/Y_r$ is also a group.

Lemma 3.2 ([20, Lemma 4.5])**.** *For $r \geq n \geq 1$, the group $\mathcal{C}/Y_r$ contains $J_n\mathcal{C}/Y_r$ and $J_n^L\mathcal{C}/Y_r$ as subgroups.*

We can state now our comparison result.

Theorem 3.3 ([20, Theorem 4.9])**.** *For all $n, l \geq 1$, we have*

$$\frac{J_n^L\mathcal{C}}{Y_{n+l}} = \frac{J_n\mathcal{C}}{Y_{n+l}} \cdot q_{n+l}\left(\mathcal{HC} \cap \mathcal{IC}^L\right), \tag{19}$$

where $q_{n+l} : \mathcal{C} \to \mathcal{C}/Y_{n+l}$ is the canonical projection.

Compare this statement with Conjecture 3.1. The main application of Theorem 3.3 is that it allows to relate the Johnson-Levine homomorphisms with the tree reduction of the LMO functor, see Section 4.

3.4 Alternative Johnson homomorphisms

Habiro and Massuyeau [10] introduced a filtration for the alternative Torelli group $\mathcal{I}^{\mathfrak{a}}$, which we call *alternative Johnson filtration*,

$$\mathcal{I}^{\mathfrak{a}} = J_1^{\mathfrak{a}}\mathcal{M} \supset J_2^{\mathfrak{a}}\mathcal{M} \supset J_3^{\mathfrak{a}}\mathcal{M} \cdots$$

by using a decreasing sequence $\{K_n\}_{n\geq 1}$ of subgroups of π. This sequence is defined by

$$K_1 = \pi, \qquad K_2 = [\pi, \pi] \cdot \mathbb{A} \quad \text{and} \quad K_n = [K_1, K_{n-1}] \cdot [K_2, K_{n-1}] \quad \text{for } n \geq 2.$$

The n-the term of the alternative Johnson filtration is given by

$$J_n^{\mathfrak{a}}\mathcal{M} = \left\{ h \in \mathcal{L} \;\middle|\; \begin{array}{c} \text{for } x \in \pi : \quad h_\#(x)x^{-1} \in K_{1+n} \\ \text{and for } y \in K_2 : \\ h_\#(y)y^{-1} \in K_{2+n} \end{array} \right\}$$

for $n \geq 1$.

The main properties of the alternative Johnson filtration and its relations with the Johnson and Johnson-Levine filtration are given in the following result.

Theorem 3.4 ([21, Theorem A]). *We have*

(i) $\bigcap_{n\geq 1} J_n^{\mathfrak{a}}\mathcal{M} = \{\mathrm{Id}_\Sigma\}$.

(ii) *For all $n \geq 1$ the group $J_n^{\mathfrak{a}}\mathcal{M}$ is residually nilpotent, that is, $\bigcap_k \Gamma_k J_n^{\mathfrak{a}}\mathcal{M} = \{\mathrm{Id}_\Sigma\}$.*

Besides, for every $n \geq 1$, we have

(iii) $J_{2n}^{\mathfrak{a}}\mathcal{M} \subset J_n\mathcal{M}$. (iv) $J_n\mathcal{M} \subset J_{n-1}^{\mathfrak{a}}\mathcal{M}$. (v) $J_n^{\mathfrak{a}}\mathcal{M} \subset J_{n+1}^{L}\mathcal{M}$,

where $J_0^{\mathfrak{a}}\mathcal{M} = \mathcal{L}$. In particular, the Johnson filtration and the alternative Johnson filtration are cofinal.

Habiro and Massuyeau also introduced the respective family of Johnson-type homomorphisms

$$\tau_n^{\mathfrak{a}} : J_n^{\mathfrak{a}}\mathcal{M} \longrightarrow D_n(H', A) \tag{20}$$

such that $\ker(\tau_n^{\mathfrak{a}}) = J_{n+1}^{\mathfrak{a}}\mathcal{M}$, to which we refer as *alternative Johnson homomorphisms*. In this case the abelian group $D_n(H', A)$ is a subgroup of $(H' \oplus A) \otimes \mathfrak{Lie}(H', A)$, where $\mathfrak{Lie}(H', A)$ is the graded free Lie algebra generated by H' in degree 1 and A in degree 2.

