

On the Resolution of Hypersurface Singularities

By Mutsuo OKA

岡 睦雄

東京工業大学 理学部 数学教室

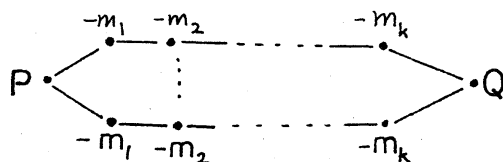
§1. Introduction.

Let $f(z_1, \dots, z_{n+1})$ be a germ of an analytic function at the origin such that $f(0) = 0$ and f has an isolated critical point at the origin. We assume that f has a non-degenerate Newton boundary. Let V be a germ of hypersurface $f^{-1}(0)$ at the origin. We will study the resolution of V . Let $\Gamma^*(f)$ be the dual Newton diagram and let Σ^* be a simplicial subdivision. It is well-known that there is a canonical resolution $\pi : \tilde{V} \rightarrow V$ which is associated with Σ^* . ([5]). However the process to get Σ^* from $\Gamma^*(f)$ is not unique and a "bad" Σ^* produces unnecessary exceptional divisors.

The purpose of this paper is to study the topology of the exceptional divisors using a "canonical" subdivision Σ^* . For example, if $n = 2$, there is a "unique" subdivision Σ^* so that the resolution graph is obtained by a canonical surgery from $S_2 \Gamma^*(f)$ (= 2 skeleton of $\Gamma^*(f)$ which we consider as a graph by a plane section). Namely we have:

(I). For a vertex P of $S_2 \Gamma^*(f)$ which corresponds to a compact two dimensional face $\Delta(P)$ of the Newton boundary $\Gamma(f)$, there is an exceptional divisor $E(P)$ which is a Riemann surface of genus $g(P)$. Here $g(P)$ is equal to the number of integral points of $\Delta(P) - \partial\Delta(P)$.

(II). A line segment $\overline{PQ}$ of $S_2 \Gamma^*(f)$ is replaced by $r(P, Q) + 1$ copies of a chain of rational curves



where $r(P, Q)$ is the number of integral points on $E - \partial E$,

$E = \Delta(P) \cap \Delta(Q)$. The continuous fraction $m_1 - \frac{1}{m_2 - \dots - \frac{1}{m_k}}$

is canonically associated to P and Q . (See Theorem (6.1).)

This paper consists of the following sections.

§2. Newton boundary and the dual Newton diagram.

§3. Canonical simplicial subdivision.

§4. The resolution of V .

§5. Topology of the exceptional divisors.

§6. The surface singularity.

§7. Three dimensional singularity.

§2. Newton boundary and the dual Newton diagram.

Let $f(z_1, \dots, z_{n+1}) = \sum a_v z^v$ be the Taylor expansion of f where $z^v = z_1^{v_1} \dots z_{n+1}^{v_{n+1}}$ as usual. Recall that the Newton boundary $\Gamma(f)$ is the union of compact faces of $\Gamma_+(f)$ where $\Gamma_+(f)$ is the convex hull in $\mathbb{R}^{n+1}$ of the union of the subsets $v + (\mathbb{R}^+)^{n+1}$ for v such that $a_v \neq 0$.

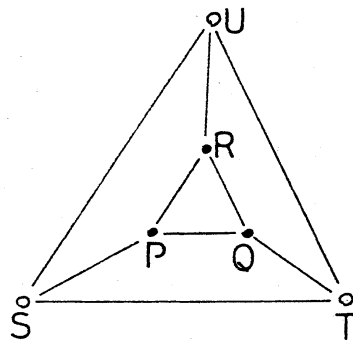
$\Gamma(f)$ has a natural polyhedral decomposition. For any closed face Δ (of any dimension) of $\Gamma(f)$, we associate the polynomial $f_\Delta(z) = \sum_{v \in \Delta} a_v z^v$.

We say that f is non-degenerate on Δ if $\frac{\partial f_\Delta}{\partial z_1} = \dots = \frac{\partial f_\Delta}{\partial z_{n+1}} = 0$

has no solution in $(\mathbb{C}^*)^{n+1}$. We say that f is non-degenerate if f is non-degenerate on any face Δ of $\Gamma(f)$. ([6], [14]).

Let N^+ be the space of positive vectors of the dual space of $\mathbb{R}^{n+1}$. For any vector $a = \begin{pmatrix} a_1 \\ \vdots \\ a_{n+1} \end{pmatrix}$ of N^+ , we associate the linear function $a(x) = \sum a_i x_i$ on $\Gamma_+(f)$. Let $d(a)$ be the minimum value of a on $\Gamma_+(f)$ and let $\Delta(a) = \{x \in \Gamma_+(f); a(x) = d(a)\}$. We introduce an equivalent relation $\sim$ on N^+ by $a \sim b$ if and only $\Delta(a) = \Delta(b)$. For any face Δ of dimension k of $\Gamma_+(f)$, there is an equivalence class Δ^* of dimension $(n+1-k)$ which is defined by $\Delta^* = \{a \in N^+; \Delta(a) = \Delta\}$. The collection of Δ^* gives a polyhedral decomposition $\Gamma^*(f)$ of N^+ which we call the dual Newton diagram of f . We identify $\Gamma^*(f)$ and its projection on the hyperplane $L = \{x_1 + \dots + x_{n+1} = 1\}$. Thus a vertex P of $\Gamma^*(f)$ is a primitive integral vector such that $\Delta(P)$ is of dimension n . If P is strictly positive, i.e. each coordinate of P is positive, $\Delta(P)$ is a compact face of $\Gamma(f)$.

Example (2.1). Let $f(x,y,z) = x^4 + y^4 + z^4 + xyz$. Then $\Gamma(f)$ has three faces of maximal dimension and $\Gamma^*(f)$ is the following.



$$\begin{aligned} P &= (2, 1, 1), S = (1, 0, 0) \\ Q &= (1, 2, 1), T = (0, 1, 0) \\ R &= (1, 1, 2), U = (0, 0, 1) \end{aligned}$$

Figure (2.1)

For any $P \in N^+$, we define the support of P by $\Delta(P)^*$.

We say a polyhedral subdivision Σ^* of $\Gamma^*(f)$ is a simplicial subdivision if the following conditions are satisfied. ([5]).

- (i) Σ^* is a subdivision by the cones over the simplexes $\sigma = (P_1, \dots, P_{k+1})$ where $P_1, \dots, P_{k+1}$ are primitive vectors and they can be extended to a basis of $\mathbb{Z}^{n+1}$. We identify the cone over a simplex σ and σ itself. In particular, any n -simplex $\sigma = (P_1, \dots, P_{n+1})$ with $P_i = {}^t(p_{1,i}, \dots, p_{n+1,i})$ satisfies $\det(p_{i,j}) = \pm 1$. The intersection of two simplexes is a simplex.
- (ii) For any n -simplex as above, there exists a permutation τ of $\{1, \dots, n+1\}$ such that

$$(2.2) \quad \Delta(P_{\tau(1)}) \supset \Delta(P_{\tau(2)}) \supset \dots \supset \Delta(P_{\tau(n+1)}).$$

Remark (2.3). The condition (2.2) can be replaced by a weaker condition but (2.2) makes the intersection of exceptional divisors easier to be understood. (See Lemma (4.7).)

§3. Canonical simplicial subdivision.

Let $P_i = {}^t(p_{1,i}, \dots, p_{n+1,i})$ ($i=1, \dots, k$) be given integral vectors of N^+ . We define a non-negative integer $\det(P_1, \dots, P_k)$ by the greatest common divisor of all $k \times k$ minors of the matrix $(p_{j,i})$ and we call $\det(P_1, \dots, P_k)$ the determinant of $P_1, \dots, P_k$.

Lemma (3.1). Let $A = (a_{i,j})$ be a unimodular matrix. Then we have $\det(P_1, \dots, P_k) = \det(AP_1, \dots, AP_k)$.

The proof is an easy exercise of linear algebra.

Lemma (3.2). Let $P_1, \dots, P_k$ be given integral vectors such that $\det(P_1, \dots, P_k) = \pm 1$. Then there exist integral vectors $P_{k+1}, \dots, P_{n+1}$ such that $\det(P_1, \dots, P_{n+1}) = \pm 1$.

Proof. Let M be the subgroup of $\mathbb{Z}^{n+1}$ generated by $P_1, \dots, P_k$. Then by the structure theorem of a finitely generated abelian group, there is a

subgroup M' of rank k such that $M \subset M'$ and M' is a direct summand of $\mathbb{Z}^{n+1}$. Then the assumption that $\det(P_1, \dots, P_k) = 1$ clearly implies $M' = M$.

(I). Division of a line segment.

Let $P = \begin{pmatrix} p_1 \\ \vdots \\ p_{n+1} \end{pmatrix}$ and $Q = \begin{pmatrix} q_1 \\ \vdots \\ q_{n+1} \end{pmatrix}$ be given distinct primitive ^{integral} vectors in N^+ .

Lemma (3.3). Let $c = \det(P, Q)$.

(i) Any integral vector P_1 on the segment $\overline{PQ}$ such that $\det(P, P_1) = 1$ can be written as

$$P_1 = \frac{1}{c} Q + \frac{c_1}{c} P$$

for some integer $c_1 \geq 0$.

(ii) There exists a unique c_1 such that $0 \leq c_1 < c$.

(iii) P_1 is necessarily primitive.

Proof. By Lemmas (3.1) and (3.2), we may assume that $Q = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

Then c is nothing but $\text{g.c.d.}(p_2, \dots, p_{n+1})$. Let $P_1 = \lambda P + \mu Q$ ($\lambda \geq 0, \mu \geq 0$) and assume that P_1 is an integral and P_1 satisfies $\det(P, P_1) = 1$. As $\det(P, P_1) = \mu \cdot \det(P, Q) = \mu c = 1$, we have $\mu = \frac{1}{c}$. As P_1 is integral, $\lambda p_i \in \mathbb{Z}$ for $i = 2, \dots, n+1$ and this implies that $\lambda = \frac{c_1}{c}$ for some $c_1 \geq 0$ and $c_1 p_1 + 1 = 0$ modulo c . The last equation has a unique solution in $0 \leq c_1 < c$ as $\text{g.c.d.}(p_1, c) = \text{g.c.d.}(p_1, \dots, p_{n+1}) = 1$. The assertion (iii) is immediate from the assumption $\det(P, P_1) = 1$. Note that $c_1 = 0$ if and only if $c = 1$.

Remark (3.4). By the abuse of language, we say that $P_1 = \lambda P + \mu Q$ is on the line segment $\overline{PQ}$ if the normalized vector $P'_1 = \frac{\lambda}{\lambda + \mu} P + \frac{\mu}{\lambda + \mu} Q$ is on $\overline{PQ}$. This is the case if and only if $\lambda \geq 0, \mu \geq 0$.

Definition (3.5). Let $\overline{PQ}$ be a line segment of $S_2 \Gamma^*(f)$. Here $S_k \Gamma^*(f)$ is the k -skeleton of $\Gamma^*(f)$. We say that

the primitive vectors $P_1, \dots, P_k$ on $\overline{PQ}$ are the canonical primitive sequences if the following is satisfied.

- (i) If $c = \det(P, Q) = 1$, $k = 1$ and $P_1 = P + Q$.
- (ii) If $c > 1$, there are non-negative integers $c = c_0 > c_1 > \dots > c_k = 1$, $c_{k+1} = 0$ such that

$$P_{i+1} = \frac{1}{c_i} Q + \frac{c_{i+1}}{c_i} P_i \quad (i = 0, \dots, k)$$

where $P_0 = P$ and $P_{k+1} = Q$.

The condition (i) is to satisfy (2.2).

Lemma (3.6). Assume that $c = \det(P, Q) > 1$ and let $P_1, \dots, P_k$ be the canonical primitive sequence of the line segment $\overline{PQ}$. Let c_i ($i=0, \dots, k+1$) be as in (3.5) and let $m_i = \frac{c_{i-1} + c_{i+1}}{c_i}$ ($i = 1, \dots, k$). Then m_i is an integer such that $m_i \geq 2$ and the continuous fraction

$$m_1 - \frac{1}{m_2 - \frac{1}{m_3 - \frac{1}{m_4 - \dots - \frac{1}{m_k}}}}$$

is equal to $\frac{c}{c_1}$.

Let $P_i = \begin{pmatrix} p_{1i} \\ \vdots \\ p_{n+1,i} \end{pmatrix}$. Then m_i is equal to $\frac{p_{j,i-1} + p_{j,i+1}}{p_{j,i}}$ for any $j = 1, \dots, n+1$.

Proof. We prove the assertion by the induction on k . Assume that $k = 1$.

Then $P_1 = \frac{1}{c} Q + \frac{1}{c} P$. Thus $m_1 = \frac{c+0}{c_1} = c$. As $c p_{j,1} = q_j + p_j$, we have

$c = \frac{p_j + q_j}{p_{j,1}}$. Assume that $k > 1$. Then as $P_1 = \frac{Q}{c} + \frac{c_1}{c} P$ and $P_2 = \frac{Q}{c_1} + \frac{c_2}{c_1} P_1$,

we have that $\det(P, P_2) = \frac{1}{c_1} \det(P, Q + c_2 P_1) = \frac{1}{c_1} (1 + \frac{c_2}{c}) \det(P, Q) = \frac{c+c_2}{c_1} = m_1$.

