

Finite support iteration of c.c.c forcing notions and Parametrized $\diamond$ -principles

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概要

We present several models which satisfy some $\diamond$ -like principles by using the ω_2 -stage finite support iteration of Suslin forcing notions.

1 Introduction

In [10] Jensen showed $V = L$ implies Suslin's Hypothesis doesn't hold. To prove this he introduced the $\diamond$ -principle:

$\diamond$ There exists a sequence $\langle A_\alpha \subset \alpha : \alpha < \omega_1 \rangle$ such that for all $X \subset \omega_1$ the set $\{\alpha < \omega_1 : X \cap \alpha = A_\alpha\}$ is stationary.

In [9] Hrušák gave a partial solution to a question of J. Roitman who asked whether $\mathfrak{d} = \omega_1$ implies $\mathfrak{a} = \omega_1$ and answered a question of Brendle who asked whether $\mathfrak{a} = \omega_1$ in any model obtained by adding a single Laver real. To prove those he introduced the $\diamond$ -like principle $\diamond_{\mathfrak{d}}$:

$\diamond_{\mathfrak{d}}$ There exists a sequence $\langle g_\alpha : \omega \leq \alpha < \omega_1 \rangle$ such that g_α is a function from α to ω and for every $f : \omega_1 \rightarrow \omega$ there is an $\alpha \geq \omega$ with $f \restriction \alpha \leq^* g_\alpha$.

In [16] Moore, Hrušák, and Džamonja provided a broad framework of “parametrized $\diamond$ -principles” and they presented the following methods to construct parametrized $\diamond$ -principles:

Theorem 1.1. *Let $\mathbb{C}(\omega_1)$ and $\mathbb{B}(\omega_1)$ be the Cohen and random algebras corresponding to the product space 2^{ω_1} with its usual topological and measure theoretic structures. The orders $\mathbb{C}(\omega_1)$ and $\mathbb{B}(\omega_1)$ force $\diamond(\text{non}(\mathcal{M}))$ and $\diamond(\text{non}(\mathcal{N}))$ respectively.*

Theorem 1.2. *Suppose that $\langle Q_\alpha : \alpha < \omega_2 \rangle$ is a sequence of Borel partial orders such that for each $\alpha < \omega_2$ Q_α is equivalent to $\wp(2)^+ \times Q_\alpha$ as a forcing notion and let $\mathcal{P}_{\omega_2}$ be the countable support iteration of this sequence. If $\mathcal{P}_{\omega_2}$ is proper and (A, B, E) is a Borel invariant then $\mathcal{P}_{\omega_2}$ forces $\langle A, B, E \rangle \leq \omega_1$ iff $\mathcal{P}_{\omega_2}$ forces $\diamond(A, B, E)$.*

In [15] by using ω_1 -stage finite support iteration of c.c.c forcing notions, several models were presented which satisfy some parametrized $\diamond$ -principles while others fail. The purpose of this paper is to provide several models satisfying some parametrized $\diamond$ -principles by using ω_2 -stage finite support iteration of Suslin forcing notions.

2 Definition and properties of Parametrized Diamonds

In [20] Vojtáš introduced a framework to describe many cardinal invariants.

Definition 2.1. [20][16] The triple (A, B, E) is an *invariant* if

- (1) $|A|, |B| \leq |\mathbb{R}|$,
- (2) $E \subset A \times B$,
- (3) For each $a \in A$ there exists $b \in B$ such that $(a, b) \in E$ and
- (4) For each $b \in B$ there exists $a \in A$ such that $(a, b) \notin E$.

We will write aEb instead of $(a, b) \in E$. If A and B are Borel subsets of some Polish spaces and E is a Borel subset of their product, we call the triple (A, B, E) Borel invariant.

Borel invariants were introduced in [3]. In the present paper we are interested only in Borel invariants.

Definition 2.2. Suppose (A, B, E) is an invariant. Then its *evaluation* is defined by

$$\langle A, B, E \rangle = \min\{|X| : X \subset B \text{ and } \forall a \in A \exists b \in X (aEb)\}.$$

If $A = B$, we will write (A, E) and $\langle A, E \rangle$ instead of (A, B, E) and $\langle A, B, E \rangle$.

Example 2.3. The following Borel invariants $(\mathcal{N}, \not\subset)$, $(\mathcal{N}, \subset)$, $(\mathbb{R}, \mathcal{M}, \in)$, $(\mathcal{M}, \mathbb{R}, \not\subset)$, $(\omega^\omega, <^*)$, $(\omega^\omega, \not\subset^*)$ and $([\omega]^\omega, \text{is split by})$ have the evaluations $\text{add}(\mathcal{N})$, $\text{cof}(\mathcal{N})$, $\text{cov}(\mathcal{M})$, $\text{non}(\mathcal{M})$, $\mathfrak{d}$, $\mathfrak{b}$ and $\mathfrak{s}$ respectively.

Definition 2.4. Suppose A is a Borel subset in some Polish space. Then $F : 2^{<\omega_1} \rightarrow A$ is *Borel* if for every $\alpha < \omega_1$ $F \upharpoonright 2^\alpha$ is a Borel function.

In [7] the principle “weak diamond principle” was introduced by Devlin and Shelah. This was the starting point for the parametrized diamond principles introduced by Moore, Hrušák and Džamonja [16].

Definition 2.5. [16](Parametrized diamond principle)

Suppose (A, B, E) is a Borel invariant. Then $\diamond(A, B, E)$ is the following statement:

$\diamond(A, B, E)$ For all Borel $F : 2^{<\omega_1} \rightarrow A$ there exists $g : \omega_1 \rightarrow B$ such that for every $f : \omega_1 \rightarrow 2$ the set $\{\alpha \in \omega_1 : F(f \upharpoonright \alpha)Eg(\alpha)\}$ is stationary.

The witness g for a given F in this statement will be called $\diamond(A, B, E)$ -sequence for F .

$\diamond(A, B, E)$ and $\diamond$ are related as follows:

Proposition 2.6. [16] Let (A, B, E) be a Borel invariant. $\diamond$ implies $\diamond(A, B, E)$.

$\diamond(A, B, E)$ and $\langle A, B, E \rangle$ are related as follows:

Proposition 2.7. [16] Suppose (A, B, E) is a Borel invariant and $\diamond(A, B, E)$ holds. Then $\langle A, B, E \rangle \leq \omega_1$ holds.