Example 3.5. *For the Dehn twists $t_{\alpha_i}, t_{\alpha_{kl}} \in \mathcal{I}^{\mathfrak{a}}$ from Example 2.1, we have*

$$\tau_1^{\mathfrak{a}}(t_{\alpha_i}) = -a_i \otimes a_i$$

and

$$\tau_1^{\mathfrak{a}}(t_{\alpha_{kl}}) = -(a_k \otimes a_k) - (a_l \otimes a_l) - (a_k \otimes a_l) - (a_l \otimes a_k).$$

Lemma 3.6 ([21, Lemma 5.10]). *For $n \geq 1$, there is a well-defined homomomorphism*

$$\iota_* : D_n(H'; A) \longrightarrow D_{n+1}(H')$$

induced by $\iota_* : H \to H'$.

Proposition 3.7 ([21, Proposition 5.11]). For $n \geq 1$, the diagram

$$\begin{tikzcd}
J_n^{\mathfrak{a}}\mathcal{M} \arrow[r, "\subset"] \arrow[d, "\tau_n^{\mathfrak{a}}"'] & J_{n+1}^{L}\mathcal{M} \arrow[d, "\tau_{n+1}^{L}"] \\
D_n(H'; A) \arrow[r, "\iota_*"] & D_{n+1}(H')
\end{tikzcd}$$

is commutative. In other words, for $J_n^{\mathfrak{a}}\mathcal{M}$, the homomorphism τ_{n+1}^{L} is determined by the homomorphism $\tau_n^{\mathfrak{a}}$.

4 Johnson-type homomorphisms and the LMO functor

4.1 Jacobi diagrams and diagrammatic Johnson-type homomorphisms

A *Jacobi diagram* is a finite unitrivalent graph such that the trivalent vertices are *oriented*, that is, its incident edges are endowed with a cyclic order. Let C be a finite set. We call a

Jacobi diagram C*-colored* if its univalent vertices are colored with elements of the $\mathbb{Q}$-vector space spanned by C.

The *internal degree* of a Jacobi diagram is the number of trivalent vertices, we denote it by i-deg. We use dashed lines to represent Jacobi diagrams and, in pictures, we assume that the orientation of trivalent vertices is counterclockwise. See Figure 5 for some examples.

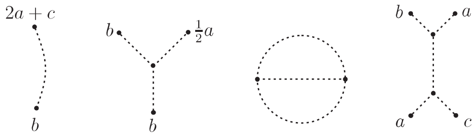


Figure 5: C-colored Jacobi diagrams of i-deg $0, 1, 2$ and 2, respectively. Here $C = \{a, b, c\}$

The space of C-colored Jacobi diagrams is defined as

$$\mathcal{A}(C) := \frac{\mathrm{Vect}_{\mathbb{Q}}\{C\text{-colored Jacobi diagrams}\}}{\text{AS, IHX, } \mathbb{Q}\text{-multilinearity}},$$

where the relations AS, IHX are local and the multilinearity relation applies to the C-colored vertices, see Figure 6.

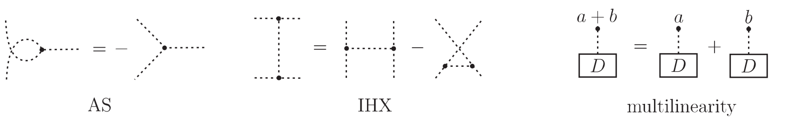


Figure 6: Relations in $\mathcal{T}(C)$. Here $a, b \in C$.

A Jacobi diagram in $\mathcal{A}(C)$ is *looped* if it has a non-contractible component, for instance the third diagram in Figure 5 is looped. We denote by $\mathcal{A}^t(C)$ the quotient of $\mathcal{A}(C)$ by the subspace generated by looped diagrams. We refer to the elements in $\mathcal{A}^t(C)$ as *tree-like* Jacobi diagrams. We denote by $\mathcal{A}^{t,c}(C)$ the subspace generated by connected tree-like Jacobi diagrams.

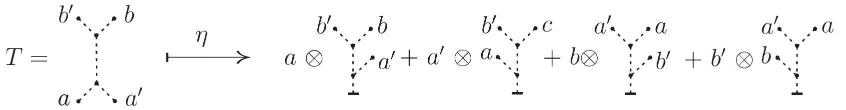
If G is a finitely generated free abelian group, we define the space $\mathcal{A}(G)$ of G-colored Jacobi diagrams by $\mathcal{A}(G) = \mathcal{A}(C)$ where C is any set of free generators of G. We are interested particularly in $G = H$, $G = H'$ or $G = H' \oplus A$.