Thus m_1 is an integer and $m_1 \geq 2$. We can write $P_1 = P/m_1 + P_2/m_1$. This implies that P_1 is the canonical primitive sequence for $\overline{PP_2}$. Now we have

$$m_1 - \frac{1}{m_2 - \dots - \frac{1}{m_k}} = \frac{c + c_2}{c_1} - \frac{1}{c_1/c_2} = c/c_1, \text{ completing the proof of the}$$

first assertion. The second assertion is obvious from the equality:

$$(c_{i-1} + c_{i+1})P_i = c_i(P_{i-1} + P_{i+1}).$$

Remark (3.7). Lemma (3.6) is true for every primitive sequence $P_1, \dots, P_s$ on $\overline{PQ}$ such that $\det(P_{i-1}, P_i) = 1$ for $i = 1, \dots, s+1$ ($P_0 = P$ and $P_{s+1} = Q$) except that $m_i \geq 1$ in this case. For example, $[m_1+1, 1, m_2+1, m_3, \dots, m_k] = [m_1, \dots, m_k]$. Here we write $r_1 - \frac{1}{r_2 - \dots - \frac{1}{r_t}} = [r_1, \dots, r_t]$ for brevity's

sake. Note that if $c = \det(P, Q) > 1$, a primitive sequence $P_1, \dots, P_s$ is canonical if and only if $m_i \geq 2$ for each i , by the second expression of m_i . In particular, if $P_1, \dots, P_k$ is ^(the) canonical primitive sequence of $\overline{PQ}$, $P_k, \dots, P_1$ is the canonical primitive sequence of $\overline{QP}$. The ambiguity of the continuous fraction expression $[r_1, \dots, r_s]$ for $r_i \geq 1$ corresponds to the existence of the exceptional curve of the first kind. (See §6.)

(II) Division of $S_3 \Gamma^*(f)$.

Let ξ be a closed cell of dimension 2 with vertices $P, \dots, S$. There is no unique way to divide ξ into simplexes with determinant 1. We first take canonical primitive sequence on the boundaries and divide ξ into triangles without adding any other vertices. The determinant need not be equal to one.

Lemma (3.8). Let Δ be a triangle with primitive integral vectors P, Q, R as vertices. Let $c = \det(P, Q, R)$ and assume that $c > 1$ and $\det(P, Q) = \det(P, R) = 1$. Then there are unique c_1 and d_1 such that

(i) $0 < c_1 < c$ and $0 \leq d_1 < c$. (ii) The point defined by $T_1 = \frac{1}{c} R + \frac{c_1}{c} Q + \frac{d_1}{c} P$ is an integral point which satisfies $\det(P, Q, T_1) \stackrel{=1}{=}$ 1. (iii) T_1 divide Δ into three (two if $d_1 = 0$) triangles with $\det(P, Q, T_1) = 1$, $\det(P, T_1, R) = c_1$ and $\det(Q, T_1, R) = d_1$.

Proof. By Lemma (3.1) and Lemma (3.2), we may assume that $P = {}^t(1, 0, \dots, 0)$, $Q = {}^t(0, 1, 0, \dots, 0)$ and $R = {}^t(r_1, r_2, r_3, 0, \dots, 0)$. Thus $c = r_3$. T_1 is an integral point if and only if $c_1 + r_2 \equiv 0$, $d_1 + r_1 \equiv 0$ modulo c . Clearly this has a unique solution in $0 \leq c_1 < c$ and $0 \leq d_1 < c$. If $c_1 = 0$, T_1 is on $\overline{PR}$ and we have a contradiction $1 = \det(P, R) = c \det(P, T_1)$. The other assertions are obvious.

Note that $\det(P, T_1) = 1$ by $\det(P, Q, T_1) = 1$. Thus we can do the same operation on the triangle $\Delta_1 = \Delta(P, T_1, R)$ if $c_1 > 1$. By the induction, we get

Lemma (3.9). Assume that $c > 1$. There exists unique integers $k, c_1, \dots, c_k$ and $d_1, \dots, d_k$ such that $c = c_0 > c_1 > \dots > c_k = 1$ and $0 \leq d_i < c_{i-1}$ and $T_i = R/c_{i-1} + c_i T_{i-1}/c_{i-1} + d_i P/c_{i-1}$ ($i = 1, \dots, k$) are integral points with $\det(P, T_{i-1}, T_i) = 1$ where $T_0 = Q$. (See Figure (3.10).)

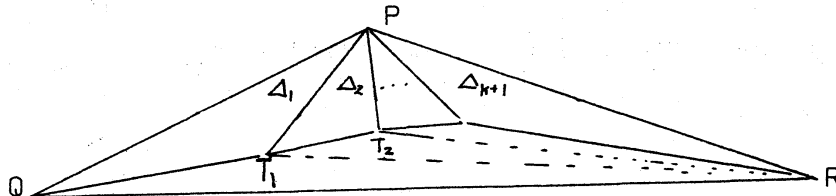


Figure (3.10).

We say that $\Delta_i = (P, T_{i-1}, T_i)$ ($i = 1, \dots, k+1$) are the canonical simplicial subdivision of the triangle (P, Q, R) at P . ($T_{k+1} = R$). In the case $c = 1$, we take

one vertex $T = Q + R$ if $\Delta(Q) \not\leq \Delta(R)$ and $\Delta(Q) \not\geq \Delta(R)$. Otherwise we do not need any subdivision.

We return to the subdivision of ξ . Take any triangle $\Delta(P', Q', R')$. If $\det(P', Q') \neq 1$, we take canonical primitive sequences on $\overline{P'Q'}$, say $P_1, \dots, P_k$, and divide $\Delta(P', Q', R')$ into $k+1$ triangles with the property $\det(P_i, P_{i+1}, R') < \det(P', Q', R')$. If $\det(P', Q') = 1$ and $\Delta(P')$ and $\Delta(Q')$ are not compatible with (2.2), we take one vertex $P_1 = P' + Q'$. Thus we may assume that $\det(P', Q') = 1$ and $\Delta(P')$ and $\Delta(Q')$ are compatible with (2.2). Then we do the same operation on $\overline{P'R'}$. As the determinants of the triangles are monotone decreasing under the above operation, after a finite number of operations we come to the situation of Lemma (3.8). Then we apply Lemma(3.8) and so on. Thus by the finite operations we can divide ξ into simplicial triangles i.e. with determinant one and the condition (2.2). By this argument, we can divide $S_3 \Gamma^*(f)$ into simplicial triangles so that the restriction of the subdivision to the $S_2 \Gamma^*(f)$ is the canonical one which is constructed in (3.5).

The subdivision of $\Gamma^*(f)$ can be done by the inductive argument. Suppose we have a simplicial subdivision of $S_m \Gamma^*(f)$. First we divide $S_{m+1} \Gamma^*(f)$ into simplexes with determinant not necessarily one without adding any vertex on $S_m \Gamma^*(f)$. Then the following Lemma plays the same role as above.

Lemma (3.1). Let Δ be a simplex with vertices $P_1, \dots, P_{m+1}$ and assume that $\det(P_1, \dots, \underset{\downarrow}{P_i}, \dots, P_{m+1}) = 1$ and $c = \det(P_1, \dots, P_{m+1}) > 1$ ($i=2, \dots, m+1$). Then there exists unique integers $c_1, \dots, c_m$ such that the vertex T defined by

$$T = c_1 P_1 / c + \dots + c_m P_m / c + P_{m+1} / c$$

is an integral point and $0 < c_i < c$ for $i \neq 1$ and $c > c_1 \geq 0$. T divides

Δ into $m+1$ simplexes with respective determinant $c_1, \dots, c_m$ and 1.

The proof is completely parallel to that of Lemma (3.8). The details are left for the reader. See also [5] for a further information.

§4. The resolution of V .

Let f be an analytic function with an isolated critical point at the origin. We assume that f has a non-degenerate Newton boundary and let Σ^* be a given simplicial subdivision of $\Gamma^*(f)$.

For any n -simplex $\sigma = (P_1, \dots, P_{n+1}) = (p_{ij})$, we associate $(n+1)$ -dimensional Euclidean space $\mathbb{C}_\sigma^{n+1}$ with coordinate $(y_{\sigma,1}, \dots, y_{\sigma,n+1})$ and holomorphic mapping $\pi_\sigma : \mathbb{C}_\sigma^{n+1} \rightarrow \mathbb{C}^{n+1}$ which is defined by

$z_i = y_{\sigma,1}^{p_{i,1}} y_{\sigma,2}^{p_{i,2}} \dots y_{\sigma,n+1}^{p_{i,n+1}}$. By the abuse of the notation, we write

$z = (y_\sigma)^\sigma$. Let X be the union of $\mathbb{C}_\sigma^{n+1}$ which are glued along the images of π_σ . Let $\pi : X \rightarrow \mathbb{C}^{n+1}$ be the projection map and let $\tilde{V}$ be the closure of $\pi^{-1}(V \cap (\mathbb{C}^*)^{n+1})$. It is well-known that $\pi : \tilde{V} \rightarrow V$ is a resolution of V ([5]). ($n \geq 2$)

Let $d_i = d(P_i)$ and $\Delta_i = \Delta(P_i)$ be as in §2. The ordering of $P_1, \dots, P_{n+1}$ is so chosen that

$$(4.1). \quad \Delta_1 > \Delta_2 > \dots > \Delta_{n+1}.$$

We define $g_{\Delta_i}(y_{\sigma,1}, \dots, y_{\sigma,n+1}) = f_{\Delta_i}(\pi_\sigma(y_\sigma)) / \prod_{j=1}^{n+1} y_{\sigma,j}^{d_j}$. It is easy

to see that g_{Δ_i} is a function of $n+1-i$ variables $y_{\sigma,i+1}, \dots, y_{\sigma,n+1}$.

If P_i is strictly positive, g_{Δ_i} is a polynomial. We can write

$\pi_{\sigma}^* f(y_{\sigma}) = y_{\sigma,1}^{d_1} \cdots y_{\sigma,n+1}^{d_{n+1}} f_{\sigma}(y_{\sigma})$. By the definition of $\tilde{V}$, $\tilde{V} \cap \mathbb{C}_{\sigma}^{n+1} = \{f_{\sigma} = 0\}$ and $f_{\sigma}(y_{\sigma,1}, \dots, \overset{j}{0}, \dots, y_{\sigma,n+1}) = g_{\Delta_i}(y_{\sigma,j+1}, \dots, y_{\sigma,n+1})$.

Thus we have

$$(4.2). \quad \tilde{V} \cap \{y_{\sigma,j} = 0\} \neq \emptyset \quad \text{if and only if} \quad \dim \Delta_i > 0.$$

Recall that $\bigcup S_k \Gamma^*(f)$ is the union of cells of $\Gamma^*(f)$ whose dimension is less than $k+1$, and we call $S_k \Gamma^*(f)$ the k -skeleton of $\Gamma^*(f)$.

((4.3))
Remark. The dimension of a cell decreases by 1 if we consider the section with the hyperplane $x_1 + \dots + x_{n+1} = 1$.

Corollary (4.4). Assume that $\sigma \cap S_n \Gamma^*(f) = \emptyset$. Then $\tilde{V} \cap \mathbb{C}_{\sigma}^{n+1} \subset (\mathbb{C}_{\sigma}^*)^{n+1}$. Thus no exceptional divisor is in this chart.

Let σ be as above and let $E(P_i; \sigma) = \tilde{V} \cap \{y_{\sigma,i} = 0\} = \{(y_{\sigma,1}, \dots, y_{\sigma,n+1}); y_{\sigma,i} = 0, g_{\Delta_i}(y_{\sigma,i+1}, \dots, y_{\sigma,n+1}) = 0\}$. The smoothness of $\tilde{V}$ and $E(P_i; \sigma)$ in a neighborhood of $\pi^{-1}(0)$ is easily derived from the non-degeneracy assumption of the Newton boundary. By the definition of π_{σ} , we have

$$(4.5). \quad \pi_{\sigma}(E(P_i; \sigma)) = \{0\} \quad \text{if and only if} \quad P_i \text{ is strictly positive.}$$

Now we study the glueing map between $\mathbb{C}_{\sigma}^{n+1}$ and $\mathbb{C}_{\tau}^{n+1}$. Let $\sigma = (P_1, \dots, P_{n+1})$ and $\tau = (Q_1, \dots, Q_{n+1})$. We can write

$$Q_i = \sum_{j=1}^{n+1} \lambda_{ji} P_j \quad \text{for any } i = 1, \dots, n+1.$$

Then $y_{\sigma} = \pi_{\sigma}^{-1} \circ \pi_{\tau}(y_{\tau}) = \pi_{\sigma^{-1} \circ \tau}(y_{\tau})$ and $\sigma^{-1} \tau$ is the matrix $\Lambda = (\lambda_{ij})$.

Namely we have

$$(4.6). \quad y_{\sigma,i} = y_{\tau,1}^{\lambda_{i,1}} \cdots y_{\tau,n+1}^{\lambda_{i,n+1}} \quad (i = 1, \dots, n+1).$$

In particular, if $P_j = Q_i$ we have

$$E(P_i; \sigma) = \{ y_\sigma ; y_{\sigma, v_i} = 0, g_{\Delta(Q_{v_i})}(y_{\sigma, v_{i+1}}, \dots, y_{\sigma, n+1}) = 0 \}.$$

Thus $E(P_1; \sigma) \cap \dots \cap E(P_k; \sigma) =$

$$\{ y_\sigma ; y_{\sigma, v_i} = 0 \text{ for } i=1, \dots, k \text{ and } g_{\Delta(Q_{v_k})}(y_{\sigma, v_{k+1}}, \dots, y_{\sigma, n+1}) = 0 \}$$

which is clearly smooth of dimension $n - k$. It is compact because

$E(P_1) \cap \dots \cap E(P_k)$ is a closed subspace of a compact divisor $E(P_k)$.