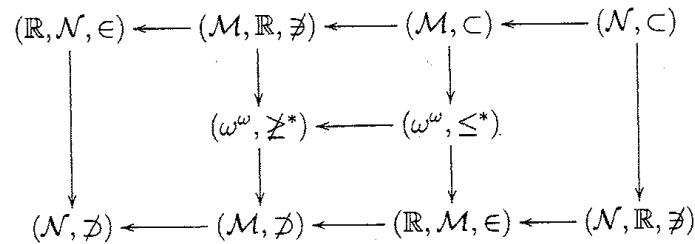
If two Borel invariants $(A_1, B_1, E_1), (A_2, B_2, E_2)$ are comparable in the Borel Tukey order, then $\diamond(A_1, B_1, E_1)$ and $\diamond(A_2, B_2, E_2)$ are related as follows:

Definition 2.8. (Borel Tukey ordering [3]) Given a pair of Borel invariants (A_1, B_1, E_1) and (A_2, B_2, E_2) , we say that $(A_1, B_1, E_1) \leq_T^B (A_2, B_2, E_2)$ if there exist Borel maps $\phi : A_1 \rightarrow A_2$ and $\psi : B_2 \rightarrow B_1$ such that $(\phi(a), b) \in E_2$ implies $(a, \psi(b)) \in E_1$.

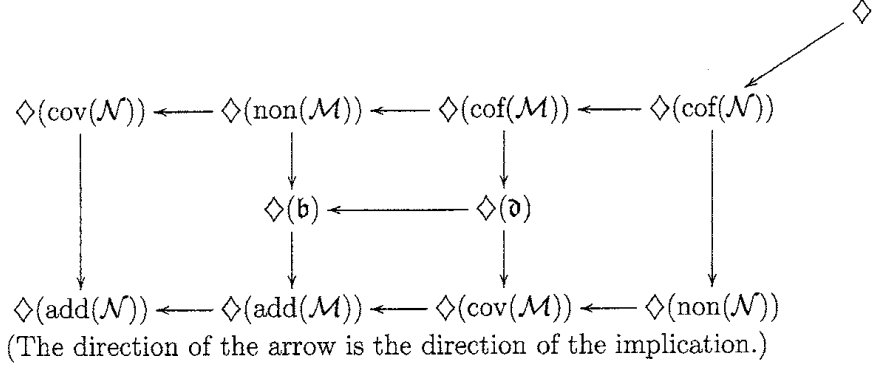
Proposition 2.9. [16] Let (A_1, B_1, E_1) and (A_2, B_2, E_2) be Borel invariants. Suppose $(A_1, B_1, E_1) \leq_T^B (A_2, B_2, E_2)$ and $\diamond(A_2, B_2, E_2)$ holds. Then $\diamond(A_1, B_1, E_1)$ holds.

Concerning $\leq_T^B$, we know the following diagram holds.

(Cichoń’s diagram)



(The direction of the arrow is from larger to smaller in the Borel Tukey order).
Hence the following holds:



We call this diagram “Cichoń’s diagram for parametrized diamonds”.

Note When we deal with Borel invariants in Cichoń’s diagram, we will use the standard notation for their evaluations to denote the Borel invariants themselves (e.g., we will use $\diamond(\text{add}(\mathcal{N}))$ to denote $\diamond(\mathcal{N}, \mathcal{D})$).

3 Construction of Parametrized Diamonds

By using ω_2 -stage finite support iteration of Suslin forcing notions we present several model which satisfies some parametrized $\diamond$ -principles.

3.1 Suslin forcing

Firstly we will introduce Suslin forcings and their properties.

Definition 3.1. [2, p.168] A forcing notion $\mathbb{P} = \langle \mathbb{P}, \leq_{\mathbb{P}} \rangle$ has a Suslin definition if $\mathbb{P} \subset \omega^\omega$, $\leq_{\mathbb{P}} \subset \omega^\omega \times \omega^\omega$ and $\perp_{\mathbb{P}} \subset \omega^\omega \times \omega^\omega$ are Σ_1^1 .

$\mathbb{P}$ is Suslin if $\mathbb{P}$ is c.c.c and has a Suslin definition.

Definition 3.2. [2, p.168] Let $M \models \text{ZFC}^*$. A Suslin forcing $\mathbb{P}$ is in M if all the parameters used in the definition of $\mathbb{P}$, $\leq_{\mathbb{P}}$ and $\perp_{\mathbb{P}}$ are in M .

For convenience we will interpret Suslin forcing notion in forcing extensions using its code rather than taking the ground model forcing notion.

Definition 3.3. Let $\mathbb{A}$ and $\mathbb{B}$ be forcing notions. Then $i : \mathbb{A} \rightarrow \mathbb{B}$ is a complete embedding if

$$(1) \forall a, a' \in \mathbb{A} (a \leq a' \rightarrow i(a) \leq i(a')),$$

$$(2) \forall a_1, a_2 \in \mathbb{A} (a_1 \perp a_2 \leftrightarrow i(a_1) \perp i(a_2)),$$

$$(3) \forall \mathcal{A} \subset \mathbb{A} (\mathcal{A} \text{ is a maximal antichain in } \mathbb{A} \rightarrow i[\mathcal{A}] \text{ is a maximal antichain in } \mathbb{B}).$$

If there is complete embedding from $\mathbb{A}$ to $\mathbb{B}$, then we write $\mathbb{A} \triangleleft \mathbb{B}$.

Suslin forcing notion has the following good property:

Lemma 3.4. Assume $\mathbb{A} \triangleleft \mathbb{B}$ and $\mathcal{P}$ is a Suslin forcing notion. Then $\mathbb{A} * \dot{\mathcal{P}} \triangleleft \mathbb{B} * \dot{\mathcal{P}}$.

Proof. Let $i : \mathbb{A} \rightarrow \mathbb{B}$ be a complete embedding. Then define $\hat{i} : \mathbb{A} * \dot{\mathcal{P}} \rightarrow \mathbb{B} * \dot{\mathcal{P}}$ by $\hat{i}(\langle a, \dot{p} \rangle) = \langle i(a), i_*(\dot{p}) \rangle$ where i_* is the class function from $\mathbb{A}$ -names to $\mathbb{B}$ -names induced by i (see [12, p.222]). It is enough to show following claim.