The rational versions $D_m(H) \otimes \mathbb{Q}$, $D_m(H') \otimes \mathbb{Q}$ and $D_m(H'; A) \otimes \mathbb{Q}$ of the target spaces of the Johnson-type homomorphisms can be interpreted as subspaces of $\mathcal{A}^{t,c}(H)$, $\mathcal{A}^{t,c}(H')$ and $\mathcal{A}^{t,c}(H' \oplus A)$, respectively, as follows. For a connected tree-like Jacobi diagram T in one of these spaces, set

$$\eta(T) = \sum_{v} \mathrm{color}(v) \otimes (T \text{ rooted at } v), \tag{21}$$

where the sum ranges over the set of univalent vertices of T and we interpret a rooted tree as a Lie commutator.

Example 4.1. *Let $a, a' \in A$ and $b, b' \in H'$. Hence,*



$$= a \otimes [[b', b], a'] + a' \otimes [a, [b', b]] + b \otimes [[a', a], b'] + b' \otimes [b, [a', a]].$$

We have that $\eta(T) \in H \otimes \mathfrak{Lie}_3(H)$ and $\eta(T) \in \big(A \otimes \mathfrak{Lie}_4(H'; A)\big) \oplus \big(H' \otimes \mathfrak{Lie}_5(H'; A)\big)$.

For $n \geq 1$ and $G = H$ or $G = H'$, denote by $\mathcal{A}_n^{t,c}(G)$ the subspace of $\mathcal{A}^t(G)$ generated by connected diagrams of i-deg $= n$. If $T \in \mathcal{A}_n^{t,c}(G)$, then $\eta(T) \in D_m(H)$, see [14, Lemma 3.1]. The following result is well known.

Theorem 4.2. *For $n \geq 1$ the map*

$$\eta : \mathcal{A}_n^{t,c}(H) \longrightarrow D_n(H) \otimes \mathbb{Q}, \tag{22}$$

defined in (21) is an isomorphism of $\mathbb{Q}$-vector spaces.

We refer to [14, Corollary 3.2] or [7, Theorem 1] for a proof of Theorem 4.2.

In particular we have an isomorphism of graded $\mathbb{Q}$-vector spaces

$$\eta : \bigoplus_{n \geq 1} \mathcal{A}_n^{t,c}(H) \longrightarrow \bigoplus_{n \geq 1} D_n(H) \otimes \mathbb{Q}. \tag{23}$$

The same statements hold replacing H by H'. We define a degree for connected tree-like Jacobi diagrams with univalent vertices colored by $H' \oplus A$, which we call *alternative degree* and denote by $\mathfrak{a}$-deg, such that if $T \in \mathcal{A}^{t,c}(H' \oplus A)$ is such that $\mathfrak{a}\text{-deg}(T) = m$ then $\eta(T) \in D_m(H'; A) \otimes \mathbb{Q}$.

Definition 4.3. Let T be a $H' \oplus A$-colored connected tree-like Jacobi diagram. The *alternative degree* of T, denoted $\mathfrak{a}\text{-deg}(T)$, is defined as

$$\mathfrak{a}\text{-deg}(T) = 2\#\{A\text{-colored vertices of } T\} + \#\{H'\text{-colored vertices of } T\} - 3.$$

Here $\#S$ denotes the cardinal of the set S.

For $n \geq 1$, let $\mathcal{T}_n^{\mathfrak{a}}(H' \oplus A)$ denote the subspace of $\mathcal{A}^{t,c}(H' \oplus A)$ generated by diagrams of alternative degree n.

Proposition 4.4 ([21, Proposition 5.28])**.** For $n \geq 1$ the map η defined in (21) induces an isomorphism

$$\eta : \mathcal{T}_m^{\mathfrak{a}}(H' \oplus A) \longrightarrow D_m(H'; A) \otimes \mathbb{Q} \tag{24}$$

of $\mathbb{Q}$-vector spaces.

Theorem 4.2 and Proposition 4.4 allow to define diagrammatic versions of the Johnson-type homomorphisms.