The divisor $E(P)$ with $\dim \Delta(P) = 1$ is special by the following reason.

Lemma (4.8). Assume that $\dim(P) = 1$. Then $E(P)$ has $r(\Delta(P)) + 1$ connected components where $r(\Delta(P))$ is the number of integral points on $\Delta(P) - \partial \Delta(P)$.

Proof. We can find a simplex $\sigma = (P_1, \dots, P_{n+1})$ such that $P_n = P$ and $\Delta(P_{n+1})$ is one of the end points $\partial \Delta(P)$. Let $f_{\Delta(P)}(z) =$

$$\sum_{i=0}^{r+1} a_i z^{v_i} \text{ where } v_0, v_1, \dots, v_{r+1} \text{ are integral points of } \Delta(P) \text{ in the}$$

order and $r = r(\Delta(P))$. $E(P; \sigma)$ is defined by $y_{\sigma, n} = 0$ and

$g_{\Delta(P_n)}(y_{\sigma, n+1}) = 0$. As the number of the integral points on $\Delta(P)$ and on

$S(g_{\Delta(P_n)})$ (= the support of $g_{\Delta(P_n)}$) are equal, we may assume that

$$g_{\Delta(P_n)}(y_{\sigma, n+1}) = \sum_{i=0}^{r+1} a_i y_{\sigma, n+1}^i. \text{ Thus by the non-degeneracy condition,}$$

$g_{\Delta(P_n)}(y_{\sigma, n+1}) = 0$ is the disjoint union of $r + 1$ planes

$$L(\sigma)_i = \{ y_{\sigma, n+1} = \xi_i \text{ and } y_{\sigma, n} = 0 \} \text{ where } \xi_i \neq 0 \text{ for } i = 1, \dots, r+1.$$

Let $\tau = (Q_1, \dots, Q_{n+1})$ be a simplex such that $\Delta(Q_1) > \dots > \Delta(Q_{n+1})$

and $Q_i = P$ for some i and $\Delta(Q_{i+1}) = \dots = \Delta(Q_{n+1}) = \text{one point}$. We

can find a simplex σ as above such that $Q_1 = P_{i_1}, Q_2 = P_{i_2}, \dots, Q_i = P_n$.

Watching the glueing map carefully, we can see that $E(P; \tau) \subset E(P; \sigma)$.

Thus $E(P)$ is covered by $E(P; \sigma)$ where σ is of the above type. Let

$\sigma' = (P'_1, \dots, P'_{n+1})$ be another simplex with $P'_n = P$. Then $y_{\sigma'}$ and y_{σ} are identified through the matrix

$$\Lambda = \begin{pmatrix} & & & 0 & * \\ & \lambda & & \vdots & \\ & & & \vdots & \\ & & & 0 & \\ * & \dots & * & 1 & * \\ 0 & \dots & \dots & 0 & \pm 1 \end{pmatrix}$$

The sign ± 1 means whether P'_{n+1} is on the same side of P_{n+1} or not

with respect to $\Delta(P)^*$. Thus the component $y_{\sigma', n+1}^{\pm 1} = \xi_i$ is glued with the component $y_{\sigma, n+1} = \xi_i$ by $y_{\sigma, 1} = y_{\sigma', 1}^{\lambda_{11}} \dots y_{\sigma', n-1}^{\lambda_{1, n-1}} y_{\sigma', n+1}^{\lambda_{1, n+1}}, \dots,$

$y_{\sigma, n-1} = y_{\sigma', 1}^{\lambda_{n-1, 1}} \dots y_{\sigma', n-1}^{\lambda_{n-1, n-1}} y_{\sigma', n+1}^{\lambda_{n-1, n+1}} \dots$. Thus the union of

$y_{\sigma', n+1}^{\pm 1} = \xi_i$ for each i make a connected component.

Remark (4.9). The above components are birationally equivalent to P^{n-1} .

§5. Topology of the exceptional divisors.

Let $g(u_1, \dots, u_n)$ be a polynomial with support $S(g)$. We say that g is globally non-degenerate (= 0-non-degenerate in [19]) if

$$g_{\Delta}(u) = \frac{\partial g_{\Delta}}{\partial u_1}(u) = \dots = \frac{\partial g_{\Delta}}{\partial u_n}(u) = 0 \text{ has no solution in } (\mathbb{C}^*)^n \text{ for}$$

any face Δ of $S(g)$. ([15]). In [15], we have proved

Theorem (5.1). Let g be a globally non-degenerate polynomial. Then

- (i) $\chi((\mathbb{C}^*)^n - g^{-1}(0)) = (-1)^n n!$ n -dim. volume $S(g)$
- (ii) If $\dim S(g) \geq 3$, $\pi_1((\mathbb{C}^*)^n - g^{-1}(0))$ is a free abelian group of rank $n + 1$.

By the additivity of the Euler characteristics and (i), we get

Corollary (5.2). ([6], [19]). Let g be as above and let $V^* = g^{-1}(0) \cap (\mathbb{C}^*)^n$. Then $\chi(V^*) = (-1)^{n+1} n! \text{ n-dim. volume } S(g)$.

In this section, we study the topology of exceptional divisors of $\pi: \tilde{V} \rightarrow V$ which is constructed in §4. Let $\sigma = (P_1, \dots, P_{k+1})$ be an (unordered) k -simplex of Σ^* . We define $E(\sigma) = \bigcap_i E(P_i)$ and $E(\sigma)^* = E(\sigma) - \bigcup \{E(Q); Q \neq P_1, \dots, P_{k+1}\}$. We also use the notation $E(\sigma) = E(P_1, \dots, P_{k+1})$ etc. Define $\Delta(\sigma)$ by the intersection of $\Delta(P_i)$. For a fixed strictly positive vertex P , the collection of $E(P, P_1, \dots, P_k)^*$ gives a canonical stratification of $E(P)$ where P and $P_1, \dots, P_k$ are adjacent. Let Δ be a face of $\Delta(P)$ of dimension $(n - k)$ and let $a(\Delta; P)$ be the number of k -simplexes $\sigma = (P, P_1, \dots, P_k)$ of Σ^* on the closure $(\Delta)^*$.

The main result of this section is

Theorem (5.3). (i) Assume that $\sigma = (P, P_1, \dots, P_k)$ be a k -simplex of Σ^* . The Euler characteristic $\chi(E(\sigma)^*)$ is non-zero if and only if $\dim \Delta(\sigma) = n - k$. Let $\tau = (Q_1, \dots, Q_{n+1})$ be an n -simplex such that $\{P, P_1, \dots, P_k\} = \{Q_1, \dots, Q_{k+1}\}$. Then $\chi(E(\sigma)^*) = (-1)^{n-k+1} (n-k)! \text{ volume } S(g_{\Delta(\sigma)}(y_{\tau, k+2}, \dots, y_{\tau, n+1}))$.
 (ii) The isomorphism class of $E(\sigma)^*$ depends only on $\Delta(\sigma)$ if $\dim \Delta(\sigma) = n-k$. It does not depend on the particular choice of Σ^* either.
 (iii) Let Δ be a $(n-k)$ -dimensional face of $\Delta(P)$ and let $\chi(\Delta; P)$ be $\chi(E(\sigma)^*)$ where σ is a k -simplex on the closure $(\Delta)^*$ with $P \in \sigma$. Then

$$\chi(E(P)) = \sum a(\Delta; P) \chi(\Delta; P) \text{ where the sum is taken for every face } \Delta \text{ of } \Delta(P).$$

Corollary (5.4). (i) The Euler characteristic of the top dimensional stratum $E(P)^*$ is $(-1)^{n+1} (n+1)\text{-dim. volume } C(0, \Delta(P))$ where $C(0, \Delta(P))$ is the cone of $\Delta(P)$ with the origin. $\left(\frac{\Delta}{d(P)} \right)$

(ii) The birational class of the exceptional divisor $E(P)$ depends only on $\Delta(P)$. If $\dim \Delta(P) = r < n$, there exists a compact complex manifold $M(P)$ of dimension $r - 1$ such that $E(P)$ is birationally equivalent to $P^{n-r} \times M(P)$.

The proof of Theorem (5.3) and Corollary (5.4) occupies the rest of of this section.

Let $\sigma = (P_1, \dots, P_{n+1})$ be a simplex with $\Delta(P_1) \supset \dots \supset \Delta(P_{n+1})$. By the definition, $\overline{E(P_{i_1}, \dots, P_{i_k})}^* \subset \mathbb{C}_\sigma^{n+1}$ and $E(P_{i_1}, \dots, P_{i_k})^*$ is defined by $y_{\sigma, i_1} = \dots = y_{\sigma, i_k} = 0$ and $g_{\Delta(P_{i_k})}(y_{\sigma, i_{k+1}}, \dots, y_{\sigma, n+1}) = 0$ with $y_{\sigma, j} \neq 0$ for $j \neq i_1, \dots, i_k$. Here we have assumed that $i_1 < i_2 < \dots < i_k$.

The polynomial $g_{\Delta(P_{i_k})}$ is defined by $f_{\Delta(P_{i_k})}(\pi_\sigma(y_\sigma)) = \left\{ \prod_{i=1}^{n+1} y_{\sigma, i}^{d(P_i)} \right\} g_{\Delta(P_{i_k})}$.

Thus it is easy to see that $g_{\Delta(P_{i_k})}$ is globally non-degenerate because f is

non-degenerate on $\Delta(P_{i_k})$. (Compare with Lemma (5.2) of [15]). As

$\dim S(g_{\Delta(P_{i_k})}) = \dim \Delta(P_{i_k})$, the assertion (i) of Theorem (5.3) is immediate

from Corollary (5.2). Assume that $P = P_1$ and $\dim \Delta(P) = n$. Then $E(P)^*$

is defined by $g_{\Delta(P)}(y_{\sigma, 2}, \dots, y_{\sigma, n+1}) = 0$ where $f_{\Delta(P)}(\pi_\sigma(y_\sigma)) =$

$y_{\sigma, 1}^{d(P_1)} \dots y_{\sigma, n+1}^{d(P_{n+1})} g_{\Delta(P)}(y_{\sigma, 2}, \dots, y_{\sigma, n+1})$. Thus we have the equalities

$$(n+1)! \text{ volume } C(0, \Delta(P)) = (n+1)! \text{ volume } C(0, S(\pi_\sigma^* f_{\Delta(P)}))$$

$$= (n+1)! \text{ volume } S(g_{\Delta(P)}) \cdot \frac{d(P)}{n+1}$$

$$= n! \text{ volume } S(g_{\Delta(P)}) \cdot d(P).$$

This proves the assertion (i) of Corollary (5.4).

Let Σ^* be another simplicial subdivision of $\Gamma^*(f)$ and let $\pi' : \tilde{V}' \rightarrow V$ be the associated resolution. We denote the exceptional divisors in $\pi' : \tilde{V}' \rightarrow V$ by $E'(P)$, $E'(\sigma)$ etc. Let $\sigma = (P_1, \dots, P_{n+1})$ be a simplex of Σ^* and let $\sigma' = (P'_1, \dots, P'_{n+1})$ be a simplex of $\Sigma^{*'}.$ We assume that

$$(5.5). \quad \begin{cases} \Delta(P_1) > \dots > \Delta(P_{n+1}) \\ \Delta(P'_1) > \dots > \Delta(P'_{n+1}) \end{cases} \quad \text{and} \quad \Delta(P_k) = \Delta(P'_k) \\ \text{and} \quad \dim \Delta(P_k) = n + 1 - k \quad \text{for some } k.$$

$E(P_1, \dots, P_k)$ is defined in $\mathbb{C}_{\sigma}^{n+1}$ by

$$y_{\sigma,1} = \dots = y_{\sigma,k} = 0 \quad \text{and} \quad g_{\Delta_k}(y_{\sigma,k+1}, \dots, y_{\sigma,n+1}) = 0$$

where $\Delta_k = \Delta(P_k)$ and $f_{\Delta_k}(\pi_{\sigma}(y_{\sigma})) = \prod y_{\sigma,i}^{d(P_i)} \cdot g_{\Delta_k}(y_{\sigma,k+1}, \dots, y_{\sigma,n+1})$.

$E'(P'_1, \dots, P'_k)$ is defined in $\mathbb{C}_{\sigma'}^{n+1}$ by

$$y_{\sigma',1} = \dots = y_{\sigma',k} = 0 \quad \text{and} \quad g'_{\Delta_k}(y_{\sigma',k+1}, \dots, y_{\sigma',n+1}) = 0$$

where $f_{\Delta_k}(\pi_{\sigma'}(y_{\sigma'})) = \prod y_{\sigma',i}^{d(P'_i)} g'_{\Delta_k}(y_{\sigma',k+1}, \dots, y_{\sigma',n+1})$.

By the assumption that $\Delta(P_k) = \Delta(P'_k)$ and $\dim \Delta(P_k) = n + 1 - k$, $\mathbb{Z}$ -submodules generated by $P_1, \dots, P_k$ and $P'_1, \dots, P'_k$ are equal and they are equal to the submodules generated by the integral points of $\overline{\text{closure}(\Delta_k^*)}$. Therefore the matrix $\sigma^{-1} \cdot \sigma'$ can be written as

$$\Lambda = \begin{pmatrix} \overset{k}{\Lambda_1} & \Lambda_3 \\ \hline 0 & \Lambda_2 \end{pmatrix}$$

where $\Lambda_i (i=1,2)$ are unimodular. Write $\mathbb{C}_\sigma^{n+1}$ as $\mathbb{C}_\sigma^k \times \mathbb{C}_\sigma^{n+1-k}$ and $y_\sigma = (y_1, y_2)$ where $y_1 \in \mathbb{C}_\sigma^k$ and $y_2 \in \mathbb{C}_\sigma^{n+1-k}$. y_2 is the coordinate of the subspace $\mathbb{C}_\sigma^{n+1-k}$ where $E(P_1, \dots, P_k)$ are defined by $g_{\Delta_k}(y_2) = 0$.