Claim 3.4.1. If $\mathcal{A} \subset \mathbb{A} * \dot{\mathcal{P}}$ is a maximal antichain, then $\hat{i}[\mathcal{A}]$ is also a maximal antichain in $\mathbb{B} * \dot{\mathcal{P}}$.

Proof of Claim. Let $\mathcal{A} = \{(a_\alpha, \dot{p}_\alpha) : \alpha < \kappa\}$ be a maximal antichain of $\mathbb{A} * \dot{\mathcal{P}}$. Assume there exists $(b, \dot{p}) \in \mathbb{B} * \dot{\mathcal{P}}$ such that $(b, \dot{p})$ is compatible with all $\hat{i}((a_\alpha, \dot{p}_\alpha))$. Let G be $\mathbb{B}$ -generic over V such that $b \in G$ and let $H = i^{-1}[G]$. Look at $\{\dot{p}_\alpha[H] : i(a_\alpha) \in G\} = \mathcal{A}' \in V[H]$.

Subclaim 3.4.1. $V[H] \models \mathcal{A}'$ is maximal antichain of $\mathcal{P} = \dot{\mathcal{P}}[H]$.

antichain: Suppose $\alpha \neq \beta$ and $i(a_\alpha), i(a_\beta) \in G$. Since $(a_\alpha, \dot{p}_\alpha) \perp (a_\beta, \dot{p}_\beta)$, $\dot{p}_\alpha[H] \perp \dot{p}_\beta[H]$.

maximality: Assume to the contrary, there exists $p \in \mathcal{P}$ such that $p \perp \dot{p}_\alpha[H]$ for any $\dot{p}_\alpha[H] \in \mathcal{A}'$. Then there exists $a \in H$ such that

$$a \Vdash \forall \alpha < \kappa (a_\alpha \in \dot{H} \rightarrow \dot{p} \perp \dot{p}_\alpha).$$

Hence $(a, \dot{p}) \perp (a_\alpha, \dot{p}_\alpha)$. This is a contradiction to the maximality of $\mathcal{A}$.

Subclaim ■

Since $V[H] \models \mathcal{A}'$ is maximal antichain in $\mathcal{P}$ and " $\mathcal{A}'$ is maximal antichain of $\mathcal{P}$ " is a $\Pi_1^1(\mathcal{A}', \mathcal{P}, \leq_{\mathcal{P}}, \perp_{\mathcal{P}})$ -formula, $V[G] \models \mathcal{A}' = \{i_*(\dot{p}_\alpha)[G] : i(a_\alpha) \in G\}$ is maximal antichain of $\mathcal{P}$ by Π_1^1 -absoluteness. But this is a contradiction to the fact $V[G] \models \dot{p}[G] \perp i_*(\dot{p}_\alpha)[G]$ for $i(a_\alpha) \in G$.

Claim ■

Hence $\mathbb{A} * \dot{\mathcal{P}} \triangleleft \mathbb{B} * \dot{\mathcal{P}}$.

□

Corollary 3.5. Let $\langle \mathcal{Q}_\alpha : \alpha < \kappa \rangle$ be a sequence of Suslin forcing notions. Let $\mathbb{P}_\kappa$ be the finite support iteration of $\langle \mathbb{P}_\alpha, \dot{Q}_\alpha : \alpha < \kappa \rangle$ where $\Vdash_{\mathbb{P}_\alpha} \dot{Q}_\alpha = \mathcal{Q}_\alpha^{V^{\mathbb{P}_\alpha}}$. If $\mathbb{A} \triangleleft \mathbb{B}$, then $\mathbb{A} * \dot{\mathbb{P}}_\kappa \triangleleft \mathbb{B} * \dot{\mathbb{P}}_\kappa$.

Proof. We shall show that if $\mathcal{A}$ is a maximal antichain of $\mathbb{A} * \dot{\mathbb{P}}_\kappa$, then $\hat{i}[\mathcal{A}]$ is also a maximal antichain of $\mathbb{B} * \dot{\mathbb{P}}_\kappa$ where $\hat{i} : \mathbb{A} * \dot{\mathbb{P}}_\kappa \rightarrow \mathbb{B} * \dot{\mathbb{P}}_\kappa$ is induced by the complete embedding $i : \mathbb{A} \rightarrow \mathbb{B}$. It is enough to prove the following claim.

Claim 3.5.1. Let $\mathcal{A} \subset \mathbb{A} * \dot{\mathbb{P}}_\kappa$. If for each $p \in \mathbb{A} * \dot{\mathbb{P}}_\kappa$ there exists $q \in \mathcal{A}$ such that $q \parallel p$, then for each $r \in \mathbb{B} * \dot{\mathbb{P}}_\kappa$ there exists $q \in \mathcal{A}$ such that $\hat{i}(q) \parallel r$.

Proof of Claim. We shall show this by induction on κ .

The successor Step is as in Lemma 3.4.

Limit step. Let κ be a limit ordinal and for $\alpha < \kappa$ the induction hypothesis holds. Let $\mathcal{A} \subset \mathbb{A} * \dot{\mathbb{P}}_\kappa$ such that for each $p \in \mathbb{A} * \dot{\mathbb{P}}_\kappa$ there exists $q \in \mathcal{A}$ such that $p \parallel q$. Assume to the contrary there exists $p \in \mathbb{B} * \dot{\mathbb{P}}_\kappa$ such that $p \perp \hat{i}(q)$ for any $q \in \mathcal{A}$. Let $\alpha = \sup\{\beta < \kappa : \Vdash_{\mathbb{P}_\beta} p(\beta) \neq 1\} < \kappa$. Since for each $r \in \mathbb{A} * \dot{\mathbb{P}}_\kappa$ there exists $q \in \mathcal{A}$ such that $r \parallel q$, for each $r' \in \mathbb{A} * \dot{\mathbb{P}}_\alpha$ there exists $q \in \mathcal{A}$ such that $q \restriction \alpha \parallel r'$. By induction hypothesis there exists $q \in \mathcal{A}$ such that $p \restriction \alpha \parallel \hat{i}_\alpha(q \restriction \alpha)$ where $\hat{i}_\alpha : \mathbb{A} * \dot{\mathbb{P}}_\alpha \rightarrow \mathbb{B} * \dot{\mathbb{P}}_\alpha$ is induced by i . By $\hat{i}_\alpha(q \restriction \alpha) = \hat{i}(q) \restriction \alpha$, $p \restriction \alpha \parallel \hat{i}(q) \restriction \alpha$. So $p \parallel \hat{i}(q)$. It is a contradiction.

