

The Fekete-Szegő problem for p -valently Janowski starlike and convex functions

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Abstract

For p -valently Janowski starlike and convex functions defined by applying subordination for the generalized Janowski function, the sharp upper bounds of a functional $|a_{p+2} - \mu a_{p+1}^2|$ related to the Fekete-Szegő problem are given.

1 Introduction

Let $\mathcal{A}_p$ denote the family of functions $f(z)$ normalized by

$$(1.1) \quad f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p = 1, 2, 3, \dots)$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Furthermore, let $\mathcal{W}$ be the class of functions $w(z)$ of the form

$$(1.2) \quad w(z) = \sum_{k=1}^{\infty} w_k z^k$$

which are analytic and satisfy $|w(z)| < 1$ in $\mathbb{U}$. Then, a function $w(z) \in \mathcal{W}$ is called the Schwarz function. If $f(z) \in \mathcal{A}_p$ satisfies the following condition

$$\operatorname{Re} \left[1 + \frac{1}{b} \left(\frac{zf'(z)}{f(z)} - p \right) \right] > 0 \quad (z \in \mathbb{U})$$

for some complex number b ($b \neq 0$), then $f(z)$ is said to be p -valently starlike function of complex order b . We denote by $\mathcal{S}_b^*(p)$ the subclass of $\mathcal{A}_p$ consisting of all functions $f(z)$ which are p -valently starlike functions of complex order b . Similarly, we say that $f(z)$ is a member of the class $\mathcal{K}_b(p)$ of p -valently convex functions of complex order b in $\mathbb{U}$ if $f(z) \in \mathcal{A}_p$ satisfies the following inequality

$$\operatorname{Re} \left[1 + \frac{1}{b} \left(\frac{zf''(z)}{f'(z)} - (p-1) \right) \right] > 0 \quad (z \in \mathbb{U})$$

for some complex number b ($b \neq 0$).

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Next, let $F(z) = \frac{zf'(z)}{f(z)} = u + iv$ and $b = \rho e^{i\varphi}$ ($\rho > 0$, $0 \leq \varphi < 2\pi$). Then, the condition of the definition of $\mathcal{S}_b^*(p)$ is equivalent to

$$(1.3) \quad \operatorname{Re} \left[1 + \frac{1}{b} \left(\frac{zf'(z)}{f(z)} - p \right) \right] = 1 + \frac{\cos \varphi}{\rho}(u - p) + \frac{\sin \varphi}{\rho}v > 0.$$

We denote by $d(l_1, p)$ the distance between the boundary line $l_1 : (\cos \varphi)u + (\sin \varphi)v + \rho - p \cos \varphi = 0$ of the half plane satisfying the condition (1.3) and the point $F(0) = p$. A simple computation gives us that

$$d(l_1, p) = \frac{|\cos \varphi \times p + \sin \varphi \times 0 + \rho - p \cos \varphi|}{\sqrt{\cos^2 \varphi + \sin^2 \varphi}} = \rho,$$

that is, that $d(l_1, p)$ is always equal to $|b| = \rho$ regardless of φ . Thus, if we consider the circle C_1 with center at p and radius ρ , then we can know the definition of $\mathcal{S}_b^*(p)$ means that $F(\mathbb{U})$ is covered by the half plane separated by a tangent line of C_1 and containing C_1 . For $p = 1$, the same things are discussed by Hayami and Owa [3].