Respectively we write $\mathbb{C}_{\sigma'}^{n+1}$ as $\mathbb{C}_{\sigma'}^k \times \mathbb{C}_{\sigma'}^{n+1-k}$ and $y_{\sigma'} = (y'_1, y'_2)$. Now we define a birational mapping $\pi_{\Lambda_2} : \mathbb{C}_{\sigma'}^{n+1-k} \rightarrow \mathbb{C}_\sigma^{n+1-k}$ by $y_2 = (y'_2)^{\Lambda_2}$.

We will show that π_{Λ_2} induces a desired biholomorphic mapping from $E'(P'_1, \dots, P'_k)^*$

to $E(P_1, \dots, P_k)^*$. By the definition of Λ , we have $y_\sigma = (y_{\sigma'})^{\Lambda}$. In

particular, $y_2 = (y'_2)^{\Lambda_2}$. Thus we have $f_{\Delta_k}(y_\sigma^{\Lambda}) = f_{\Delta_k}(\pi_{\sigma'}(y_{\sigma'}))$ which

implies that

$$g_{\Delta_k}(y_2^{\Lambda_2}) = g'_{\Delta_k}(y'_2) \cdot \prod_{i=k+1}^{n+1} (y_{\sigma',i})^{\alpha_i} \text{ for some } \alpha_{k+1}, \dots, \alpha_{n+1} \in \mathbb{Z}.$$

The last equality says that $\pi_{\Lambda_2}(E'(P'_1, \dots, P'_k)^*) = E(P_1, \dots, P_k)^*$. This

completes the proof of the assertion (ii) of Theorem (5.3). The assertion (iii) of Theorem (5.3) is immediate from (ii) and the additivity of the Euler characteristics.

To prove (ii) of Corollary (5.4), we take a strictly positive vertex P

such that $\dim \Delta(P) = r$ and $0 < r < n$. Let $\sigma = (P_1, \dots, P_{n+1})$ be a simplex with $P_{n-r+1} = P$ and $\Delta(P_1) \supset \dots \supset \Delta(P_{n+1})$. $E(P, \sigma)$ is the

product $\mathbb{C}_\sigma^{n-r} \times \{E(P_1, \dots, P_{n-r+1}) \cap \mathbb{C}_\sigma^{n+1}\}$. Thus we can take

$E(P_1, \dots, P_{n-r+1})$ as $M(P)$. The isomorphism $E(P; \sigma) \rightarrow \mathbb{C}_\sigma^{n-r} \times (E(P_1, \dots, P_{n-r+1}) \cap$

$\mathbb{C}_\sigma^{n+1})$ extends to a desired birational mapping. This completes the proof.

Corollary (5.6). Let P be a strictly positive vertex with $\dim \Delta(P) = 1$.

Then each component is birationally equivalent to P^{n-1} and the Euler

characteristic is equal to a_n = the number of $(n-1)$ -simplexes $\Delta = (P_1, \dots, P_n)$ such that $P_n = P$ and $\Delta(P_1) > \dots > \Delta(P_n)$.

Remark (5.7). Two simplexes with the same vertices are counted once.

§6. The surface singularity.

In this section, we study the case $n = 2$ in detail. Let $\pi: \widetilde{V} \rightarrow V$ be a good resolution of V which is constructed in §4. Let E_i ($i=1, \dots, k$) be the irreducible components of $\pi^{-1}(0)$. The resolution graph Γ is defined in the following way. For any E_i , we associate a vertex v_i with weight m_i which is the self-intersection number of E_i . When E_i is not a rational curve, we also put the genus g_i of E_i to v_i . If $E_i \cap E_j \neq \emptyset$, v_i and v_j are joined by a line segment. A lot of information about the resolution is derived from this weighted graph Γ . ([13], [4], [3], [7].)

The 2-skeleton $S_2 \Gamma^*(f)$ can be expressed by a graph through a hyperplane section. (See Figure (2.1) in §2.) We will show that Γ is obtained from $S_2 \Gamma^*(f)$ by a canonical surgery. For a 2-dimensional face Δ of $\Gamma(f)$, we define an integer $g(\Delta)$ by the number of the integral points on $\Delta - \partial\Delta$. Similarly for a 1-dimensional face Ξ of $\Gamma(f)$, we define $r(\Xi)$ by the number of integral points on $\Xi - \partial\Xi$. For any line segment $\overline{PQ}$ of $\Gamma^*(f)$, $r(P, Q)$ is defined by $r(\Delta)$ where Δ^* is the interior of $\overline{PQ}$. Our main result in this section is

Theorem (6.1). Let $\pi: \widetilde{V} \rightarrow V$ be the resolution associated with Σ^* .

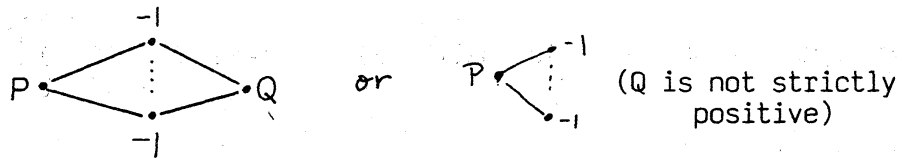
Then we have

- (i) If P is strictly positive and $\dim \Delta(P) = 2$, $E(P)$ has genus $g(\Delta(P))$.
- (ii) If P is strictly positive and $\dim \Delta(P) = 1$, $E(P)$ is $r(\Delta(P)) + 1$ disjoint copies of rational curves.

Assume that Σ^* is canonical (in the sense of (3.5)). Then

- (iii) the resolution graph is obtained by the following surgery of $S_2(\Gamma^*(f))$.

Let $\overline{PQ}$ be a line segment of $S_2 \Gamma^*(f)$ and assume that P is strictly positive. Let $c = \det(P, Q)$. If $c = 1$, we put $r(P, Q) + 1$ copies of rational curves with self-intersection -1 between P and Q as follows:



If $c > 1$, let c_1 be the unique integer such that $P_1 = \frac{1}{c} Q + \frac{c_1}{c} P$ is integral and $0 < c_1 < c$. Let $\frac{c}{c_1} = [m_1, \dots, m_k]$. We insert

$r(P, q) + 1$ copies of a chain of rational curves. $\begin{array}{c} -m_1 \quad -m_2 \quad \dots \quad -m_k \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \end{array}$ between P and Q . Those vertices which are not strictly positive are omitted after the surgery from the resolution graph.

(iv) Let P be a strictly positive vertex with $\dim \Delta(P) = 2$. Let

$Q_1, \dots, Q_s$ be the vertices of Σ^* which are adjacent to P and

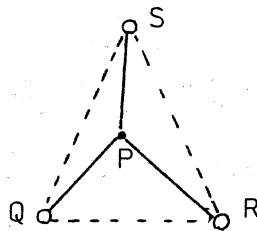
$\dim \Delta(Q_i) = 1$. Let $P = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$ and $Q_i = \begin{pmatrix} q_{1i} \\ q_{2i} \\ q_{3i} \end{pmatrix}$. Then the self-intersection number of $E(P)$ is

$$- \frac{\sum_{i=1}^s (r(P, Q_i) + 1) q_{1,i}}{p_1}$$

Before the proof of Theorem (6.1), we explain the theorem by two examples.

Example (6.2). Let $f(z_1, z_2, z_3) = z_1^2 + z_2^3 + z_3^5$. Then

$S_2 \Gamma^*(f)$ is as follows. ($\circ$ implies "not strictly positive.")



$$P = {}^t(15, 10, 6), \quad Q = {}^t(1, 0, 0) \\ R = {}^t(0, 1, 0), \quad S = {}^t(0, 0, 1)$$

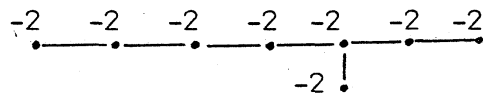
It is easy to see that $g(\Delta(P)) = 0$. $\det(P, Q) = 2$.

$$Q' = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 15 \\ 10 \\ 6 \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \\ 3 \end{pmatrix}. \text{ As } \det(P, R) = 3 \text{ and } R' = \frac{1}{3} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{2}{3} \begin{pmatrix} 15 \\ 10 \\ 6 \end{pmatrix} = \begin{pmatrix} 10 \\ 7 \\ 4 \end{pmatrix}$$

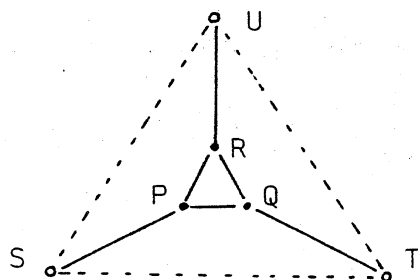
$$\text{we have } \frac{3}{2} = 2 - \frac{1}{2}. \text{ As } \det(P, S) = 5 \text{ and } S' = \frac{1}{5} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \frac{4}{5} \begin{pmatrix} 15 \\ 10 \\ 6 \end{pmatrix} = \begin{pmatrix} 12 \\ 8 \\ 5 \end{pmatrix},$$

$$\text{we have } \frac{5}{4} = [2, 2, 2, 2]. \text{ As } r(P, Q) = r(P, R) = r(P, S) = 0,$$

$$E_P^2 = -\frac{8 + 10 + 12}{15} = -2. \text{ Thus the resolution graph is}$$



Example (6.3). Let $f(z_1, z_2, z_3) = z_1^3 + z_2^4 + z_3^4 + z_1 z_2 z_3$. $\Gamma(f)$ has three 2-dimensional faces with genus 0. $S_2 \Gamma^*(f)$ is:



$$P = {}^t(2, 1, 1), S = {}^t(1, 0, 0)$$

$$Q = {}^t(4, 5, 3), T = {}^t(0, 1, 0)$$

$$R = {}^t(4, 3, 5), U = {}^t(0, 0, 1)$$

(1) $\det(P, S) = 1$, $r(P, S) = 3$. Thus we add 4 vertices of self-intersection number -1 to P.

(2) $\det(P, Q) = 2$, $r(P, Q) = 0$. As $\frac{1}{2}P + \frac{1}{2}Q = \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix}$, $c_1 = 1$. Thus we

add $\overset{-2}{\bullet}$ between P and Q.

(3) $\det(Q, T) = 1$, $r(Q, T) = 0$. Thus we add $\overset{-1}{\bullet}$ to Q.

(4) $\det(Q, R) = 8$, $r(Q, R) = 0$. As $\frac{1}{8}Q + \frac{1}{8}R = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, we add $\overset{-8}{\bullet}$

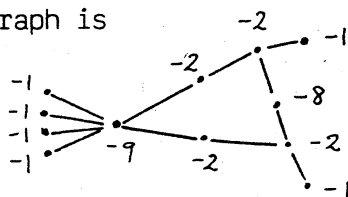
between Q and R. $\overline{PR}$ and $\overline{RU}$ are similar with $\overline{PQ}$ and $\overline{QT}$.

$$E(P)^2 = -\frac{4 \cdot 3 + 3 + 3}{2} = -9$$

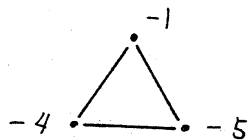
$$E(Q)^2 = -\frac{3 + 4 + 1}{4} = -2$$

$$E(R)^2 = -\frac{1 + 3 + 4}{4} = -2$$

Thus the resolution graph is

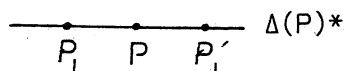


which is blown down to



Now we are ready to prove Theorem (6.1). Let P be a strictly positive vertex of Σ^* with $\dim \Delta(P) = 1$. Then P is on the line segment $\Delta(P)^*$.

Thus there are exactly two vertices P_1, P'_1 on the closure of $\Delta(P)^*$ which are adjacent to P .



By Corollary (5.6), we have $\chi(E(P)) = (r(\Delta(P)) + 1) \cdot 2$. As $E(P)$ has $r(\Delta(P) + 1)$ components which are birationally equivalent, each component must be a rational curve P^1 . We will see this directly.

Let $\sigma = (P_1, P, P_2)$ and $\sigma' = (P'_1, P, P'_2)$ be two simplexes where P_2 and P'_2 are taken on the same side with respect to $\Delta(P)^*$ for brevity's

sake. In $\mathbb{C}_\sigma^3$, $E(P; \sigma)$ is defined by $y_{\sigma,2} = 0$ and $y_{\sigma,3} = \xi_i$ ($i=1, \dots, r(\Delta(P)) + 1$). In $\mathbb{C}_{\sigma'}^3$, $E(P; \sigma')$ is defined by $y_{\sigma',2} = 0$ and $y_{\sigma',3} = \xi_i$. They are glued by the matrix

$$\sigma^{-1} \sigma' = \begin{pmatrix} -1 & 0 & b \\ a & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

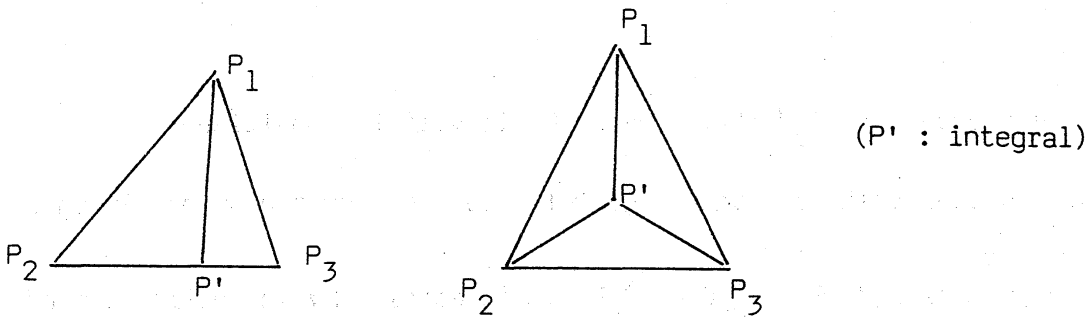
Thus $y_{\sigma,3} = y_{\sigma',3}$ and $y_{\sigma,1} = y_{\sigma',1}^{-1} \cdot y_{\sigma',3}^b$. Namely on each component $y_{\sigma,3} = y_{\sigma',3} = \xi_i$, $y_{\sigma,1} = y_{\sigma',1}^{-1} \cdot \xi_i^b$. Thus it is a rational curve. This proves (ii) of Theorem (6.1).