Definition 4.5. Let $n \geq 1$. The *diagrammatic version* of the n-th alternative Johnson homomorphism is defined as the composition

$$J_n^{\mathfrak{a}}\mathcal{M} \xrightarrow{\tau_n^{\mathfrak{a}}} D_n(B;A) \otimes \mathbb{Q} \xrightarrow{\eta^{-1}} \mathcal{T}_m^{\mathfrak{a}}(H' \oplus A). \tag{25}$$

Similarly, the *diagrammatic versions* of the n-th Johnson homomorphism and of the n-th Johnson-Levine homomorphism are defined as the compositions

$$J_n\mathcal{M} \xrightarrow{\tau_n} D_n(H) \otimes \mathbb{Q} \xrightarrow{\eta^{-1}} \mathcal{A}_n^{t,c}(H) \tag{26}$$

and

$$J_n^L\mathcal{M} \xrightarrow{\tau_n^L} D_n(H') \otimes \mathbb{Q} \xrightarrow{\eta^{-1}} \mathcal{A}_n^{t,c}(H'), \tag{27}$$

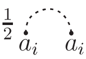
respectively.

In Equations (26) and (27) we can change $\mathcal{M}$ by $\mathcal{C}$.

Example 4.6. *In Example 3.5 we calculated* $\tau_1^{\mathfrak{a}}(t_{\alpha_i}) = -a_i \otimes a_i$ *and*

$$\tau_1^{\mathfrak{a}}(t_{\alpha_{kl}}) = -(a_k \otimes a_k) - (a_l \otimes a_l) - (a_k \otimes a_l) - (a_l \otimes a_k),$$

for the Dehn twists t_{α_i} *and* $t_{\alpha_{kl}}$ *from Example 2.1. We have*

$\eta^{-1}\tau_1^{\mathfrak{a}}(t_{\alpha_i}) = -\frac{1}{2}$  .

and

$\eta^{-1}\tau_1^{\mathfrak{a}}(t_{\alpha_{kl}}) = -\frac{1}{2}$  $-\ \frac{1}{2}$  $-$  .

4.2 Sketch of the definition of the LMO functor

This subsection is devoted to a brief description of the LMO functor $\widetilde{Z} : \mathcal{C}^L \to \mathcal{A}^2$. We refer to [3] for more details. In this subsection we also consider formal series of Jacobi diagrams. For instance, if D is a Jacobi diagram we can consider the series $\exp_{\sqcup}(D) = \sum_{k\geq 0} \frac{1}{k!} D^k$, where D^k is the disjoint union of k copies of D.

Let $M = (M, m_+, m_-) \in \mathcal{C}$. We can associate to M a particular kind of tangle whose components split in g bottom components labelled $1^-, \ldots, g^-$; and g top components labelled labelled $1^+, \ldots, g^+$ (they are called *bottom-top tangles* in [3]). The association is defined as follows. First fix a system of meridians and parallels $\{\alpha_i, \beta_i\}$ on Σ as shown in Figure 1.
Then attach g 2-handles on the bottom surface of M by sending the cores of the 2-handles to the curves $m_-(\alpha_i)$. In the same way, attach g 2-handles on the top surface of M by sending the cores to the curves $m_+(\beta_i)$. This way we obtain a pair (B, γ), called the *bottom-top tangle presentation* of M, where B is a compact connected oriented 3-manifold with $\partial([-1,1]^3) \xrightarrow{\sim} \partial B$ and γ is a tangle in B determined by the cocores of the 2-handles. In Figure 7 we illustrate the procedure to obtain the bottom-top tangle presentation of the trivial cobordism $\Sigma \times [-1,1]$, see also Example 4.7.