Then, we introduce the following function

$$(1.4) \quad p(z) = \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1)$$

which has been investigated by Janowski [4]. Therefore, the function $p(z)$ given by (1.4) is said to be the Janowski function. Furthermore, as a generalization of the Janowski function, Kuroki, Owa and Srivastava [6] have investigated the Janowski function for some complex parameters A and B which satisfy one of the following conditions

$$(1.5) \quad \begin{cases} \text{(i) } A \neq B, |B| < 1, |A| \leq 1 \text{ and } \operatorname{Re}(1 - A\bar{B}) \geq |A - B| \\ \text{(ii) } A \neq B, |B| = 1, |A| \leq 1 \text{ and } 1 - A\bar{B} > 0. \end{cases}$$

Here, we note that the Janowski function generalized by the conditions (1.5) is analytic and univalent in $\mathbb{U}$, and satisfies $\operatorname{Re}(p(z)) > 0$ ($z \in \mathbb{U}$). Moreover, Kuroki and Owa [5] discussed the fact that the condition $|A| \leq 1$ can be omitted from among the conditions in (1.5)–(i) as the conditions for A and B to satisfy $\operatorname{Re}(p(z)) > 0$. In the present paper, we consider the more general Janowski function $p(z)$ as follows:

$$(1.6) \quad p(z) = \frac{p + Az}{1 + Bz} \quad (p = 1, 2, 3, \dots)$$

for some complex parameter A and some real parameter B ($A \neq pB$, $-1 \leq B \leq 0$). Then, we don't need to discuss the other cases because for the function

$$(1.7) \quad q(z) = \frac{p + A_1 z}{1 + B_1 z} \quad (A_1, B_1 \in \mathbb{C}, A_1 \neq pB_1, |B_1| \leq 1),$$

letting $B_1 = |B_1|e^{i\theta}$ and replacing z by $-e^{-i\theta}z$ in (1.7), we see that

$$p(z) = q(-e^{-i\theta}z) = \frac{p - A_1 e^{-i\theta}z}{1 - |B_1|z} \equiv \frac{p + Az}{1 + Bz} \quad (A = -A_1 e^{-i\theta}, B = -|B_1|)$$

maps $\mathbb{U}$ onto the same circular domain as $q(\mathbb{U})$.

Remark 1.1 For the case $B = -1$ in (1.6), we know that $p(z)$ maps $\mathbb{U}$ onto the following half plane

$$\operatorname{Re} (p + \overline{A}) p(z) > \frac{p^2 - |A|^2}{2}$$

and for the case $-1 < B \leq 0$ in (1.6), $p(z)$ maps $\mathbb{U}$ onto the circular domain

$$\left| p(z) - \frac{p + AB}{1 - B^2} \right| < \frac{|A + pB|}{1 - B^2}.$$

Let $p(z)$ and $q(z)$ be analytic in $\mathbb{U}$. Then we say that the function $p(z)$ is subordinate to $q(z)$ in $\mathbb{U}$, written by

$$p(z) \prec q(z) \quad (z \in \mathbb{U}),$$

if there exists a function $w(z) \in \mathcal{W}$ such that $p(z) = q(w(z))$ ($z \in \mathbb{U}$). In particular, if $q(z)$ is univalent in $\mathbb{U}$, then $p(z) \prec q(z)$ if and only if

$$p(0) = q(0) \quad \text{and} \quad p(\mathbb{U}) \subset q(\mathbb{U}).$$

We next define the subclasses of $\mathcal{A}_p$ by applying the subordination as follows:

$$\mathcal{S}_p^*(A, B) = \left\{ f(z) \in \mathcal{A}_p : \frac{zf'(z)}{f(z)} \prec \frac{p + Az}{1 + Bz} \quad (z \in \mathbb{U}) \right\}$$

and

$$\mathcal{K}_p(A, B) = \left\{ f(z) \in \mathcal{A}_p : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{p + Az}{1 + Bz} \quad (z \in \mathbb{U}) \right\}$$

where $A \neq pB$, $-1 \leq B \leq 0$. We immediately know that

$$(1.8) \quad f(z) \in \mathcal{K}_p(A, B) \quad \text{if and only if} \quad \frac{zf'(z)}{p} \in \mathcal{S}_p^*(A, B).$$

Then, we have the next theorem.