To prove the assertion (i) of the theorem, we need

Lemma (6.4). Let Δ be a compact convex polyhedron in $\mathbb{R}^2$ with integral vertices $P_1, \dots, P_k$. Then $2 \text{ volume } \Delta = 2 g(\Delta) + \sum_{i=1}^k (r(\Delta_i) + 1) - 2$, where $\partial \Delta = \Delta_1 \cup \Delta_2 \dots \cup \Delta_k$.

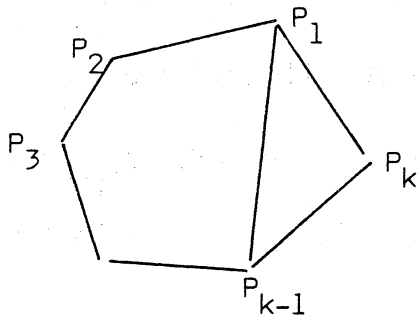
Proof. Step 1. Assume that $k = 3$, $g(\Delta) = r(\Delta_1) = r(\Delta_2) = r(\Delta_3) = 0$. By a parallel translation if necessary, we may assume that $P_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Then P_2 and P_3 are primitive vectors and $2 \cdot \text{volume } \Delta = \det(P_2, P_3)$. Assume that $c = \det(P_2, P_3) > 1$. By Lemma (3.3), we can find $0 < c_1 < c$ such that $Q = \frac{1}{c} P_2 + \frac{c_1}{c} P_1$ is integral. However this is a contradiction to the assumption $g(\Delta) = r(\Delta_1) = 0$ as $Q \in \Delta - \{P_1, P_2, P_3\}$. Thus the assertion is true in this case.

Step 2. $k = 3$. Assume that $r(\Delta_1) \neq 0$ or $g(\Delta) \neq 0$. Then we divide Δ into two or three triangles respectively.



It is easy to see that both sides of the assertion are additive under the above divisions. Therefore this case is reduced to Step 1 by a finite subdivision.

Step 3. The assertion for general $k \geq 3$ is proved by the induction on k and the additivity of the assertion under the following division:



Let P be a strictly positive vertex of Σ^* such that $\dim \Delta(P) = 2$.

Let $\Delta_1, \dots, \Delta_k$ be the boundaries of $\Delta(P)$. Then P has k -branches in

$\Gamma^*(f)$ which correspond bijectively to Δ_i ($i=1, \dots, k$). Each

branch has a unique vertex P_i which is adjacent to P in Σ^* . Let

$\sigma = (P, P_1, P_1')$ be a simplex. In $\mathbb{C}_6^3$, $E(P; \sigma)$ is defined by $y_{\sigma,1} = 0$,

$g_{\Delta(P)}(y_{\sigma,2}, y_{\sigma,3}) = 0$. By Theorem (5.3), we have $\chi(E(P)) = -2 \text{ volume}(S(g_{\Delta(P)})) +$

$\sum_{i=1}^k (r(\Delta_i) + 1)$. We use the equality $g(\Delta(P)) = g(S(g_{\Delta(P)}))$ which is

immediate from the fact that σ is a unimodular matrix. Thus by Lemma (6.4),

we get

$$\chi(E(P)) = -2 g(\Delta(P)) + 2$$

which implies that $g(\Delta(P))$ is the genus of $E(P)$.

Now we assume that Σ^* is a canonical simplicial subdivision. Let

$\overline{PQ}$ be a line segment of $S_2 \Gamma^*(f)$ where P is strictly positive.

Case 1. Assume that $c = \det(P, Q) > 1$. Let c_1 be the unique integer such that $0 < c_1 < c$ and $P_1 = \frac{1}{c} Q + \frac{c_1}{c} P$ is integral. By the definition of Σ^* , Σ^* has k vertices $P_1, \dots, P_k$ on $\overline{PQ}$ which are inductively defined by

$$P_{i+1} = \frac{1}{c_i} Q + \frac{c_{i+1}}{c_i} P_i \quad (i=0, \dots, k-1)$$

where $P_0 = P$, $0 < c_{i+1} < c_i$ for $i = 0, \dots, k-1$ and $c_0 = c$ and

$c_k = 1$. Observe that $P_1, \dots, P_k$ are strictly positive. Let $\sigma_i = (P_i, P_{i+1}, R_i)$

be a simplex of Σ^* . In $\mathbb{C}_{\sigma_i}^3$, $E(P_i)$ is defined by $y_{\sigma_i,1} = 0$ and

$y_{\sigma_i,3} = \xi_j$ ($j = 1, \dots, r(P, Q) + 1$) and $E(P_{i+1})$ is defined by $y_{\sigma_i,2} = 0$

$y_{\sigma_i,3} = \xi_j$ ($j = 1, \dots, r(P, Q) + 1$). Thus each component of $E(P_i)$ has a non-empty intersection with only one component of $E(P_{i+1})$. We note the components

of $E(P_i)$ by $E_{i,j}$ ($j = 1, \dots, r(P, Q) + 1$). We may assume that

$E_{i,j} \cdot E_{i,k} = 0$ ($j \neq k$) and $E_{i,j} \cdot E_{i+1,j} = E_{i,j} \cdot E_{i-1,j} = 1$, $E_{i,j} \cdot E_{i+1,k} =$

$E_{i,j} \cdot E_{i-1,k} = 0$ ($j \neq k$). To compute $E_{i,j}^2$, we consider the divisor of holomorphic

function $\pi^* z_1$. In $\mathbb{C}_{\sigma_i}^3$, $\pi^* z_1 =$

$$y_{\sigma_i,1}^{p_{1,i}} y_{\sigma_i,2}^{p_{1,i+1}} y_{\sigma_i,3}^{r_{1,i}} \quad \text{where} \quad P_i = \begin{pmatrix} p_{1,i} \\ p_{2,i} \\ p_{3,i} \end{pmatrix} \quad \text{and} \quad R_i = \begin{pmatrix} r_{1,i} \\ r_{2,i} \\ r_{3,i} \end{pmatrix}.$$

Thus $(\pi^* z_1) = \sum_{i=0}^{k+1} p_{1i} E(P_i) + D$ where D has no intersection with any $E(P_i)$

($i = 1, \dots, k$). By Theorem 2.6 of [8], $(\pi^* z_1) \cdot E_{ij} = 0$. Thus we get

$$-E_{ij}^2 = \frac{p_{1,i-1} + p_{1,i+1}}{p_{1i}} = m_i \quad \text{by Lemma (3.6) where } \frac{c}{c_1} = [m_1, \dots, m_k].$$

Case 2. Assume that $c = \det(P, Q) = 1$. By the definition of the canonical primitive sequence, Σ^* has a unique vertex $P_1 = P + Q$ on $\overline{PQ}$. Let $E_1, \dots, E_{r(P,Q)+1}$ be the components of $E(P_1)$. Again by Theorem (2.6) of [8], we get $E_i^2 = -1$. This proves the assertion (iii) of Theorem (6.1).

The assertion (iv) is proved by the same argument. This completes the proof of Theorem (6.1).

In practice, the following is useful for the computation of $g(\Delta)$.

Corollary (6.5). Let P be a strictly positive vertex of Σ^* with $\dim \Delta(P) = 2$. Then $2 - 2g(\Delta(P)) = -\frac{6}{d(P)} \text{ volume } c(0, \Delta(P)) + \sum_{i=1}^k (r(\Delta_i) + 1)$

where $\partial \Delta(P) = \Delta_1 \cup \dots \cup \Delta_k$.

Now we study the resolutions of several interesting singularities.

(A). Weighted homogeneous polynomials.

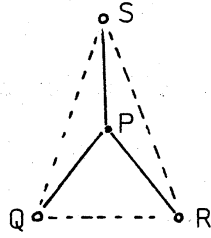
The surface singularities with $\mathbb{C}^*$ -action were studied by Orlik-Wagreich in [17]. Our method gives an alternative description of the continuous fractions which were derived in [17] from the orbit invariants of the $\mathbb{C}^*$ -action.

(I). Pham-Brieskorn variety.

Let $f_a(z) = z_1^{a_1} + z_2^{a_2} + z_3^{a_3}$ where $a_i \geq 2$. Let $s = \text{g.c.d.}(a_1, a_2, a_3)$

and let $ss_i = \text{g.c.d.}(a_{i-1}, a_{i+1})$ modulo 3. Then s_1, s_2, s_3 are mutually coprime and we can write $a_i = s s_{i-1} s_{i+1} a'_i$ modulo 3 for $i=1, 2, 3$ where

a'_1, a'_2, a'_3 are mutually coprime. The dual Newton diagram $\Gamma^*(f)$ is



$$P = \begin{pmatrix} s_1 a'_1 a'_3 \\ s_2 a'_1 a'_3 \\ s_3 a'_1 a'_2 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$S = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Surgery data are

	c	c_1	$r+1$
$\overline{PQ}$	$c_Q = a'_1$	$c_{Q,1} P_1 + 1 \equiv 0 \pmod{c_Q}$	ss_1
$\overline{PR}$	$c_R = a'_2$	$c_{R,1} P_2 + 1 \equiv 0 \pmod{c_R}$	ss_2
$\overline{PS}$	$c_S = a'_3$	$c_{S,1} P_3 + 1 \equiv 0 \pmod{c_S}$	ss_3

$$g(E(P)) = \frac{1}{2} s \{ ss_1 s_2 s_3 - (s_1 + s_2 + s_3) \} + 1.$$

We assume that $s_1 \leq s_2 \leq s_3$. Then it is easy to see that $g(E(P)) = 0$

if and only if

$$(\#_1) \quad s = s_1 = s_2 = 1 \quad \text{or}$$

$$(\#_2) \quad s = 2, s_1 = s_2 = s_3 = 1.$$

Note that $(\#_1)$ is equivalent to $\text{g.c.d.}(a_3, a_1) = \text{g.c.d.}(a_3, a_2) = 1$ and $(\#_2)$

is equivalent to $\text{g.c.d.}(a_i, a_j) = 2$ for any $i \neq j$.

Let $K_a = f_a^{-1}(0) \cap S^5$. K_a is diffeomorphic to the boundary of a tubular neighborhood of the total exceptional divisors. $g(E(P)) = 0$ is equivalent to $H_*(K_a; \mathbb{Q}) = H_*(S^3; \mathbb{Q})$ ([4]).

As a special case of (i), $H_*(K_a; \mathbb{Z}) = H_*(S^3; \mathbb{Z})$ if and only if $(\#_1)$, $(a_i, a_j) = 1$ for $i \neq j$ by [2].

In general, different polynomials may have the same resolution graph. ([1], [7]). For example, Pham-Brieskorn polynomials $x^2 + y^{57} + z^{114}$, $x^3 + y^{29} + z^{87}$, $x^8 + y^9 + z^{72}$ have the same minimal resolution: $\textcircled{28}$ (28 is the genus.)
-1

However we have

Theorem (6.6). Assume that K_a is a $\mathbb{Q}$ -homology sphere. Then

- (i) There is no other polynomial f_b which has the same resolution graph as f_a .
- (ii) The fundamental group $\pi_1(K_a)$ is ^(either) a cyclic group if $a = (2, 2, a_3)$, or a central extension of a Dyck (or a triangle) group.

Proof. If $a = (2, 2, a_3)$, the resolution is A_{a_3-1} . Thus we can assume that $a \neq (2, 2, a_3)$ i.e. f_a is not of A_k type. We assume that $a'_1 \leq a'_2 \leq a'_3$. First observe that the intersection number $E(P)^2$ (after blowing down exceptional divisors of the first kind) is equal to

$$(*) - \left\{ \frac{c_{Q,1}}{c_Q} ss_1 + \frac{c_{R,1}}{c_R} ss_2 + \frac{c_{S,1}}{c_S} ss_3 + \frac{s}{a'_1 a'_2 a'_3} \right\}.$$

(Here $c_{Q,1} = 0$ if $c_Q = 1$.)

Case 1. $a'_1 > 1$. Then the resolution graph has three distinct families

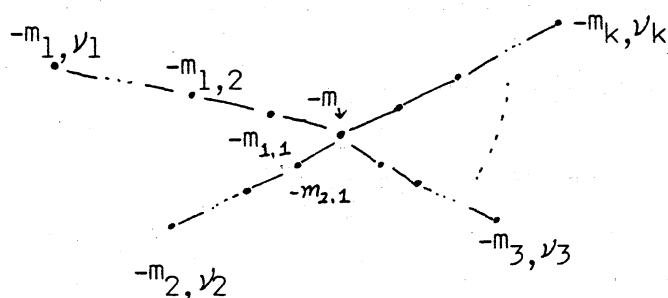
of branches which correspond to the continuous fractions $c_Q/c_{Q,1}$, $c_R/c_{R,1}$, $c_S/c_{S,1}$. Thus the resolution graph determines a_1' , a_2' , a_3' , ss_1 , ss_2 , ss_3 . Therefore $s = \text{g.c.d.}(ss_1, ss_2, ss_3)$ and s_i are also determined. Thus $a = (a_1, a_2, a_3)$ is also unique.