Claim ■

□

Let $\langle \mathcal{R}_\alpha : \alpha < \kappa \rangle$ be a sequence of Suslin forcing notions where all parameters appear in the ground model. Let $\mathbb{P}_\kappa$ be the finite support iteration of $\langle \mathbb{P}_\alpha, \dot{Q}_\alpha : \alpha < \kappa \rangle$ where $\Vdash_{\mathbb{P}_\alpha} \dot{Q}_\alpha = \mathcal{R}_\alpha^{V^{\mathbb{P}_\alpha}}$. Let $I \subset \kappa$. Recursively define $\mathbb{P}_I^\alpha$ by

(i) $\mathbb{P}_I^\alpha$ is given. Then $\mathbb{P}_I^{\alpha+1} = \mathbb{P}_I^\alpha * \dot{Q}'_\alpha$ where

$$\Vdash_{\mathbb{P}_I^\alpha} \dot{Q}'_\alpha = \begin{cases} \mathcal{R}_\alpha^{V^{\mathbb{P}_I^\alpha}} & \alpha \in I \\ \{1\} & \text{otherwise.} \end{cases}$$

(ii) Suppose α is a limit ordinal and $\mathbb{P}_I^\beta$ is given for $\beta < \alpha$. Define $\mathbb{P}_I^\alpha$ as the finite support iteration of $\langle \mathbb{P}_I^\beta, \dot{Q}'_\beta : \beta < \alpha \rangle$

Put $\mathbb{P}_I := \mathbb{P}_I^\kappa$.

Lemma 3.6. $\mathbb{P}_I \triangleleft \mathbb{P}_\kappa$.

Proof. We shall show for $\alpha \leq \kappa$ $\mathbb{P}_I^\alpha \triangleleft \mathbb{P}_\alpha$ by the induction on $\alpha \leq \kappa$.

Successor step. Suppose $\mathbb{P}_I^\alpha \triangleleft \mathbb{P}_\alpha$. If $\alpha \notin I$, it is clear that $\mathbb{P}_I^{\alpha+1} \triangleleft \mathbb{P}_{\alpha+1}$. If $\alpha \in I$,

then $\mathbb{P}_I^{\alpha+1} \triangleleft \mathbb{P}_{\alpha+1}$ is proved as in Lemma 3.4.

Limit step. Let α be a limit ordinal and for $\beta < \alpha$ the induction hypothesis holds. Define $i : \mathbb{P}_I^\alpha \triangleleft \mathbb{P}_\alpha$ by $i(p) = i_\beta(p)$ if $p \in \mathbb{P}_I^\beta$ for some $\beta < \alpha$ where $i_\beta : \mathbb{P}_I^\beta \rightarrow \mathbb{P}_\beta$ is the complete embedding. It is enough to prove the following claim.

Claim 3.6.1. *Let $\mathcal{A} \subset \mathbb{P}_I^\alpha$. If for each $p \in \mathbb{P}_I^\alpha$ there exists $q \in \mathcal{A}$ such that $q \parallel p$, then for each $r \in \mathbb{P}_\alpha$ there exists $q \in \mathcal{A}$ such that $i(q) \parallel r$.*

Proof of Claim. Let $\mathcal{A} \subset \mathbb{P}_I^\alpha$ such that for each $p \in \mathbb{P}_I^\alpha$ there exists $q \in \mathcal{A}$ such that $q \parallel p$. Let $r \in \mathbb{P}_\alpha$. Since $\mathbb{P}_\alpha$ is the finite support iteration of $\langle \mathbb{P}_\beta, \dot{Q}_\beta : \beta < \alpha \rangle$, there is $\beta < \alpha$ such that $r \in \mathbb{P}_\beta$. Since for each $p \in \mathbb{P}_I^\alpha$ there exists $q \in \mathcal{A}$ such that $q \parallel p$, for each $p' \in \mathbb{P}_I^\beta$ there exists $q \in \mathcal{A}$ such that $q \restriction \beta \parallel p'$. By induction hypothesis there exists $q \in \mathcal{A}$ such that $i_\beta(q \restriction \beta) = i(q) \restriction \beta \parallel r$. So $i(q) \parallel r$. Hence for each $r \in \mathbb{P}_\alpha$ there exists $q \in \mathcal{A}$ such that $i(q) \parallel r$.

Claim ■

Lemma □

For $\mathbb{P}_\kappa$ -name $\dot{x}$ for a real, there is following property.

Lemma 3.7. *Let $\mathbb{P}_\kappa$ is the κ -stage finite support iteration of Suslin forcing notions. If $\dot{x}$ is $\mathbb{P}_\kappa$ -name for a real. Then there exists countable $I \subset \kappa$ such that $\dot{x}$ is $\mathbb{P}_I$ -name.*

3.2 Niceness

In this paper we will force $\diamond(A, B, E)$ for Borel invariants (A, B, E) which satisfy the following properties:

There exist $\langle E_n : n \in \omega \rangle$ and $\langle U^n : n \in \omega \rangle$ such that

- (0) E_n is a Borel set for $n \in \omega$,
- (1) $E = \bigcap_{n \in \omega} E_n$,
- (2) $E_{n+1} \subset E_n$,
- (3) $U^n : A \rightarrow \wp(A)$ such that $U^n(x)$ is a Borel set
- (4) $x E_n y$ implies that there exists $m \geq n$ such that $U^m(x) \subset \{z \in A : z E_n y\}$.
- (5) $U^m(x) \subset \{z \in A : z E_n y\}$ is absolute with parameters x, y, U^m and E_n .