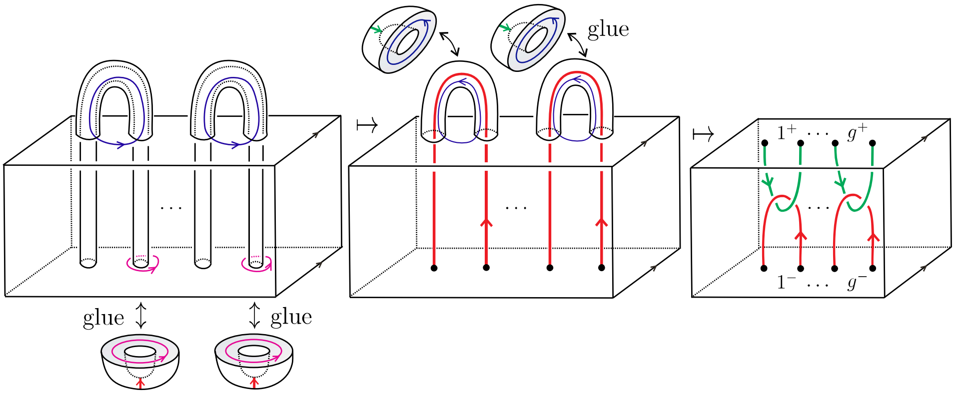
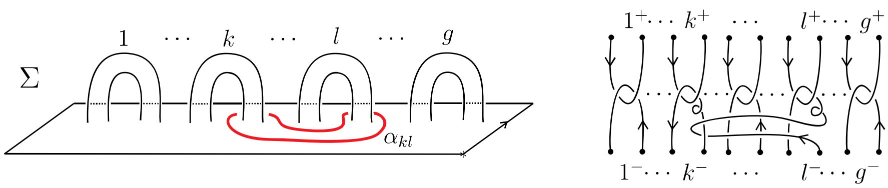


Figure 7: Obtaining the bottom-top tangle presentation of the trivial cobordism $\Sigma \times [-1, 1]$.



(a) (b)

Figure 8: (a) Curve α_{kl} and (b) bottom-top tangle presentation of $c(t_{\alpha_{kl}})$.

Example 4.7. *Figure 8 (b) shows the bottom-top tangle presentation of* $c(t_{\alpha_{kl}}) \in \mathcal{C}^L$, *where* $t_{\alpha_{kl}} \in \mathcal{I}^{\mathfrak{a}}$ *is the (left) Dehn twist about the curve* α_{kl}.

Basically the objects of the source and target category of the LMO functor are the non-negative integers[3]. The set of morphisms in the source category are Lagrangian homology cobordisms, in our case we only consider the monoid $\mathcal{C}^L$ which corresponds to the set of morphisms from g to g. The set of morphisms from g to f in the target category is a subspace $\mathcal{A}(g, f)$ of diagrams in $\mathcal{A}(C_f^g)$ where $C_f^g = \{1^+, \ldots, g^+\} \cup \{1^-, \ldots, f^-\}$.

Roughly speaking, the LMO functor $\widetilde{Z} : \mathcal{C}^L \to \mathcal{A}$ is defined as follows. Let $M \in \mathcal{C}^L$. Let (B, γ') be the bottom-top tangle presentation of M. Take a *surgery presentation* of (B, γ'), that is, a framed link $L \subset \mathrm{int}([-1, 1]^3)$ and a tangle γ in $[-1, 1]^3 \setminus L$ such that surgery along L carries $([-1, 1]^3, \gamma)$ to (B, γ'). Now, consider the Kontsevich integral of $L \cup \gamma$, which gives a series of a kind of Jacobi diagrams. To get rid of the ambiguity in the surgery presentation, it is necessary to use some combinatorial operations on the space of diagrams. Among these operations there is the so-called *Aarhus integral* (see [1, 2]), which is a kind of formal Gaussian integration on the space of diagrams. We then obtain

[2]We only consider a restriction of the LMO functor to the monoid of Lagrangin homology cobordisms $\mathcal{C}^L$.

[3]The definition of the LMO functor uses the *Kontsevich integral*, which is the universal invariant among Vassiliev (or finite-type) invariants of links. Because of this, it is necessary to modify the objects of in the source category: instead of non-negative integers, the objects are non-associative words in the single letter $\bullet$.

a series of Jacobi diagrams $\widetilde{Z}(M)$ in $\mathcal{A}(g,g)$.

The colors $1^+,\dots,g^+$ and $1^-,\dots,g^-$ in $\widetilde{Z}(M)$ refer to the curves $m_+(\beta_1),\dots, m_+(\beta_g)$ and $m_-(\alpha_1),\dots,m_-(\alpha_g)$ on the top and bottom surfaces of M respectively.

The definition of the Kontsevich integral requires the choice of a *Drinfeld associator*, and the bottom-top tangle presentation requires the choice of a system of meridians and parallels. Thus, the LMO functor also depends on these choices.