Theorem 1.2 If $f(z) \in \mathcal{S}_p^*(A, B)$ ($-1 < B \leq 0$), then $f(z) \in \mathcal{S}_b^*(p)$ where

$$b = \frac{|B(-pB + \operatorname{Re}(A)) \cos \varphi + B \operatorname{Im}(A) \sin \varphi + |A - pB||}{1 - B^2} e^{i\varphi} \quad (0 \leq \varphi < 2\pi).$$

Espesially, $f(z) \in \mathcal{S}_p^*(A, -1)$ if and only if $f(z) \in \mathcal{S}_b^*(p)$ where $b = \frac{p + A}{2}$.

Proof. Supposing that $\frac{zf'(z)}{f(z)} \prec \frac{p + Az}{1 - z}$, it follows from Remark 1.1 that

$$\operatorname{Re} \left[(p + \overline{A}) \frac{zf'(z)}{f(z)} \right] > \frac{p^2 - |A|^2}{2}$$

that is, that

$$\operatorname{Re} \left[2(p + \overline{A}) \frac{zf'(z)}{f(z)} \right] > \operatorname{Re} [2p(p + \overline{A})] - |p + A|^2.$$

This means that

$$\operatorname{Re} \left[\frac{2(p + \bar{A})}{|p + A|^2} \left(\frac{zf'(z)}{f(z)} - p \right) \right] > -1$$

which implies that

$$\operatorname{Re} \left[\frac{1}{\frac{1}{2}(p + A)} \left(\frac{zf'(z)}{f(z)} - p \right) \right] > -1.$$

Therefore, $f(z) \in \mathcal{S}_b^*$ where $b = \frac{p + A}{2}$. The converse is also completed.

Next, for the case $-1 < B \leq 0$, by the definition of the class $\mathcal{S}_b^*(A, B)$, if a tangent line l_2 of the circle C_2 containing the point p is parallel to the straight line $L : (\cos \theta)u + (\sin \theta)v = 0$ ($-\pi \leq \theta < \pi$), and the image $F(\mathbb{U})$ by $F(z) = \frac{zf'(z)}{f(z)}$ is covered by the circle C_2 , then there exists a non-zero complex number b with $\arg(b) = \theta + \pi$ and $|b| = d(l_2, p)$ such that $f(z) \in \mathcal{S}_b^*(p)$, where $d(l_2, p)$ is the distance between the tangent line l_2 and the point p . Now, for the function $p(z) = \frac{p + Az}{1 + Bz}$ ($A \neq pB$, $-1 < B \leq 0$), the image $p(\mathbb{U})$ is equivalent to

$$C_2 = \left\{ \omega \in \mathbb{C} : \left| \omega - \frac{p - AB}{1 - B^2} \right| < \frac{|A - pB|}{1 - B^2} \right\}$$

and the point ξ on $\partial C_2 = \left\{ \omega \in \mathbb{C} : \left| \omega - \frac{p - AB}{1 - B^2} \right| = \frac{|A - pB|}{1 - B^2} \right\}$ can be written by

$$\xi := \xi(\theta) = \frac{|A - pB|}{1 - B^2} e^{i\theta} + \frac{p - AB}{1 - B^2} \quad (-\pi \leq \theta < \pi).$$

Further, the tangent line l_2 of the circle C_2 through each point $\xi(\theta)$ is parallel to the straight line $L : (\cos \theta)u + (\sin \theta)v = 0$. Namely, l_2 can be represented by

$$l_2 : (\cos \theta) \left(u - \frac{|A - pB| \cos \theta + p - B \operatorname{Re}(A)}{1 - B^2} \right) + (\sin \theta) \left(v - \frac{|A - pB| \sin \theta - B \operatorname{Im}(A)}{1 - B^2} \right) = 0$$

which implies that

$$l_2 : (\cos \theta)u + (\sin \theta)v - \frac{|A - pB| + \{p - B \operatorname{Re}(A)\} \cos \theta - B \operatorname{Im}(A) \sin \theta}{1 - B^2} = 0.$$