Example

- (i) For $(2^\omega, 2^\omega, \exists^\infty n (* \restriction I_n = *' \restriction I_n))$ let $x E_n y$ if $\exists m \geq n (x \restriction I_m = y \restriction I_m)$ and $U^n(x) = [x \restriction I_n] := \{y \in 2^\omega : y \restriction I_n = x \restriction I_n\}$. Then $\langle E_n : n \in \omega \rangle$ and $\langle U^n : n \in \omega \rangle$ satisfy (0)-(5).
- (ii) For $(\omega^\omega, \not\leq^*)$ let $x E_n y$ if $\exists m \geq n (x(m) < y(n))$ and $U^n(x) = \bigcup_{m \leq x(n)} [\langle n, m \rangle]$. Then $\langle E_n : n \in \omega \rangle$ and $\langle U^n : n \in \omega \rangle$ satisfy (0)-(5).
- (iii) Let $\text{LOC} = \{\phi : \phi : \omega \rightarrow [\omega]^{<\omega} \text{ where } |\phi(n)| \leq (n+1)^2 \text{ for } n \in \omega\}$. If $\phi \in \text{LOC}$, we call ϕ slalom. Then for $f \in \omega^\omega$ and $\phi \in \text{LOC}$ $\phi \sqsupset f$ if $\forall^\infty n (f(n) \in \phi(n))$. For $(\text{LOC}, \omega^\omega, \not\sqsupset)$ let $\phi E_n f$ if $\exists m \geq n (f(m) \notin \phi(m))$ and $U^n(\phi) = \bigcup_{s \subset \phi(n)} [\langle n, s \rangle]$. Then $\langle E_n : n \in \omega \rangle$ and $\langle U^n : n \in \omega \rangle$ satisfy (0)-(5).

For a Borel invariant (A, B, E) with $\langle U^n : n \in \omega \rangle$ and $\langle E_n : n \in \omega \rangle$ which satisfies (0)-(5), we will define the notion (A, B, E) -nice and show that the ω_2 -stage finite support iteration of some Suslin forcing notions forces parametrized $\Diamond$ -principles.

Definition 3.8. Let (A, B, E) be a Borel invariant with $\langle E_n : n \in \omega \rangle$ and $\langle U^n : n \in \omega \rangle$ satisfying (0)-(5). Let $\mathbb{P}$ be a forcing notion and $\mathcal{Q}$ be a Suslin forcing notion or finite support iteration of Suslin forcing notions.

Then $\mathcal{Q}$ is (A, B, E) -nice for $\mathbb{P}$ if for all $\mathcal{Q}$ -names $\dot{x}$ for an element of A for each $(p, \dot{q}) \in \mathbb{P} * \dot{\mathcal{Q}}$ there exists $x \in A \cap V$ such that for all $r \leq_{\mathbb{P}} p$ for all but finitely many n there exists $q' \in \mathcal{Q}$ such that $(1, q') \Vdash (r, \dot{q})$ and $q' \Vdash_{\mathcal{Q}} \dot{x} \in U^n(x)$.

There are following examples of niceness.

Proposition 3.9. Suppose I is countable subset of some ordinal κ . Then

- (1) $\mathbb{D}_I$ is $(2^\omega, \mathcal{N}, \in)$ -nice for $\mathbb{D}_{\omega_1}$
- (2) $\mathbb{B}_I$ is $(\omega^\omega, \not\leq^*)$ -nice for $\mathbb{B}_{\omega_1}$.
- (3) $\mathbb{E}_I$ is $(2^\omega, \mathcal{N}, \in)$ -nice for $\mathbb{E}_\omega$ and $(\omega^\omega, \not\leq^*)$ -nice for $\mathbb{E}_{\omega_1}$.
- (4) $(\mathbb{B} * \mathbb{D})_I$ is $(\text{LOC}, \omega^\omega, \not\sqsupset)$ -nice for $(\mathbb{B} * \mathbb{D})_{\omega_1}$.

Proof.

We shall show only $|I| = 1$. The General case is similar but more complicated.

(1). Let $\langle I_n : n \in \omega \rangle$ be a partition of ω such that $I_0 = \{0\}$, $I_1 = \{1, 2\}, \dots$, $I_{n+1} = \{\max(I_n) + 1, \dots, \max(I_n) + n + 1\}$. For $x \in 2^\omega$ let

$$A_x = \{y \in 2^\omega : \exists^\infty n \in \omega (x \upharpoonright I_n = y \upharpoonright I_n)\}.$$

Then A_x is null. So If $\diamond(2^\omega, 2^\omega, \exists^\infty n (* \upharpoonright I_n = *' \upharpoonright I_n))$ holds, then $\diamond(\text{cov}(\mathcal{N}))$ holds. So instead of showing that $\mathbb{D}$ is $(2^\omega, \mathcal{N}, \in)$ -nice for $\mathbb{D}_{\omega_1}$ we shall show $\mathbb{D}$ is $(2^\omega, 2^\omega, \exists^\infty n (* \upharpoonright I_n = *' \upharpoonright I_n))$ -nice for $\mathbb{D}_{\omega_1}$.

Let $\dot{x}$ be a $\mathbb{D}$ -name such that $\Vdash_{\mathbb{D}} \dot{x} \in 2^\omega$. Let $\langle p, \dot{q} \rangle \in \mathbb{D}_{\omega_1} * \dot{\mathbb{D}}$. For $s \in \omega^{<\omega}$ define $D_s \subset \mathbb{D}$ by $p \in D_s$ if there exists $f \in \omega^\omega$ such that $p = \langle s, f \rangle$. Then $\mathbb{D} = \bigcup_{s \in \omega^{<\omega}} D_s$.

Without loss of generality we can assume $p \Vdash_{\mathbb{D}_{\omega_1}} \dot{q} = \langle \check{s}, \dot{f} \rangle$ for some $s \in \omega^{<\omega}$. Then define $x_s \in 2^\omega \cap V$ so that $\forall m \in \omega \forall p \in D_s \neg p \Vdash x_s \upharpoonright I_m \neq \dot{x} \upharpoonright I_m$. Let $r \leq p$ and $m \in \omega$. Define $\langle r_n : n \in \omega \rangle$, $f \in \omega^\omega \cap V$ so that

- (i) $r_0 \leq r$, $r_{n+1} \leq r_n$ and
- (ii) r_n decides $\dot{f}(n)$ and $r_n \Vdash \dot{f}(n) = f(n)$.

Let $q' \leq \langle s, f \rangle$ such that $q' \Vdash_{\mathbb{D}} \dot{x} \in [x_s \upharpoonright I_m] = U^m(x_s)$.

Claim 3.9.1. $\langle 1, q' \rangle \Vdash \langle r, \langle s, \dot{f} \rangle \rangle$.

Proof of Claim. Let $q' = \langle t, g \rangle$. Then $r \upharpoonright |t| \Vdash \dot{f} \upharpoonright |t| = g \upharpoonright |t|$. So $\langle r \upharpoonright |t|, \langle t, \dot{f} \rangle \rangle \leq \langle r, \langle s, \dot{f} \rangle \rangle$. Hence $\langle 1, q' \rangle \Vdash \langle r, \langle s, \dot{f} \rangle \rangle$.