For $M \in \mathcal{C}^L$, denote by $\widetilde{Z}^t(M)$ the reduction of $\widetilde{Z}(M)$ modulo looped diagrams.

Example 4.8. *Consider the cobordism $c(t_{\alpha_i}) \in \mathcal{IC}^L$ from Example 2.1. We have*

$$\widetilde{Z}^t(c(t_{\alpha_i})) = \exp_\sqcup\left(\sum_{i=1}^{g} \underset{i^-}{\overset{i^+}{\vdots}} + \frac{1}{2}\, \underset{i^-\quad i^-}{\frown}\right) \sqcup \exp_\sqcup(\text{i-deg} \geq 2),$$

which shows that there are no terms of i-deg $= 1$ *in* $\widetilde{Z}^t(c(t_{\alpha_i}))$.

Example 4.9. *Consider the cobordism $c(t_{\alpha_{kl}})$, where $t_{\alpha_{kl}}$ is like in Example 2.1. In this case we obtain*

$$\widetilde{Z}^t(c(t_{\alpha_{kl}})) = \exp_\sqcup\left(\sum_{i=1}^{g} \underset{i^-}{\overset{i^+}{\vdots}} + \frac{1}{2}\, \underset{k^-\quad k^-}{\frown} + \frac{1}{2}\, \underset{l^-\quad l^-}{\frown} + \underset{k^-\quad l^-}{\frown}\right)$$
$$\sqcup \exp_\sqcup\left(-\frac{1}{2}\, \underset{k^-\quad l^-}{\overset{k^+}{\curlywedge}} - \frac{1}{2}\, \underset{l^-\quad k^-}{\overset{l^+}{\curlywedge}} + (\text{i-deg} \geq 2)\right).$$

Comparing the results in Examples 4.8 and 4.9 with the results in Example 4.6, we can see that (after the change $a_k \mapsto k^-$), the diagrammatic diagrammatic versions of τ_{α_i} and $\tau_{\alpha_{kl}}$ appear in the LMO functor with an opposite sign. This is a more general fact which we develop in next section.

4.3 Johnson-type homomorphisms and the LMO functor

We finish this note by stating the explicit relation between Johnson-type homomorphisms and the tree reduction of the LMO functor.

Theorem 4.10 ([3, Corollary 5.11])**.** *For $M \in J_n\mathcal{C}$, we have*

$$\widetilde{Z}^t(M) \Big/ \left(\, \vdots \mapsto 0\right) = \emptyset - \eta^{-1}\tau_n(M)_{|a_j \mapsto j^-,\ b_j \mapsto j^+} + (\text{i-deg} > n).$$

Theorem 4.11 ([20, Theorem 5.4])**.** *For $M \in J_n^L\mathcal{C}$, we have*

$$\widetilde{Z}^t(M) \Big/ \left(\, \vdots \mapsto 0,\ \ \underset{i^-}{\curlyvee} \mapsto 0\right) = \emptyset - \eta^{-1}\tau_n^L(M)_{|b_j \mapsto j^+} + (\text{i-deg} > n).$$

Theorem 4.12 ([21, Theorem 6.14]). *If $h \in J_1^{\mathfrak{a}}\mathcal{M} = \mathcal{I}^{\mathfrak{a}}$, then*

$$\log\left(\widetilde{Z}^t(c(h))\right) = \left(\sum_{i=1}^{g} \begin{smallmatrix} i^+ \\ \vdots \\ i^- \end{smallmatrix}\right) - \left(\eta^{-1}\tau_1^{\mathfrak{a}}(h)\right)_{|a_j \mapsto j^-,\, b_j \mapsto j^+} + (\mathfrak{a}\text{-deg} > 1).$$

Theorem 4.13 ([21, Theorem 6.16]). *For $h \in J_n^{\mathfrak{a}}\mathcal{M}$ with $n \geq 2$, we have*

$$\left.\widetilde{Z}^t(c(h))\right/_{\vdots\ \mapsto 0} = \emptyset - \eta^{-1}\tau_n^{\mathfrak{a}}(h)_{\substack{|b_j \mapsto j^+ \\ a_j \mapsto j^-}} + (\mathfrak{a}\text{-deg} > n).$$

The above results provide a topological interpretation for the tree reduction of the LMO functor. Theorem 4.10 was generalized by Massuyeau in [16]. He proved that it is possible to read a refinement of the Johnson homomorphisms (The *Morita homomorphisms*) in the tree reduction of the LMO functor.