Then, we see that the distance $d(l_2, p)$ between the point p and the above tangent line l_2 of the circle C_2 is

$$\begin{aligned} & \left| \cos \theta \times p + \sin \theta \times 0 - \frac{|A - pB| + \{p - B \operatorname{Re}(A)\} \cos \theta - B \operatorname{Im}(A) \sin \theta}{1 - B^2} \right| \\ &= \frac{|-B(-pB + \operatorname{Re}(A)) \cos \theta - B \operatorname{Im}(A) \sin \theta + |A - pB||}{1 - B^2}. \end{aligned}$$

Therefore, if the subordination

$$\frac{zf'(z)}{f(z)} \prec \frac{p + Az}{1 + Bz} \quad (A \neq pB, -1 < B \leq 0)$$

holds true, then $f(z) \in \mathcal{S}_b^*$ where

$$b = \frac{|-B(-pB + \operatorname{Re}(A)) \cos \theta - B \operatorname{Im}(A) \sin \theta + |A - pB||}{1 - B^2} e^{i(\theta + \pi)}.$$

Finally, setting $\varphi = \theta + \pi$ ($0 \leq \varphi < 2\pi$), the proof of the theorem is completed. $\square$

Noonan and Thomas [8], [9] have stated the q -th Hankel determinant as

$$H_q(n) = \det \begin{pmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{pmatrix} \quad (n, q \in \mathbb{N} = \{1, 2, 3, \dots\}).$$

This determinant is discussed by several authors with $q = 2$. For example, we can know that the functional $|H_2(1)| = |a_3 - a_2^2|$ is known as the Fekete-Szegő problem and they consider the further generalized functional $|a_3 - \mu a_2^2|$ where $a_1 = 1$ and μ is some real number (see, [1]). The purpose of this investigation is to find the sharp upper bounds of the functional $|a_{p+2} - \mu a_{p+1}^2|$ for functions $f(z) \in \mathcal{S}_p^*(A, B)$ or $\mathcal{K}_p(A, B)$.

2 Preliminary results

We need some lemmas to establish our results. Applying the Schwarz lemma or subordination principle.

Lemma 2.1 *If a function $w(z) \in \mathcal{W}$, then*

$$|w_1| \leq 1.$$

Equality is attained for $w(z) = e^{i\theta} z$ for any $\theta \in \mathbb{R}$.

The following lemma is obtained by applying the Schwarz-Pick lemma (see, for example, [7]).

Lemma 2.2 *For any functions $w(z) \in \mathcal{W}$, the inequality*

$$|w_2| \leq 1 - |w_1|^2$$

holds true. Namely, this gives us the following representation

$$w_2 = (1 - |w_1|^2) \zeta$$

for some ζ ($|\zeta| \leq 1$).

3 p -valently Janowski starlike functions

Our first main result is contained in

Theorem 3.1 *If $f(z) \in \mathcal{S}_p^*(A, B)$, then $|a_{p+2} - \mu a_{p+1}^2| \leq$*

$$\begin{cases} \frac{|(A - pB) \{(1 - 2\mu)A - ((p+1) - 2p\mu)B\}|}{2} & (|(1 - 2\mu)A - ((p+1) - 2p\mu)B| \geq 1) \\ \frac{|A - pB|}{2} & (|(1 - 2\mu)A - ((p+1) - 2p\mu)B| \leq 1) \end{cases}$$

with equality for

$$f(z) = \begin{cases} \frac{z^p}{(1+Bz)^{\frac{pB-A}{B}}} & \text{or } z^p e^{Az} \quad (B=0) & (|(1-2\mu)A - ((p+1)-2p\mu)B| \geq 1) \\ \frac{z^p}{(1+Bz^2)^{\frac{pB-A}{2B}}} & \text{or } z^p e^{\frac{A}{2}z^2} \quad (B=0) & (|(1-2\mu)A - ((p+1)-2p\mu)B| \leq 1). \end{cases}$$