Claim ■ (1) □

(2). Let $\dot{x}$ be a $\mathbb{B}$ -name such that $\Vdash_{\mathbb{B}} \dot{x} \in \omega^\omega$. Define $x \in \omega^\omega \cap V$ so that $\mu(\{\dot{x}(n) \leq x(n)\}) \geq 1 - \frac{1}{2^{n+1}}$. Let $(p, \dot{q}) \in \mathbb{B}_{\omega_1} * \dot{\mathbb{B}}$. Without loss of generality we can assume $p \Vdash_{\mathbb{B}_{\omega_1}} \mu(\dot{q}) \geq \frac{1}{2^n}$. Then for any $r \leq p$ and $m \geq n$ $(r, \dot{q}) \Vdash (1, \{\dot{x}(m) \leq x(m)\})$ and $\{\dot{x}(m) \leq x(m)\} \Vdash_{\mathbb{B}} \dot{x} \in \bigcup_{i \leq x(m)} [\langle m, i \rangle] = U^m(x)$.

□

(3). $(2^\omega, \mathcal{N}, \in)$ -niceness is shown as (1).

$(\omega^\omega, \not\leq^*)$ -niceness: For $s \in \omega^{<\omega}$ and $k \in \omega$ let $E_{s,k} = \{p \in \mathbb{E} : p = \langle s, \dot{F} \rangle \text{ and } |\dot{F}| = k\}$. Then $\mathbb{E} = \bigcup_{s \in \omega^{<\omega}, k \in \omega} E_{s,k}$. Let $\dot{x}$ be $\mathbb{E}$ -name such that $\Vdash_{\mathbb{E}} \dot{x} \in \omega^\omega$. Let $\langle p, \dot{q} \rangle \in \mathbb{E}_{\omega_1} * \dot{\mathbb{E}}$. Without loss of generality we can assume $p \Vdash_{\mathbb{E}_{\omega_1}} \dot{q} \in E_{s,k}$. Then define $x_{s,k} \in \omega^\omega \cap V$ by

$$x_{s,k}(i) = \min\{j : \forall p \in E_{s,k} \neg(p \Vdash \dot{x} > j)\}.$$

For $j < k$ let $\dot{f}_j$ be a $\mathbb{E}_{\omega_1}$ -name such that $p \Vdash_{\mathbb{E}_{\omega_1}} \dot{q} = \langle s, \dot{F} \rangle$ and $\dot{F} = \{\dot{f}_j : j < k\}$.

Let $r \leq p$ and $m \in \omega$. Then define $\langle r_n : n \in \omega \rangle$ and $\{f_i : i < k\} \in \omega^\omega \cap V$ so that

(i) $r_0 \leq r, r_{n+1} \leq r_n$ and

(ii) r_m decides $\dot{f}_j \restriction m$ for $j < k$ and $r_m \Vdash_{\mathbb{E}_{\omega_1}} \dot{f}_j \restriction m = \dot{f}_j \restriction m$ for $j < m$.

Let $F = \{\dot{f}_j; j < k\}$ and $q' \leq \langle s, F \rangle$ such that $q' \Vdash_{\mathbb{E}} \dot{x}(m) < x_{s,k}(m)$. Then $q' \Vdash_{\mathbb{E}} \dot{x}(m) \in \bigcup_{i < x_{s,k}(m)} [\langle m, i \rangle] = U^m(x_{s,k})$.

Claim 3.9.2. $(r, \dot{q}) \parallel (1, q')$.

Proof of Claim. Let $q' = \langle t, G \rangle$. Since $r_{|t|} \Vdash_{\mathbb{E}_{\omega_1}} \dot{f}_j \restriction |t| = \dot{f}_j \restriction |t|$ for $j < k$, $r_{|t|} \Vdash_{\mathbb{E}_{\omega_1}} q' \parallel \dot{q}$. So $(1, q') \parallel (r, \dot{q})$.

Claim■ (3) □

(4) By [11] we can assume $\mathbb{A} := (\mathbb{B} * \dot{\mathbb{D}})_I$ is Boolean Algebra with strictly positive finitely additive measure μ . Let $\dot{\phi}$ is $\mathbb{A}$ -name such that $\Vdash_{\mathbb{A}} \dot{\phi} \in \text{LOC}$. For each $n \in \omega$ define $k_n \in \omega$ so that $\mu([k_n \in \dot{\phi}(n)]) < \frac{1}{n}$. Then define $\phi \in \text{LOC} \cap V$ by $\phi(n) = \{k_n\}$. Let $\langle p, \dot{q} \rangle \in (\mathbb{B} * \dot{\mathbb{D}})_{\omega_1}$. Without loss of generality we can assume $p \Vdash_{(\mathbb{B} * \dot{\mathbb{D}})_{\omega_1}} \mu(\dot{q}) > \frac{1}{k}$. Let $r \leq q$. Since $\mu([k_n \notin \dot{\phi}(n)]) \geq 1 - \frac{1}{k}$ for $n \geq k$, $r \Vdash_{(\mathbb{B} * \dot{\mathbb{D}})_{\omega_1}} \mu(\dot{q} \cap [k_n \notin \dot{\phi}]) \geq 0$ for $n \geq k$. Since $[\dot{\phi} \in U^n(\phi)] = [k_n \notin \dot{\phi}(n)]$, $r \Vdash_{(\mathbb{B} * \dot{\mathbb{D}})_{\omega_1}} \mu(\dot{q} \cap [\dot{\phi} \in U^n(\phi)]) > 0$. Hence $(r, \dot{q}) \parallel (1, [\dot{\phi} \in U^n(\phi)])$.

□

If $\mathcal{Q}$ is (A, B, E) -nice for $\mathbb{P}$, then elements of $A \cap V^{\mathcal{Q}}$ have a following property.

Theorem 3.10. [Minami] Let (A, B, E) be a Borel invariant with $\langle E_n : n \in \omega \rangle$ and $\langle U^n : n \in \omega \rangle$ which satisfy (0)-(5). Let $\mathbb{P}$ be a forcing notion such that there exists $\mathbb{P}$ -name $\dot{r}$ for an element of B such that $\Vdash_{\mathbb{P}} \dot{x} E \dot{r}$ for $x \in A \cap V$ and let $\mathcal{Q}$ be a Suslin forcing notion or the finite support iteration of Suslin forcing notions. If $\mathcal{Q}$ is (A, B, E) -nice for $\mathbb{P}$ and $\dot{x}$ is a $\mathcal{Q}$ -name for an element of $A \cap V^{\mathbb{P}}$, then $\Vdash_{\mathbb{P} * \dot{\mathcal{Q}}} \dot{x} E \dot{r}$.