Case 2. Assume that $a_1' = 1$ and $a_2' > 1$. Then $s = s_2 = s_3 = 1$ is impossible as $a_1 > 1$. Thus the graph has $s(s_2 + s_3) \geq 3$ branches and $E(P)$ can not be blown down. By the graph a_2' , a_3' , ss_2 , ss_3 are determined. Thus $s = \text{g.c.d.}(ss_2, ss_3)$ is also determined. By $(\#_1)$ and $(\#_2)$, $s_1 = 1$ and a is also recovered by these data.

Case 3. Assume that $a_1' = a_2' = 1$ and $a_3' > 1$. As $a \neq (2, 2, a_3)$ by the assumption, we must have $s = s_1 = s_2 = 1$ by $(\#_1)$ and $(\#_2)$. Thus $a_1 = s_3$ which implies $s_3 > 2$. The graph has s_3 branches at $E(P)$ and $E(P)$ can not be blown down. Thus a_3' and s_3 are determined from the graph, which makes $a = \overbrace{(a_1, a_2, a_3)}$ to be unique.

The case that $a_i' = 1$ for $i=1, \dots, 3$ are impossible from the assumption and $(\#_1)$ and $(\#_2)$. The assertion (ii) is immediate from

Lemma (6.7). Let Γ be a star of rational curves in a complex surface M



where $k \geq 3$ and $-m_{i,j} \leq -2$ and $-m \overset{(\leq -1)}{\text{are}}$ self-intersection numbers. Let $\partial \mathcal{N}$ be the boundary of a sufficiently small tubular neighborhood of

Γ in M . Let $\frac{c_i}{c'_i} = [m_{i,1}, \dots, m_{i,\nu_i}]$. Then $\pi_1(\partial N)$ is a central extension of a Dyck group $\langle x_1, \dots, x_k; x_1^{c_1} = \dots = x_k^{c_k} = x_1 x_2 \dots x_k = e \rangle$.

Proof. By [13] or [4], $\pi_1(\partial N)$ can be represented by $\langle x, x_{ij}$
 $i = 1, \dots, k, 1 \leq j \leq \nu_i; R_1: x_{i,j-1} x_{ij}^{-m_{ij}} x_{i,j+1} = e, i = 1, \dots, k,$
 $j = 1, \dots, \nu_i$ and $R_2: x_{11} x_{21} \dots x_{k,1} x^{-m} = e \rangle$ where $x_{i,0} = x$ and
 $x_{i,\nu_i+1} = e$. By R_1 , we can cancel generators x_{ij} ($1 \leq j \leq \nu_i - 1$) so that
 $\pi_1(\partial N) \cong \langle x_1, \dots, x_k, x; x = x_1^{c_1} = \dots = x_k^{c_k}, x^m = x_1^{c'_1} x_2^{c'_2} \dots x_k^{c'_k} \rangle$.

Here $x_i = x_{i,\nu_i}$. Clearly x is an element of the center of $\pi_1(\partial N)$

and $\pi_1(\partial N)/\langle x \rangle$ is isomorphic to

$$\langle x_1, \dots, x_k; x_1^{c_1} = \dots = x_k^{c_k} = e, x_1^{c'_1} \dots x_k^{c'_k} = e \rangle.$$

As c_i and c'_i are coprime, we can take $y_i = x_j^{c'_i}$ as generators. Thus

$\pi_1(\partial N)/\langle x \rangle$ is isomorphic to the Dyck group

$$\langle y_1 \dots y_k; y_1^{c_1} = \dots = y_k^{c_k} = y_1 \dots y_k = e \rangle.$$

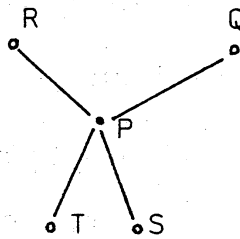
For the other geometry of K_a , see [17] and [11].

(II). Let $f = z_1^{a_1} + z_1(z_2^{a_2} + z_3^{a_3}) + bz_2^s z_3^t$ where b is a generic constant and $s > 0, t > 0$. We assume that f is a weighted homogeneous by the weights $P = (p_1, p_2, p_3)$. Thus we have

$$(*) \quad p_1 a_1 = p_1 + p_2 a_2 = p_1 + p_3 a_3 = s p_2 + t p_3.$$

This class of weighted homogeneous polynomials is omitted in the classification

table of Orlik-Wagreich [17] and this class is special in the sense that the corresponding S^1 -action on $K = f^{-1}(0) \cap S^5$ has four singular orbit types. (For the other classes, there are three orbit types.) In fact, the dual Newton diagram is



$$\begin{aligned} T &= {}^t(t, 0, 1) \\ S &= {}^t(s, 1, 0) \\ Q &= {}^t(0, 1, 0), R = {}^t(0, 0, 1) \end{aligned}$$

Surgery data are

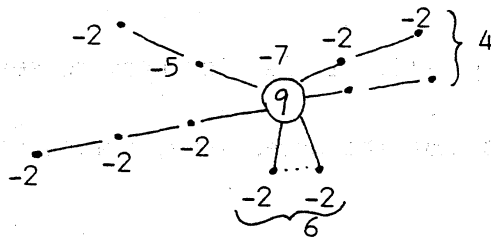
	c	c_1	$r + 1$
PQ	(p_1, p_3)	$1 + c_1 p_2 \equiv 0 \pmod{c}$	$\frac{a_3 c}{p_1}$
PR	(p_1, p_2)	$1 + c_1 p_3 \equiv 0 \pmod{c}$	$\frac{a_2 c}{p_1}$
PT	p_2	$1 + c_1 p_3 \equiv 0 \pmod{c}$	1
PS	p_3	$1 + c_1 p_2 \equiv 0 \pmod{c}$	1

$$g(E(P)) = \frac{a_1 a_2 a_3 + s a_3 + t a_2}{2 p_1 a_1} - \frac{1}{2} \left\{ \frac{a_3}{p_1} (p_1, p_3) + \frac{a_2}{p_1} (p_1, p_2) \right\} + 1.$$

Example (6.8). $f = z_1^{13} + z_1(z_2^{18} + z_3^8) + z_2^6 z_3^6$.

f is ^(a)weighted homogeneous by the weight $P = (6, 4, 9)$.
(polynomial)

The resolution graph is



The other classes of weighted homogeneous polynomials can be treated in the same way.

(B). $T_{p,q,r}$ singularity. $([1])$.

Let $f = z_1^p + z_2^q + z_3^r + z_1 z_2 z_3$ where $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. Let

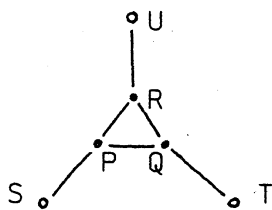
$P = {}^t(p_1, p_2, p_3)$, $Q = {}^t(q_1, q_2, q_3)$ and $R = {}^t(r_1, r_2, r_3)$ be the vertices

of $\Gamma^*(f)$ which correspond to the supports of $z_2^q + z_3^r + z_1 z_2 z_3$,

$z_1^p + z_3^r + z_1 z_2 z_3$ and $z_1^p + z_2^q + z_1 z_2 z_3$ respectively. They satisfy

$$(*) \begin{cases} p_2 q = p_3 r = p_1 + p_2 + p_3 \\ q_1 p = q_3 r = q_1 + q_2 + q_3 \\ r_1 p = r_2 q = r_1 + r_2 + r_3 \end{cases}$$

Thus $S_2 \Gamma^*(f)$ is



$$S = {}^t(1, 0, 0)$$

$$T = {}^t(0, 1, 0)$$

$$U = {}^t(0, 0, 1)$$

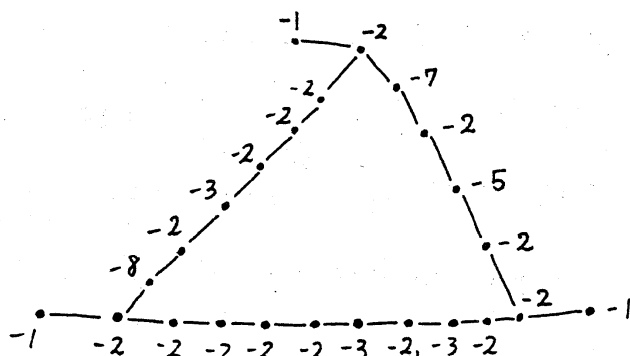
First observe that $g(\Delta(P)) = g(\Delta(Q)) = g(\Delta(R)) = 0$. Next we have

$\det(P, S) = \text{g.c.d}(p_2, p_3) = 1$ by (*). It is also easy to see that

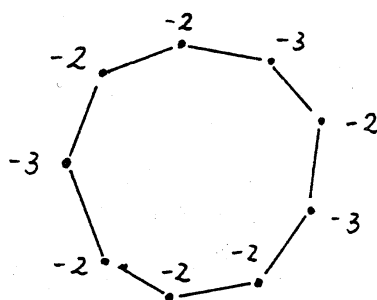
$r(P, Q) = 0$. Thus after blowing down the exceptional curves of the first kind, we get

Proposition (6.9). The resolution graph of T_{pqr} is a cyclic chain of rational curves.

Example (6.10). $T_{5,6,7}: x^5 + y^6 + z^7 + xyz = 0$. $P = (29, 7, 6)$,
 $Q = (7, 23, 5)$, $R = (6, 5, 19)$. The resolution graph by the theorem is



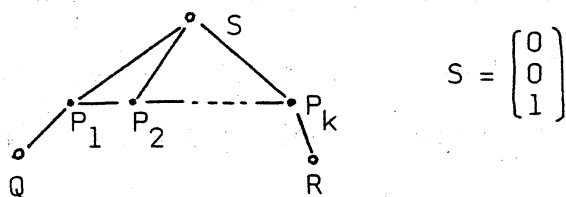
By blowing down, the graph reduces to



(C). Join type singularity. ([18], [16])

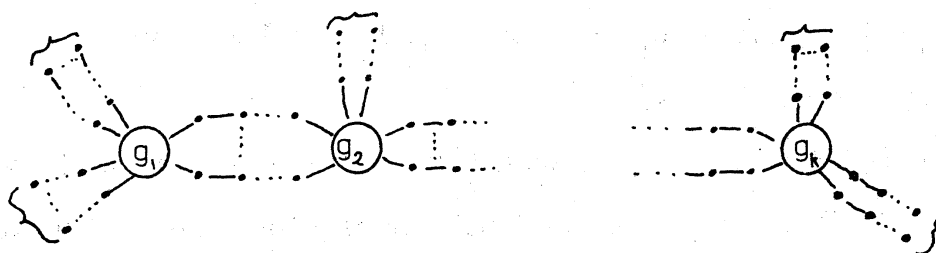
Let $g(x, y)$ be an analytic function with a non-degenerate Newton boundary. We assume that $g^{-1}(0)$ has an isolated singularity at the origin. Let $Q_i = (a_i, b_i)$ ($i = 1, \dots, k$) be the vertices of $\Gamma(g)$ and let

$\Delta_i = \overline{Q_{i-1}Q_i}$ ($i = 1, \dots, k$). We assume that $a_0 < a_1 < \dots < a_k$ and $b_0 > b_1 > \dots > b_k$. Let $f(x, y, z) = g(x, y) + z^a$ for an integer $a > 1$. The Newton boundary $\Gamma(f)$ has k 2-simplexes $\widetilde{\Delta}_i$ ($i = 1, \dots, k$) which correspond to $\{\Delta_i\}$ and $\Gamma^*(f)$ looks like this:



(Q is either $t(1, 0, 0)$ or $t(a, 0, 1)$ depending on whether $a_0 = 0$ or 1 .)

Thus the resolution graph Γ of $V = f^{-1}(0)$ is of the following type.



We look for the necessary and sufficient condition for Γ to be a tree of rational curves. We must have

- (i) $r(P_i, P_{i+1}) = 0$ for $i = 1, \dots, k-1$ and
- (ii) $g(P_i) = 0$ for $i = 1, \dots, k$.

The condition (i) is equivalent to

- (i)' $\text{g.c.d.}(a, a_i, b_i) = 1$ for $i = 1, \dots, k-1$.

By an easy calculation using Corollary (6.5), we have

$$g(P_i) = \frac{1}{2} \{ m_i \text{g.c.d.}(a, d_i) - \text{g.c.d.}(a, a_{i-1}, b_{i-1}) - \text{g.c.d.}(a, a_i, b_i) - m_i \} + 1$$

where $m_i = \text{g.c.d.}(a_i - a_{i-1}, b_{i-1} - b_i)$ and $d_i = (a_i b_{i-1} - a_{i-1} b_i) / m_i$.

Note that $g_{\Delta_i}(x, y)$ is a weighted homogeneous polynomial of degree d_i with weights $(\frac{b_{i-1} - b_i}{m_i}, \frac{a_i - a_{i-1}}{m_i})$. Thus for $2 \leq i \leq k-1$, $g(P_i) = 0$ (under

(i)) if and only if $\text{g.c.d}(a, d_i) = 1$. As g has an isolated critical point at the origin, $a_0 = 0$ or 1 and $b_k = 0$ or 1 . We assume that $(a_0, b_k) \neq (0, 0)$ if $k = 1$ by (I). We consider $g(P_1)$. If $a_0 = 1$, we have $\text{g.c.d}(a, a_0, b_0) = 1$ and therefore $\text{g.c.d}(a, d_1) = 1$. Assume that $a_0 = 0$.

