

Weak BAB conjecture for log del Pezzo surfaces and Mori fiber spaces

代数幾何学シンポジウム記録
2013年度 pp.157-157

Chen Jiang

Graduate School of Mathematical Sciences, The University of Tokyo

Definitions

We work over complex number field.

Let X be a normal projective variety and Δ be an $\mathbb{Q}$ -divisor on X with coefficients in $[0, 1]$ such that $K_X + \Delta$ is $\mathbb{Q}$ -Cartier. We say that (X, Δ) is a *log Fano variety* if $-(K_X + \Delta)$ is ample. In dimension two, it is called log del Pezzo surface. Let $f : Y \rightarrow X$ be a log resolution of (X, Δ) , write

$$K_Y = f^*(K_X + \Delta) + \sum a_i F_i,$$

where F_i is a prime divisor. For some $\epsilon \in (0, 1]$, the pair (X, Δ) is called ϵ -kawamata log terminal (ϵ -klt, for short) if $a_i > -1 + \epsilon$ for all i , or ϵ -log canonical (ϵ -lc, for short) if $a_i \geq -1 + \epsilon$ for all i .

Boundedness on log Fano varieties

One of the most interesting conjecture in minimal model theory is the following B-A-B Conjecture due to A. Borisov, L. Borisov and V. Alexeev.

(Birational) BAB conjecture

Fix $0 < \epsilon < 1$, an integer $n > 0$, and consider the set of all n -dimensional ϵ -klt log Fano varieties (X, Δ) . The set of underlying varieties $\{X\}$ is (birationally) bounded.

The BAB Conjecture is still open in dimension three and higher. We are mainly interested in the following weak conjecture for anti-canonical volumes which is a consequence of BAB Conjecture.

Weak BAB conjecture

Fix $0 < \epsilon < 1$, an integer $n > 0$, and consider the set of all n -dimensional ϵ -klt log Fano varieties (X, Δ) . The volume $\text{Vol}(-(K_X + \Delta)) = (-K_X + \Delta)^n$ is bounded from above by a fixed number $M(n, \epsilon)$ depending only on n and ϵ .

Weak BAB conjecture in dimension two

In dimension two, the conjecture is well-researched in history. We give an optimal bound.

Theorem A

Let (X, Δ) be an ϵ -lc weak log del Pezzo surface. Then the anti-canonical volume $\text{Vol}(-(K_X + \Delta)) = (K_X + \Delta)^2$ satisfies

$$(K_X + \Delta)^2 \leq \max \left\{ 9, \lfloor 2/\epsilon \rfloor + 4 + \frac{4}{\lfloor 2/\epsilon \rfloor} \right\},$$

where $\lfloor \cdot \rfloor$ means round down.

Moreover, the equality holds if and only if one of the following holds:

- (1) $\epsilon > \frac{2}{5}$ and (X, Δ) is $(\mathbb{P}^2, 0)$;
- (2) $\epsilon \leq \frac{1}{2}$ and (X, Δ) is $(\mathbb{F}_n, (1 - \frac{2}{n})S_n)$ or $(\text{PC}_n, 0)$, where $n = \lfloor 2/\epsilon \rfloor$, $\mathbb{F}_n$ is the n -th Hirzebruch surface, $S_n \subset \mathbb{F}_n$ is the unique curve with negative self-intersection and PC_n is the projective cone over a rational normal curve of degree n .

BAB conjecture for Mori fiber spaces

A variety X is said to be with a *Mori fiber space* structure if X has $\mathbb{Q}$ -factorial terminal singularities and there is a morphism $f : X \rightarrow S$ such that $-K_X$ is ample on fibers, $\rho(X/S) = 1$ and $\dim X > \dim S$. We have the following facts.

Fact

BAB $\implies$ BAB for Mfs $\implies$ birational BAB
"weak BAB" for Mfs $\implies$ weak BAB

So it is very interesting to investigate the boundedness of Mori fiber spaces.

3-fold Mori fiber spaces

There are three possible Mori fiber space structures on a 3-fold log Fano pair (X, Δ) .

Case 0: $\dim S = 0$

In this case X is a terminal Fano 3-fold with Picard number one. The BAB conjecture is true for this class of varieties by Kawamata. And the optimal bound for the anti-canonical volume is given by Prokhorov.

Case 1: $\dim S = 1$

In this case S is just $\mathbb{P}^1$ and the general fiber is smooth del Pezzo surface. We can prove BAB conjecture for $\mathbb{P}^2$ -bundles over $\mathbb{P}^1$, or more general, $\mathbb{P}^n$ -bundles over $\mathbb{P}^1$. The method is to compute discrepancies for some special centers to get some bound.

Theorem B

$X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n))$ with $0 \leq a_1 \leq \cdots \leq a_n$, and (X, Δ) is ϵ -klt log Fano for some Δ . Then $a_n < 2/\epsilon$.

Case 2: $\dim S = 2$

In this case X is a conic bundle over S . We can prove weak BAB conjecture for the case $S = \mathbb{P}^2$. The idea is that if the volume is big then we can construct non-klt centers and using connected lemma to get special non-klt centers which is very big on the fibers.

Theorem C

Let $X \rightarrow \mathbb{P}^2$ be a conic bundle, and (X, Δ) is ϵ -klt log Fano for some Δ . Then $-(K_X + \Delta)^3 \leq (12/\epsilon)^3$.