Proof. Let $f(z) \in \mathcal{S}_p^*(A, B)$. Then, there exists the function $w(z) \in \mathcal{W}$ such that

$$\frac{zf'(z)}{f(z)} = \frac{p + Aw(z)}{1 + Bw(z)}$$

which means that

$$(n-p)a_n = \sum_{k=p}^{n-1} (A - kB)a_k w_{n-k} \quad (n \geq p+1)$$

where $a_p = 1$. Thus, by the help of the relation in Lemma 2.2, we see that

$$\begin{aligned} |a_{p+2} - \mu a_{p+1}^2| &= \left| \frac{1}{2}(A - pB) \{w_2 + (A - (p+1)B)w_1^2\} - \mu(A - pB)^2 w_1^2 \right| \\ &= \frac{|A - pB|}{2} |(1 - w_1^2)\zeta + \{(A - (p+1)B) - 2\mu(A - pB)\}w_1^2|. \end{aligned}$$

Then, by Lemma 2.1, supposing that $0 \leq w_1 \leq 1$ without loss of generality, and applying the triangle inequality, it follows that

$$\begin{aligned} |(1 - w_1^2)\zeta + \{(A - (p+1)B) - 2\mu(A - pB)\}w_1^2| &\leq 1 + \{|(A - (p+1)B) - 2\mu(A - pB)| - 1\}w_1^2 \\ &\leq \begin{cases} |(A - (p+1)B) - 2\mu(A - pB)| & (|(A - (p+1)B) - 2\mu(A - pB)| \geq 1; w_1 = 1) \\ 1 & (|(A - (p+1)B) - 2\mu(A - pB)| \leq 1; w_1 = 0). \end{cases} \end{aligned}$$

□

Especially, taking $\mu = \frac{p+1}{2p}$ in Theorem 3.1, we obtain

Corollary 3.2 If $f(z) \in \mathcal{S}_p^*(A, B)$, then

$$\left| a_{p+2} - \frac{p+1}{2p} a_{p+1}^2 \right| \leq \begin{cases} \frac{|A(A - pB)|}{2p} & (|A| \geq p) \\ \frac{|A - pB|}{2} & (|A| \leq p) \end{cases}$$

with equality for

$$f(z) = \begin{cases} \frac{z^p}{(1+Bz)^{\frac{pB-A}{B}}} & \text{or } z^p e^{Az} \quad (B=0) & (|A| \geq p) \\ \frac{z^p}{(1+Bz^2)^{\frac{pB-A}{2B}}} & \text{or } z^p e^{\frac{A}{2}z^2} \quad (B=0) & (|A| \leq p). \end{cases}$$

Furthermore, putting $A = p - 2\alpha$ and $B = -1$ for some α ($0 \leq \alpha < p$) in Theorem 3.1, we arrive at the following result by Hayami and Owa [2, Theorem 3].

Corollary 3.3 *If $f(z) \in \mathcal{S}_p^*(\alpha)$, then*

$$|a_{p+2} - \mu a_{p+1}^2| \leq \begin{cases} (p - \alpha) \{ (2(p - \alpha) + 1) - 4(p - \alpha)\mu \} & \left(\mu \leq \frac{1}{2} \right) \\ p - \alpha & \left(\frac{1}{2} \leq \mu \leq \frac{p - \alpha + 1}{2(p - \alpha)} \right) \\ (p - \alpha) \{ 4(p - \alpha)\mu - (2(p - \alpha) + 1) \} & \left(\mu \geq \frac{p - \alpha + 1}{2(p - \alpha)} \right) \end{cases}$$

with equality for

$$f(z) = \begin{cases} \frac{z}{(1 - z)^{2(p - \alpha)}} & \left(\mu \leq \frac{1}{2} \text{ or } \mu \geq \frac{p - \alpha + 1}{2(p - \alpha)} \right) \\ \frac{z}{(1 - z^2)^{p - \alpha}} & \left(\frac{1}{2} \leq \mu \leq \frac{p - \alpha + 1}{2(p - \alpha)} \right). \end{cases}$$

4 p -valently Janowski convex functions

Similarly, we consider the functional $|a_{p+2} - \mu a_{p+1}^2|$ for p -valently Janowski convex functions.