As $m_1 = (a_1, b_0 - b_1)$ and $d_1 = \frac{a_1 b_0}{m_1}$, we have

$g(P_1) \cong \frac{1}{2}(m_1 - 1) \{ \text{g.c.d}(a, d_1) - 1 \}$. Thus $g(P_1) = 0$ if and only if either $\text{g.c.d}(a, b_0) = \text{g.c.d}(a, d_1) = 1$ or $(a_1, b_0 - b_1) = 1$ and $(a, b_0) = (a, a_1 b_0)$.

Thus we obtain

Theorem (6.11). (Compare with [9]). Assume that g is not of Brieskorn type. Then Γ is a tree of rational curves if and only if the following are satisfied.

- (i) $\text{g.c.d}(a, a_i, b_i) = 1$ for $i = 1, \dots, k-1$.
- (ii) $\text{g.c.d}(a, d_i) = 1$ for $i = 2, \dots, k-1$.
- (iii) If $\text{g.c.d}(a, a_0, b_0) = 1$, $\text{g.c.d}(a, d_1) = 1$.

If $\text{g.c.d}(a, a_0, b_0) > 1$, $\text{g.c.d}(a, b_0) = \text{g.c.d}(a, a_1 \cdot b_0)$ and

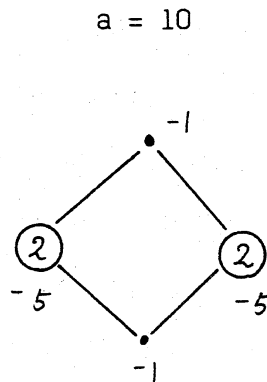
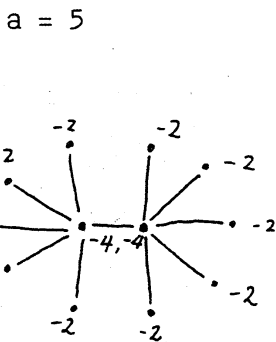
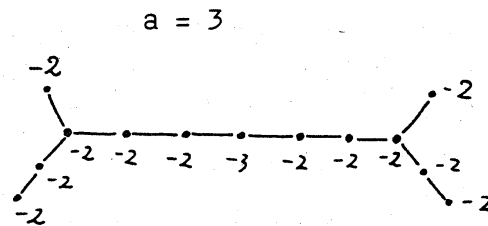
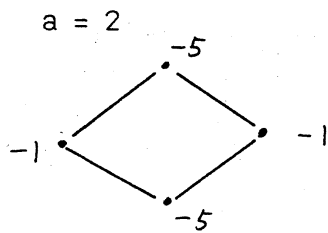
$$(a_1, b_0 - b_1) = 1.$$

- (iv) If $\text{g.c.d}(a, a_k, b_k) = 1$, $\text{g.c.d}(a, d_k) = 1$.

If $\text{g.c.d}(a, a_k, b_k) > 1$, $\text{g.c.d}(a, a_k) = \text{g.c.d}(a, b_{k-1}a_k)$ and $\text{g.c.d}(a, a_k - a_{k-1}) = 1$.

Remark (6.12). Γ is a tree of rational curves if and only if the neighborhood boundary K of the exceptional divisors is a $\mathbb{Q}$ -homology sphere ([4]). The latter condition is equivalent to: 1 is not an eigenvalue of the monodromy of f . Thus Theorem (6.11) can also be proved through [19].

Example (6.12). Let $g(x, y) = x^5 + x^2y^2 + y^5$. Then we have $m_1 = m_2 = 1$ and $d_1 = d_2 = 10$. Γ is a tree of rational curves if and only if $\text{g.c.d}(a, 10) = 1$ or 5. Here are some examples.



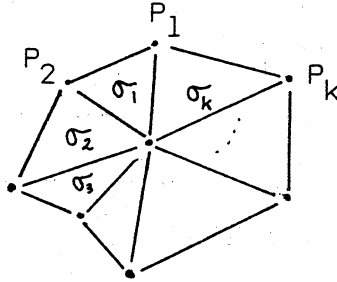
§7. Three dimensional singularity.

Let $V = f^{-1}(0)$ be as in §4 and we assume that $n = 3$. In this section, we study the topology of the exceptional divisors $E(P)$. We have three types

of exceptional divisors: The dimension of $\Delta(P)$ is either 1, 2, or 3. We are particularly interested in the case that $\dim \Delta(P) = 3$.

(I). Assume that $\dim \Delta(P) = 1$ or $\dim \Delta(P)^* = 3$.

Let $\sigma_1, \dots, \sigma_k$ be two simplexes in $\Delta(P)^*$ which contain P as a vertex.



By Lemma (4.8) and Theorem (5.3), $E(P)$ has $r(\Delta(P)) + 1$ connected components $E(P)_i$ ($i = 1, \dots, r(\Delta(P)) + 1$) which are birationally equivalent to P^2 and

$$(7.1) \quad \chi(E(P)_i) = k.$$

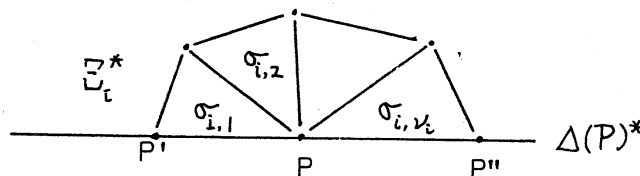
Note that k is greater than or equal to 3.

(II). Assume that $\dim \Delta(P) = 2$ ($\dim \Delta(P)^* = 2$).

Let P' and P'' be vertices of Σ^* which are on $\Delta(P)^*$ and are adjacent to P . Let $C(P)$ be the curve defined by $E(P) \cap E(P')$. By Lemma (6.4) and the calculation in the proof of Theorem (6.1), we have

$$(7.2) \quad g(C(P)) = g(\Delta(P)).$$

Let $\Xi_1, \dots, \Xi_t$ be 1-dimensional faces of $\Delta(P)$. Then $\Delta(P)^*$ is in the boundaries of Ξ_i^* for $i = 1, \dots, t$ in $\Gamma^*(f)$. Let $\sigma_{i1}, \dots, \sigma_{i, \nu_i}$ be the 2-simplexes in Σ^* which are adjacent to P and $\sigma_{ij} \subset \text{closure}(\Xi_i^*)$.



Thus by Theorem (6.1), we have

$$(7.3) \quad \chi(E(P)) = -4g(\Delta(P)) + 4 + \sum_{i=1}^t (\nu_i - 2)(r(\Xi_i) + 1).$$

Note that $\nu_i \geq 2$ in general so that $\chi(E(P)) \geq 2(2 - 2g(\Delta(P)))$.

If $\nu_i = 2$ for $i = 1, \dots, t$, the equality holds and $E(P)$ canonically projects to $C(P)$ with the fiber P^1 .

(III). Assume that $\dim \Delta(P) = 3$.

Let $\Delta_1, \dots, \Delta_s$ be 2-dimensional faces of $\Delta(P)$ and let

$\Xi_1, \dots, \Xi_q$ be 1-dimensional faces of $\Delta(P)$. Let ν_i be the number of 2-simplexes in Ξ_i^* which contain P . By Theorem (5.3) and the calculations in (I) and (II), we have

$$(7.4) \quad \chi(E(P)) = \frac{4! \text{vol}((0, \Delta(P)))}{d(P)} - 2 \sum_{i=1}^s g(\Delta_i) + 2s + \sum_{j=1}^q (\nu_j - 2)(r(\Xi_j) + 1).$$

Lemma (7.5). Assume that $\Delta(P)$ is a simplex. Let $\sigma = (P, P_1, P_2, P_3)$ be a simplex of Σ^* and let $j: E(P)^* \rightarrow (\mathbb{C}_\sigma^*)^3$ be the inclusion map. Then $j_\# : \pi_1(E(P)^*) \rightarrow \pi_1(\mathbb{C}_\sigma^*)^3$ is an isomorphism.

Proof. When we move the coefficients of f keeping the non-degeneracy condition, then $E(P)^*$ moves diffeomorphically as there are simultaneous smooth compactifications $E(P)$. Thus we may assume that

$$g_{\Delta}(y_{\sigma,1}, y_{\sigma,2}, y_{\sigma,3}) = c + \sum_{i=1}^3 a_i y_{\sigma}^{\mu_i} \quad \text{where } c \neq 0, \quad a_i \neq 0$$

($i = 1, 2, 3$) and $0, \mu_1, \mu_2, \mu_3$ are vertices of $S(g_{\Delta})$. Let

$h(y_{\sigma,1}, y_{\sigma,2}, y_{\sigma,3})$ be the weighted homogeneous polynomial defined by

$$\sum_{i=1}^3 a_i y_{\sigma}^{\mu_i}. \quad \text{By Theorem (5.3) of [15], the map } b: (\mathbb{C}^*)^3 - h^{-1}(0) \rightarrow (\mathbb{C}^*)^3 \times \mathbb{C}^*$$

defined by $b(y_{\sigma}) = (y_{\sigma}, h(y_{\sigma}))$ gives an isomorphism of the fundamental groups. On the other hand, $E(P)^* = h^{-1}(-c)$ and $(\mathbb{C}^*)^3 - h^{-1}(0)$ is a fiber bundle over $\mathbb{C}^*$ with fiber $h^{-1}(-c)$. Thus the assertion is immediate from the homotopy exact sequence of the above fiber bundle.

Remark (7.6). Lemma (7.5) will presumably hold for any $\Delta(P)$ with $\dim \Delta(P) = 3$.

Theorem (7.8). Assume that $\Delta(P)$ is a simplex. Then the fundamental group of $E(P)$ is a finite cyclic group.

Proof. Let $\sigma = (P, P_1, P_2, P_3)$ be a simplex of Σ^* such that

$$\Delta(P) \supset \Delta(P_1) \supset \Delta(P_2) \supset \Delta(P_3) \quad \text{and} \quad P_1 \in \Delta_1^* \quad \text{and} \quad P_2 \in \Xi_1^*.$$

(We assume that $\Xi_1 \subset \Delta_1$). Then by Lemma (7.5), $\pi_1(E(P; \sigma))$ is a free

abelian group generated by $e_3 = (0, 0, 1)$. Let $\sigma' = (P, P'_1, P'_2, P'_3)$ be another simplex of Σ^* with $\Delta(P) \supset \Delta(P'_1) \supset \Delta(P'_2) \supset \Delta(P'_3)$ and

$$\dim \Delta(P'_2) = 1. \quad \text{Let } \sigma^{-1} \sigma' = \begin{bmatrix} 1 & * & \cdots & * \\ - & - & - & - \\ 0 & & & \\ 0 & \Lambda' & & \\ 0 & & & \end{bmatrix}. \quad \text{Write } \Lambda' = (\lambda_{ij}).$$

Then by ^(the) Van Kampen Theorem, $\pi_1(E(P; \sigma) \cup E(P; \sigma'))$ is a cyclic group generated

by e_3 with relations $\lambda_{3,1} e_3 = \lambda_{3,2} e_3 = 0$. Note that $\lambda_{3,1} = \lambda_{3,2} = 0$ if and only if P'_1, P'_2 are on the $\text{closure}(\Xi_1^*)$. Thus the assertion is immediate from the existence of σ' such that $\lambda_{3,1} \neq 0$ or $\lambda_{3,2} \neq 0$. group

Remark (7.9). In fact, the order of the fundamental group does not depend on the choice of Σ^* . The verification will be done in [21].

Now we study the divisor curves on $E(P)$ which are the intersection with $E(P')$ such that P' are adjacent to P . We denote by $C(P')$ the divisor $E(P, P')$. On each Δ_i^* , there is a point P_i which is adjacent to P .

Δ_i^* and Δ_j^* are on the boundaries of Ξ_k^* if $\Delta_i \cap \Delta_j = \Xi_k$.

We assume that Δ_i^* and Δ_j^* are on the boundaries of Ξ_k^* . Let

$P_{k,1}, \dots, P_{k,\nu_{k-1}}$ be vertices of Ξ_k^* which are adjacent to P .

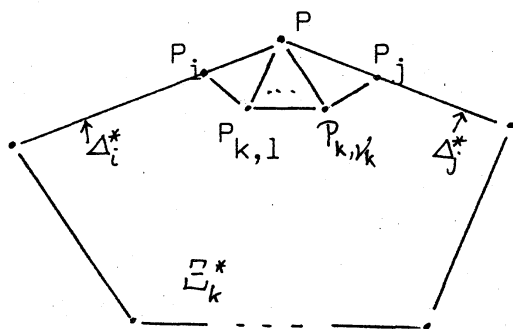


Figure (7.10)

By the calculation of the Euler characteristics, it is easy to see that each connected component of $C(P_{k,m})$ is a rational curve and $C(P_i)$ has

genus $g(\Delta_i)$. Let $P = \begin{pmatrix} P_1 \\ \vdots \\ P_4 \end{pmatrix}$. To calculate the self-intersection numbers

of $C(P_i)$ and $C(P_{k,m})$, we consider the rational function $\mathcal{P} = \pi^*(z_1^{p_2}/z_2^{p_1})$

on $E(P)$. For any vector $Q = \begin{pmatrix} q_1 \\ \vdots \\ q_4 \end{pmatrix}$, we define $|Q, P|$ by $\overbrace{q_1 p_2 - q_2 p_1}^{\text{the } 2 \times 2 \text{ minor}}$.