Proof. Suppose $\mathcal{Q}$ is (A, B, E) -nice for $\mathbb{P}$. Let $\dot{r}$ be a $\mathbb{P}$ -name for an element of $B \cap V^{\mathbb{P}}$ such that $\Vdash \dot{x} E \dot{r}$ for $x \in A \cap V$. Let $\dot{x}$ be a $\mathcal{Q}$ -name for an element of $A \cap V^{\mathcal{Q}}$. It suffices to show for each $(p, \dot{q}) \in \mathbb{P} * \dot{\mathcal{Q}}$ there exists $(r, \dot{s}) \leq (p, \dot{q})$ such that

$$(r, \dot{s}) \Vdash \dot{x} E \dot{r}.$$

Let $(p, \dot{q}) \in \mathbb{P} * \dot{\mathcal{Q}}$. Since $\mathbb{P}$ is (A, B, E) -nice for $\mathcal{Q}$, there exists $x \in A \cap V$ such that

$$\forall r \leq_{\mathbb{P}} p \forall^{\infty} n \exists q' \in \mathcal{Q} ((1, q') \parallel (r, \dot{q}) \text{ and } q' \Vdash_{\mathcal{Q}} \dot{x} \in U^n(x)).$$

Let $r \leq p$ and $n \in \omega$ such that $r \Vdash_{\mathbb{P}} "xE_n\dot{r}"$ and if $m \geq n$, there exists $q' \in \mathcal{Q}$ $(1, q') \Vdash (r, \dot{q})$ and $q' \Vdash_{\mathcal{Q}} \dot{x} \in U^m(x)$. Since $r \Vdash xE_n\dot{r}$, there exists $m \geq n$ such that

$$r \Vdash U^m(x) \subset \{z \in A : zE_n\dot{r}\}.$$

Pick $q' \in \mathcal{Q}$ such that $(1, q') \Vdash (r, \dot{q})$ and $q' \Vdash_{\mathcal{Q}} \dot{x} \in U^m(x)$. Let $(p', \dot{q}^*) \leq (1, q'), (r, \dot{q})$. Then

$$(p', \dot{q}^*) \Vdash \dot{x} \in U^m(x) \subset \{z \in A : zE_n\dot{r}\}.$$

Hence $(p', \dot{q}^*) \Vdash \dot{x}E_n\dot{r}$. Therefore $\Vdash \dot{x}E\dot{r}$.

□

Theorem 3.11. *Let (A, B, E) be a Borel invariant with $\langle E_n, n \in \omega \rangle$ and $\langle U^n : n \in \omega \rangle$ satisfying (0)-(5). Let $\mathbb{P}_{\omega_2}$ be a ω_2 -stage finite support iteration of Suslin forcing notion and*

(1) *for all $\beta < \omega_2$ there exists a $\mathbb{P}_{\beta+\omega_1}$ -name $\dot{r}$ for an element of A such that $\Vdash_{\mathbb{P}_{\beta+\omega_1}} "xE\dot{r}"$ for $x \in A \cap V^{\mathbb{P}_{\beta}}$.*

(2) *for all $\beta < \omega_2$ for all I countable subset of $\omega_2 \setminus (\beta + \omega_1)$ $V^{\mathbb{P}_{\beta}} \models "$ $\mathbb{P}_I$ is (A, B, E) - nice for $\mathbb{P}_{[\beta, \beta+\omega_1]}$ $"$.*

Then $\mathbb{P}_{\omega_2} \models \Diamond(A, B, E)$.

Proof. Let $\dot{F}$ be a $\mathbb{P}_{\omega_2}$ -name for a Borel function. Since $\mathbb{P}_{\omega_2}$ has c.c.c and $\mathbb{P}_{\omega_2}$ is the finite support iteration of $\langle \mathbb{P}_{\alpha}, \dot{Q}_{\alpha} : \alpha < \omega_2 \rangle$ without loss of generality we can assume F is in ground model. By (1) let $\dot{r}_{\alpha}$ be a $\mathbb{P}_{\omega_1}$ -name such that $\Vdash_{\mathbb{P}_{\omega_1}} xE\dot{r}_{\alpha}$ for $x \in A \cap V^{\mathbb{P}_{\alpha}}$ for $\alpha < \omega_1$. We shall show $\Vdash_{\mathbb{P}_{\omega_2}} "\langle \dot{r}_{\alpha} : \alpha < \omega_1 \rangle$ is a $\Diamond(A, B, E)$ -sequence for F ".

Claim 3.11.1. *Let $\dot{f}$ be a $\mathbb{P}_{\omega_2}$ -name such that $\Vdash_{\mathbb{P}_{\omega_2}} \dot{f} : \omega_1 \rightarrow 2$. Then*

$$\{\alpha \in \omega_1 : \dot{f} \restriction \alpha \text{ is } \mathbb{P}_I\text{-name where } I \cap \omega_1 \subset \alpha \text{ and } I \text{ is countable}\}$$

contains a club.

■

Let $\dot{x} = F(\dot{f} \restriction \alpha)$ such that $\dot{x}$ is a $\mathbb{P}_I$ -name, I is countable and $I \cap \omega_1 \subset \alpha$. In $V^{\mathbb{P}_{\alpha}}$ we can assume $\dot{r}_{\alpha}$ is $\mathbb{P}_{[\alpha, \omega_1]}$ -name and $\dot{x}$ is $\mathbb{P}_{I \cap [\omega_1, \omega_2]}$ -name. Hence to show $\Vdash_{\mathbb{P}_{\omega_2}} "\langle \dot{r}_{\alpha} : \alpha < \omega_1 \rangle$ is $\Diamond(A, B, E)$ -sequence for F ", it suffices to show that $\Vdash_{\mathbb{P}_{\omega_1} * \mathbb{P}_I} "xE\dot{r}_{\alpha}"$ where $\dot{x}$ is $\mathbb{P}_I$ -name for an element of $A \cap V^{\mathbb{P}_I}$.