Theorem 4.1 *If $f(z) \in \mathcal{K}_p(A, B)$, then*

$$|a_{p+2} - \mu a_{p+1}^2| \leq \begin{cases} \frac{p|(A - pB) \{ ((p + 1)^2 - 2p(p + 2)\mu)A - ((p + 1)^3 - 2p^2(p + 2)\mu)B \}|}{2(p + 1)^2(p + 2)} & (|((p + 1)^2 - 2p(p + 2)\mu)A - ((p + 1)^3 - 2p^2(p + 2)\mu)B| \geq (p + 1)^2) \\ \frac{p|A - pB|}{2(p + 2)} & (|((p + 1)^2 - 2p(p + 2)\mu)A - ((p + 1)^3 - 2p^2(p + 2)\mu)B| \leq (p + 1)^2) \end{cases}$$

with equality for

$$f(z) = \begin{cases} z {}^p_2F_1 \left(p, p - \frac{A}{B}; p + 1; -Bz \right) \quad \text{or} \quad z {}^p_1F_1 \left(p, p + 1; Az \right) & (B \neq 0) \\ (|((p + 1)^2 - 2p(p + 2)\mu)A - ((p + 1)^3 - 2p^2(p + 2)\mu)B| \geq (p + 1)^2) \\ z {}^p_2F_1 \left(\frac{p}{2}, \frac{pB - A}{2B}; 1 + \frac{p}{2}; -Bz^2 \right) \quad \text{or} \quad z {}^p_1F_1 \left(\frac{p}{2}, 1 + \frac{p}{2}; \frac{A}{2}z^2 \right) & (B = 0) \\ (|((p + 1)^2 - 2p(p + 2)\mu)A - ((p + 1)^3 - 2p^2(p + 2)\mu)B| \leq (p + 1)^2) \end{cases}$$

where ${}_2F_1(a, b; c; z)$ represents the ordinary hypergeometric function and ${}_1F_1(a, b; z)$ represents the confluent hypergeometric function.

Proof. By the help of the relation (1.8) and Theorem 3.1, if $f(z) \in \mathcal{K}_p(A, B)$, then

$$\left| \frac{p+2}{p} a_{p+2} - \mu \frac{(p+1)^2}{p^2} a_{p+1}^2 \right| = \frac{p+2}{p} \left| a_{p+2} - \frac{(p+1)^2}{p(p+2)} \mu a_{p+1}^2 \right| \leq C(\mu)$$

where $C(\mu)$ is one of the values in Theorem 3.1. Then, dividing the both sides by $\frac{p+2}{p}$ and replacing $\frac{(p+1)^2}{p(p+2)} \mu$ by μ , we obtain the theorem. $\square$

Now, letting $\mu = \frac{(p+1)^3}{2p^2(p+2)}$ in Theorem 4.1, we have

Corollary 4.2 If $f(z) \in \mathcal{K}_p(A, B)$, then

$$\left| a_{p+2} - \frac{(p+1)^3}{2p^2(p+2)} a_{p+1}^2 \right| \leq \begin{cases} \frac{|A(A-pB)|}{2(p+2)} & (|A| \geq p) \\ \frac{p|A-pB|}{2(p+2)} & (|A| \leq p) \end{cases}$$

with equality for

$$f(z) = \begin{cases} z^p {}_2F_1\left(p, p - \frac{A}{B}; p+1; -Bz\right) & \text{or } z^p {}_1F_1(p, p+1; Az) \quad (B \neq 0) \quad (|A| \geq p) \\ z^p {}_2F_1\left(\frac{p}{2}, \frac{pB-A}{2B}; 1 + \frac{p}{2}; -Bz^2\right) & \text{or } z^p {}_1F_1\left(\frac{p}{2}, 1 + \frac{p}{2}; \frac{A}{2}z^2\right) \quad (B = 0) \quad (|A| \leq p) \end{cases}$$

where ${}_2F_1(a, b; c; z)$ represents the ordinary hypergeometric function and ${}_1F_1(a, b; z)$ represents the confluent hypergeometric function.