References

[1] Bar-Natan, D., Garoufalidis, S., Rozansky, L., Thurston, D., *The Aarhus integral of rational homology 3-spheres I. A highly non trivial flat connection on S^3*, Selecta Math. (N.S.) **8 (3)** (2002) 315–339.

[2] Bar-Natan, D., Garoufalidis, S., Rozansky, L., Thurston, D., *The Aarhus integral of rational homology 3-spheres II. Invariance and universality*, Selecta Math. (N.S.) **8 (3)** (2002) 341–371.

[3] Cheptea, D., Habiro, K., Massuyeau, G., *A functorial LMO invariant for Lagrangian cobordisms*, Geom. Topol. **12** (2008) 1091–1170.

[4] Garoufalidis, S., Levine, J., *Tree-level invariants of three-manifolds, Massey products and the Johnson homomorphism*, Graphs and patterns in mathematics and theoretical physics, Proc. Sympos. Pure Math. **73** (2005) 173–203.

[5] Goussarov, M., *Variations of knotted graphs. The geometric technique of n-equivalence*, Algebra i Analiz **12 (4)** (2000) 79–125.

[6] Goussarov, M., *Finite type invariants and n-equivalence of* 3*-manifolds*, C. R. Acad. Sci. Paris Sér. I Math. **329 (6)** (1999) 517–522.

[7] Habegger, N., Pitsch, W., *Tree level Lie algebra structures of perturbative invariants*, J. Knot Theory Ramifications **12 (3)** (2003) 333–345.

[8] Habiro, K., *Claspers and finite type invariants of links*, Geom. Topol. **4** (2000) 1–83.

[9] Habiro, K., Massuyeau, G., *From mapping class groups to monoids of homology cobordisms: a survey*, Handbook of Teichmüller theory Volume III. IRMA Lect. Math. Theor. Phys. **17** (2012) 465–529.

[10] Habiro, K., Massuyeau, G., *Generalized Johnson homomorphisms for extended N-series*, J. Algebra **510** (2018) 205–258.

[11] Johnson, D., *A survey of the Torelli group*, Low-dimensional topology (San Francisco, Calif., 1981) Contemp. Math., **20**, Amer. Math. Soc., Providence, RI (1983) 165–179.

[12] Johnson, D., *The structure of the Torelli group I. A finite set of generators for $\mathcal{T}$*, Ann. of Math. (2) **118 (3)** (1983) 423–442.

[13] Levine, J., *Homology cylinders: an enlargement of the mapping class group*, Algebr. Geom. Topol. **1** (2001) 243–270.

[14] Levine, J., *Addendum and correction to: "Homology cylinders: an enlargement of the mapping class group" [Algebr. Geom. Topol.* **1** *(2001), 243–270]*, Algebr. Geom. Topol. **2** (2002) 1197–1204.

[15] Levine, J., *The Lagrangian filtration of the mapping class group and finite-type invariants of homology spheres*, Math. Proc. Cambridge Philos. Soc. **141** (2006) 303–315.

[16] Massuyeau, G., *Infinitesimal Morita homomorphisms and the tree-level of the LMO invariant*, Bull. Soc. Math. France **140** (2012) 101–161.

[17] Massuyeau, G., Meilhan, J.B., *Equivalence relations for homology cylinders and the core of the Casson invariant*, Trans. Amer. Math. Soc. **365 (10)** (2013) 5431–5502.

[18] Morita, S., *Casson invariant, signature defect of framed manifolds and the secondary characteristic classes of surface bundles*, J. Differential Geom. **47** (1997) 560–599.

[19] Stallings, J., *Homology and central series of groups*, J. Algebra **2** (1965) 170–181.

[20] Vera, A., *Johnson-Levine homomorphisms and the tree reduction of the LMO functor*, Mathematical Proceedings of the Cambridge Philosophical Society, (2019) 1-35. doi:10.1017/S0305004119000410

[21] Vera, A., *Alternative versions of the Johnson homomorphisms and the LMO functor*, arXiv:1902.10012. Algebr. Geom. Topol. (to appear).

Department of Mathematics
Kyoto University
Kyoto 606-8502
JAPAN
E-mail address: `aaveraa@unal.edu.co`