By (2) $\mathbb{P}_I$ is (A, B, E) -nice for $\mathbb{P}_{\omega_1}$. By Theorem 3.10 $\Vdash \dot{x}E\dot{r}_{\alpha}$. Hence $\langle \dot{r}_{\alpha} : \alpha < \omega_1 \rangle$ is a $\Diamond(A, B, E)$ -sequence for F .

□

Remark 3.11.2. Same argument holds for $\mathbb{P}_\kappa$ if $\text{cf}(\kappa) \geq \omega_2$.

Corollary 3.12. Each of the following are relatively consistent with ZFC:

- (i) $\mathfrak{c} = \text{add}(\mathcal{M}) = \omega_2 + \diamond(\text{cov}(\mathcal{N}))$ (see Diagram 1).
- (ii) $\mathfrak{c} = \text{cov}(\mathcal{N}) = \text{cov}(\mathcal{M}) = \omega_2 + \diamond(\mathfrak{b})$ (see Diagram 2).
- (iii) $\mathfrak{c} = \text{non}(\mathcal{M}) = \text{cov}(\mathcal{M}) = \omega_2 + \diamond(\mathfrak{b}) + \diamond(\text{cov}(\mathcal{N}))$ (see Diagram 3).
- (iv) $\mathfrak{c} = \text{cov}(\mathcal{N}) = \text{add}(\mathcal{M}) = \omega_2 + \diamond(\text{add}(\mathcal{N}))$ (see Diagram 4).

Proof. (i) Suppose $V \models \text{CH}$. By Theorem 3.11 and Proposition 3.9 (1) $V^{\mathbb{D}_{\omega_2}} \models \diamond(\text{cov}(\mathcal{N}))$. Since $\mathbb{D}_{\omega_2}$ adds ω_2 -many dominating reals and Cohen reals, $V^{\mathbb{D}_{\omega_2}} \models \mathfrak{c} = \mathfrak{b} = \text{cov}(\mathcal{M}) = \omega_2$. Since $\text{add}(\mathcal{M}) = \min\{\mathfrak{b}, \text{cov}(\mathcal{M})\}$ (see [19], [14]),

$$V^{\mathbb{D}_{\omega_2}} \models \diamond(\text{cov}(\mathcal{N})) + \mathfrak{c} = \text{add}(\mathcal{M}) = \omega_2.$$

Cichoń's diagram for parametrized diamond looks as follows where a ω_2 means the corresponding evaluation of Borel invariant is ω_2 while parametrized diamonds principle for the others hold.

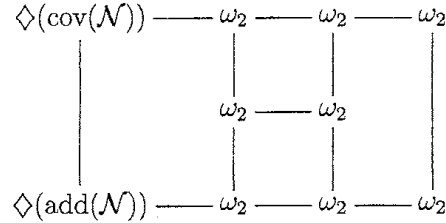


Diagram 1.

(ii) Suppose $V \models \text{CH}$. By Theorem 3.11 and Proposition 3.9 (2) $V^{\mathbb{B}_{\omega_2}} \models \diamond(\mathfrak{b})$. Since $\mathbb{B}_{\omega_2}$ adds ω_2 many Cohen and random reals, $V^{\mathbb{B}_{\omega_2}} \models \mathfrak{c} = \text{cov}(\mathcal{N}) = \text{cov}(\mathcal{M}) = \omega_2$. Hence

$$V^{\mathbb{B}_{\omega_2}} \models \diamond(\mathfrak{b}) + \mathfrak{c} = \text{cov}(\mathcal{N}) = \text{cov}(\mathcal{M}) = \omega_2.$$

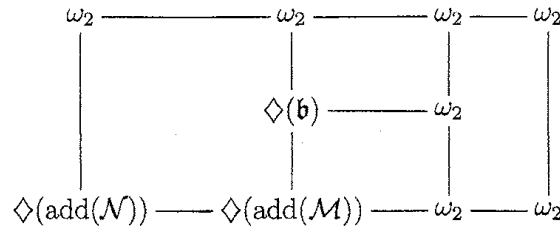


Diagram 2.

(iii) Suppose $V \models \text{CH}$. By Theorem 3.11 and Proposition 3.9 (3) $V^{\mathbb{E}_{\omega_2}} \models \diamond(\text{cov}(\mathcal{N})) + \diamond(\mathfrak{b})$. Since $\mathbb{E}_{\omega_2}$ adds ω_2 many Cohen and almost different reals, $\mathfrak{c} = \text{non}(\mathcal{M}) = \text{cov}(\mathcal{M}) = \omega_2$. Hence

$$V^{\mathbb{E}_{\omega_2}} \models \diamond(\text{cov}(\mathcal{N})) + \diamond(\text{cov}(\mathcal{M})) + \mathfrak{c} = \text{non}(\mathcal{M}) + \text{cov}(\mathcal{M}).$$

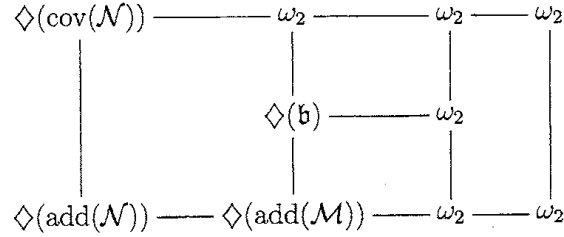


Diagram 3.

(iv) Suppose $V \models \text{CH}$. By Theorem 3.11 and Proposition 3.9 (4) $V^{(\mathbb{B} * \dot{\mathbb{D}})_{\omega_2}} \models \diamond(\text{add}(\mathcal{N}))$. Since $(\mathbb{B} * \dot{\mathbb{D}})_{\omega_2}$ adds ω_2 many random, Cohen and dominating reals, $\mathfrak{c} = \text{cov}(\mathcal{N}) = \text{add}(\mathcal{M}) = \min\{\mathfrak{b}, \text{cov}(\mathcal{N})\} = \omega_2$. Hence

$$V^{(\mathbb{B} * \dot{\mathbb{D}})_{\omega_2}} \models \diamond(\text{add}(\mathcal{N})) + \mathfrak{c} = \text{cov}(\mathcal{N}) = \text{add}(\mathcal{M}) = \omega_2.$$

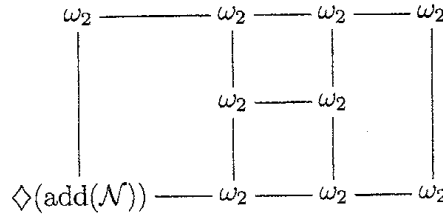


Diagram 4

□

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