Moreover, we suppose that $A = p - 2\alpha$ and $B = -1$ for some α ($0 \leq \alpha < p$). Then, we arrive at the result by the Hayami and Owa [2, Theorem 4].

Corollary 4.3 If $f(z) \in \mathcal{K}_p(\alpha)$, then

$$|a_{p+2} - \mu a_{p+1}^2| \leq \begin{cases} \frac{p(p-\alpha) \{(p+1)^2(2(p-\alpha)+1) - 4p(p+2)(p-\alpha)\mu\}}{(p+1)^2(p+2)} & \left(\mu \leq \frac{(p+1)^2}{2p(p+2)}\right) \\ \frac{p(p-\alpha)}{p+2} & \left(\frac{(p+1)^2}{2p(p+2)} \leq \mu \leq \frac{(p+1)^2(p-\alpha+1)}{2p(p+2)(p-\alpha)}\right) \\ \frac{p(p-\alpha) \{4p(p+2)(p-\alpha)\mu - (p+1)^2(2(p-\alpha)+1)\}}{(p+1)^2(p+2)} & \left(\mu \geq \frac{(p+1)^2(p-\alpha+1)}{2p(p+2)(p-\alpha)}\right) \end{cases}$$

with equality for

$$f(z) = \begin{cases} z^p {}_2F_1(p, 2(p-\alpha); p+1; z) & \left(\mu \leq \frac{(p+1)^2}{2p(p+2)} \text{ or } \mu \geq \frac{(p+1)^2(p-\alpha+1)}{2p(p+2)(p-\alpha)}\right) \\ z^p {}_2F_1\left(\frac{p}{2}, p-\alpha; 1 + \frac{p}{2}; z^2\right) & \left(\frac{(p+1)^2}{2p(p+2)} \leq \mu \leq \frac{(p+1)^2(p-\alpha+1)}{2p(p+2)(p-\alpha)}\right) \end{cases}$$

where ${}_2F_1(a, b; c; z)$ represents the ordinary hypergeometric function.

References

- [1] M. Fekete and G. Szegő, *Eine Bemerkung über ungerade schlichte Funktionen*, J. London Math. Soc. **8**(1933), 85–89.
- [2] T. Hayami and S. Owa, *Hankel determinant for p -valently starlike and convex functions of order α* , General Math. **17**(4), (2009), 29–44.
- [3] T. Hayami and S. Owa, *New properties for starlike and convex functions of complex order*, Int. Journal of Math. Analysis, **4**(1), (2010), 39–62.
- [4] W. Janowski, *Extremal problem for a family of functions with positive real part and for some related families*, Ann. Polon. Math **23**(1970), 159–177.
- [5] K. Kuroki and S. Owa, *Some subordination criteria concerning the Sălăgean operator*, J. Inequal. Pure and Appl. Math. **10**(2), (2009), Article ID 36, 1–11.
- [6] K. Kuroki, S. Owa and H. M. Srivastava, *Some subordination criteria for analytic functions*, Bull. Soc. Sci. Lett. Lodz. **52** (2007), 27–36.
- [7] Z. Nehari, *Conformal Mappings*, McGraw-Hill, New York, 1952.
- [8] J. W. Noonan and D. K. Thomas, *On the Hankel determinant of areally mean p -valent functions*, Proc. London Math. Soc. **25**(1972), 503–524.
- [9] J. W. Noonan and D. K. Thomas, *On the second Hankel determinant of areally mean p -valent functions*, Trans. Amer. Math. Soc. **223**(2) (1976), 337–346.

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