We assume that Σ^* is canonical at P in the sense of Lemma (3.9).

Namely

- (i) $\Sigma^*/S_2 \Gamma^*(f)$ is the subdivision by the canonical primitive sequences.
- (ii) $P_{k,1}, \dots, P_{k,\nu_k-1}$ are determined by (ii-a) if $\det(P, P_i, P_j) = 1$
 $\nu_k = 2$ and $P_{k,1} = P_i + P_j$ or (ii-b) $c = \det(P, P_i, P_j) > 1$ and
- $$(*) \quad P_{k,m+1} = \frac{1}{c_m} P_j + \frac{c_{m+1}}{c_m} P_{k,m} + \frac{d_{m+1}}{c_m} P \quad (m = 0, \dots, \nu_k - 1)$$

where $c_0 = c$ and $c_0 > c_1 > \dots > c_{\nu_k-1} = 1$ and $c_{\nu_k} = 0$ and $0 \leq d_m < c_{m-1}$

$P_{k,0} = P_i$ and $P_{k,\nu_k} = P_j$. Let $P_{k,m} = \begin{pmatrix} x_{1,m} \\ \vdots \\ x_{4,m} \end{pmatrix}$.

Let $C(P_{k,m})_t$; $t = 1, \dots, r(\Sigma_k) + 1$ be the components of $C(P_{k,m})$.

Then by the same discussion as in §6, we have $(\mathcal{P}) = \sum_{k,m} |P_{k,m}, P| C(P_{k,m})$.

Therefore we have

$$(**) \quad -C(P_{k,m})_t^2 = \frac{|P_{k,m-1}, P| + |P_{k,m+1}, P|}{|P_{k,m}, P|}$$

for $m = 1, \dots, \nu_k - 1$. On the other hand, we can interpret (*) as:

$$(*)' \quad |P_{k,m+1}, P| c_m = |P_{k,m}, P| c_{m+1} + |P_j, P|$$

By the exact same discussion as in Lemma (3.6), we get

Theorem (7.10). Assume that $\sum^*$ is canonical at P . Then the continuous fraction $[s_1, \dots, s_{\nu_{k-1}}]$ is equal to $\frac{c}{c_1}$ where $s_m = -C(P_{k,m})^2_t$.
 (If $c = 1$, $\nu_k = 2$ and $s_1 = 1$. Otherwise, $s_i \geq 2$.) $-C(P_i)^2$ is equal to

$$\frac{\sum' (r(\Sigma_k) + 1) |P_{k,1}, P|}{|P_i, P|}$$

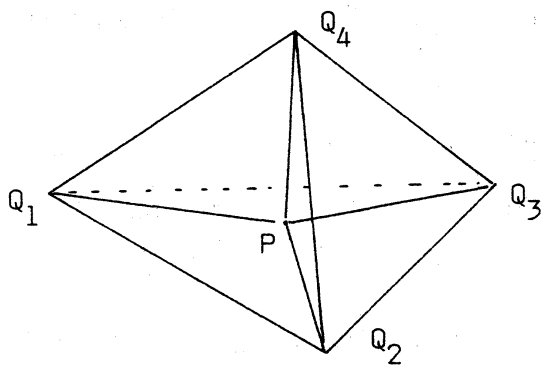
where $\sum'$ is the sum for k such that $\Sigma_k \subset \Delta_i$.

Remark (7.11). If $|P_i, P| = 0$, we take another 2×2 minor to determine the intersection numbers.

Example (7.12). Let $f(z_1, z_2, z_3, z_4) = z_1^{a_1} + z_2^{a_2} + z_3^{a_3} + z_4^{a_4}$.

We assume that $\text{g.c.d.}(a_i, a_j) = 1$ for any $i \neq j$. Then

$P = {}^t(a_2 a_3 a_4, a_1 a_3 a_4, a_1 a_2 a_4, a_1 a_2 a_3)$ is the weight vector. The dual Newton diagram $\Gamma^*(f)$ is a 3-simplex plus a barycenter P .



$$Q_1 = {}^t(1, 0, 0, 0)$$

$$Q_2 = {}^t(0, 1, 0, 0)$$

$$Q_3 = {}^t(0, 0, 1, 0)$$

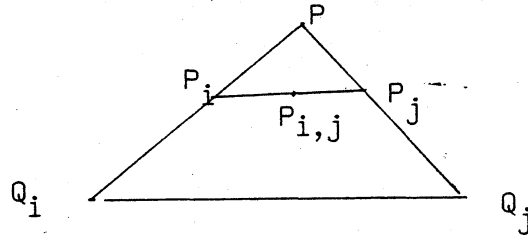
$$Q_4 = {}^t(0, 0, 0, 1)$$

Thus $\Gamma^*(f)$ has four 1-simplexes $\overline{PQ_i}$ and six 2-simplexes $\Delta(PQ_i Q_j)$

at P . Let $\sum^*$ be a canonical simplicial subdivision. As $\det(P, Q_i) = a_i$,
 at P

the vertex P_i on PQ_i which is adjacent to P can be written as

$P_i = \frac{1}{a_i} Q_i + \frac{a_i'}{a_i} P$ for some integer a_i' , $0 < a_i' < a_i$. Consider the two-simplex $\Delta(P, Q_i, Q_j)$ ($i \neq j$). As $\det(P, P_i, P_j) = 1$, we need only one vertex $P_{ij} = P_i + P_j$ on $\Delta(P, P_i, P_j)$ at P .



By Theorem (7.10),

$$\chi(E(P)) = \frac{a_1 a_2 a_3 a_4}{a_1 a_2 a_3 a_4} + 0 + 8 + 0$$

$$= 9$$

$$- C(P_{ij})^2 = 1$$

$$- C(P_i)^2 = \frac{\sum_{j \neq i} |P_{ij}, P|}{|P_i, P|} = 2$$

We blow down 6 curves of the first kind $C(P_{ij})$ ($i \neq j$) to obtain a surface M of Euler characteristic $(9 - 6) = 3$.

Assertion. M is 2-dimensional projective space.

Proof. Let $\sigma = (P, P_1, P_{12}, P')$ be a simplex of Σ^* such that

$$\Delta(P') = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}. \text{ Then, in } \mathbb{C}_\sigma^4, E(P; \sigma) \text{ is defined by } y_{\sigma,1} = 0, \text{ and}$$

$$g_\sigma = \pi_{\sigma^*} f / y_{\sigma,1}^{a_1 a_2 a_3 a_4} y_{\sigma,2}^{d(P_1)} y_{\sigma,3}^{d(P_{ij})} y_{\sigma,4}^{d(P')}$$

More precisely, as $3!$ volume $S(g_\sigma) = 1$, we have

$$g_{\sigma}(y_{\sigma,2}, y_{\sigma,3}, y_{\sigma,4}) = y_{\sigma,2}^{\alpha_3} y_{\sigma,3}^{\alpha_4} y_{\sigma,4}^{\beta_4} + y_{\sigma,3}^{\beta_4} y_{\sigma,4}^{\alpha_4} + y_{\sigma,4}^{\alpha_4} + 1,$$

where $\alpha_3, \alpha_4, \beta_4$ are positive integers.

Let L be the hyperplane in P^3 defined by $x_1 + x_2 + x_3 + x_4 = 0$.

We define the holomorphic mapping $\psi : E(P; \sigma) \rightarrow L$ by

$$(*) \quad \psi(y_{\sigma}) = [y_{\sigma,2}^{\alpha_3} y_{\sigma,3}^{\alpha_4} y_{\sigma,4}^{\beta_4} ; y_{\sigma,3}^{\beta_4} y_{\sigma,4}^{\alpha_4} ; y_{\sigma,4}^{\alpha_4} ; 1]. \quad \psi \text{ can}$$

also be written as $\psi(y_{\sigma}) = [x_1; \dots; x_4]$ where

$$(*)' \quad x_i = \pi_{\sigma}^* z_i^{a_i} / y_{\sigma,1}^{a_1 a_2 a_3 a_4} \quad (i = 1, 2, 3, 4).$$

It is easy to see that ψ maps $E(P; \sigma) \cap (y_{\sigma,3} = 0)$ to a point and ψ is bijective on the complement. By the second expression $(*)'$, ψ can be extended to any $E(P; \sigma')$, $\sigma' = (P, P'_1, P'_2, P'_3)$ by

$$x_i = \pi_{\sigma'}^* z_i^{a_i} / y_{\sigma',1}^{a_1 a_2 a_3 a_4}$$

As $\psi(C(P_{ij})) = 1$ point, ψ induces an isomorphism $\psi' : M \rightarrow L$ as is desired.

The other exceptional divisors of the resolution $\pi : \tilde{V} \rightarrow V$ are birationally equivalent to P^2 by (I), (II) and Corollary (5.4).

Thus we obtain

Theorem (7.13). Let $f(z) = z_1^{a_1} + z_2^{a_2} + z_3^{a_3} + z_4^{a_4}$ and assume that

a_i ($i = 1, \dots, 4$) are mutually coprime. Let Σ^* be a canonical simplicial subdivision of $\Gamma^*(f)$ and let $\pi : \tilde{V} \rightarrow V$ be the associated resolution of V . Then the exceptional divisors are birationally equivalent to P^2 .

Example (7.14). Let $f(z) = z_1^d + z_2^d + z_3^d + z_4^d$. This singularity can be resolved by a single blowing up with an exceptional divisor E_d which

is the projective surface of degree d in P^3 defined by $f = 0$. This corresponds to the resolution with respect to $\Sigma^* = \Gamma^*(f)$. (The condition (2.2) is not satisfied.)

If we take a canonical simplicial subdivision Σ^* at P $E(P)$ contains $6d$ curves of the first kind and $\chi(E(P)) = d^3 - 4(d-1)(d-2) + 8$. Let M be the manifold which is obtained by blowing down $6d$ curves of the first kind. Then by the similar discussion as in Example (7.12), we have $M \cong E_d$.

Remark (7.15). In general there is no unique minimal resolution of 3-dimensional singularity.

Example (7.16). Let $f(z) = z_1^{da_1} + z_2^{da_2} + z_3^{da_3} + z_4^{da_4}$ where a_i are mutually coprime. Then we can take the same Σ^* as in Example (7.12) as a canonical simplicial subdivision of $\Gamma^*(f)$. The only differences are $g(\Delta_i) = \frac{(d-1)(d-2)}{2}$ and $r(\Sigma_j) + 1 = d$. Thus $\chi(E(P)) = d^3 - 4(d-1)(d-2) + 8$. Blow down $6d$ curves of exceptional curves of the first kind and let M be the quotient manifold. We can define an isomorphism $\psi : M \rightarrow E_d \subset P^3$ by the same equation $(*)'$ of Example (7.12).

Further details about three dimensional singularity will be studied in [21].

References

- [1] Arnold, V.I., Critical Points of Smooth Functions and their Normal Forms, Uspekhi. Mat. Nauk 30:5 (1975), 3- 65, MR # 8701.
- [2] Brieskorn, E., Beispiele zur Differentialtopologie von Singularitäten. Inventiones Math. 2 (1966) 1 - 14.
- [3] Durfee, A.H., The Signature of Smoothings of Complex Surface Singularities. Math. Ann. 232, 85 - 98 (1978).
- [4] Hirzebruch, F., The Topology of Normal Singularities of an algebraic Surface. Seminaire Bourbaki, n° 250, 1962/63.
- [5] Kempf, G., Knudsen, F., Mumford, D., Siant-Donat, B., Toroidal Embeddings, Lecture Notes in Math. 339, Berlin-Heidelberg-New York, Springer 1973.
- [6] Kouchnirenko, A.G., Polyèdres de Newton et Nombres de Milnor. Inventiones Math. 32, 1 - 32 (1976).
- [7] Laufer, H.B., On μ for surface singularities, Proceedings of Symposia in Pure Math. 45 - 49, vol. 30 (1977).
- [8] Laufer, H.B., Normal Two- Dimensional Singularities. Annals of Math. Studies 71, 1971, Princeton Univ. Press.
- [9] Libgober, A., Alexander Polynomial of Plane Algebraic Curves and Cyclic Multiple Planes. Duke Math. J. vol. 49, No. 4, 833 - 851 , 1982.
- [10] Milnor, J., Singular Points of Complex Hypersurfaces. Annals of Math. Studies 61, 1968, Princeton Univ. Press.
- [11] Milnor, J., On the three dimensional Brieskorn manifolds $M(p,q,r)$. 175 - 225, in Knots, Groups, and 3-Manifolds, Annals of Math. Studies 84, 1975, Princeton Univ. Press.
- [12] Milnor, J., Orlik, P., Isolated singularities defined by weighted homogeneous polynomials. Topology 9, 285 - 293 (1970).
- [13] Mumford, D., The topology of normal singularities of an algebraic surface and a criterion for simplicity. Publ. math. No. 9, l'Inst. des hautes

études sci. Paris. 1961.

- [14] Oka, M., On the bifurcation of the multiplicity and the topology of the Newton boundary. J. Math. Soc. Japan, vol 31 (1979), 435 - 450.
- [15] Oka, M., On the topology of the Newton boundary II, J. Math. Soc. Japan, vol. 32 (1980), 65 - 92.
- [16] Oka, M., On the homotopy types of hypersurfaces defined by weighted homogeneous polynomials. Topology vol. 12, 19 - 32 (1973).
- [17] Orlik, P., Wagreich, P., Isolated singularities of algebraic surface with C^* - action. Annals of Math. 93, 205 - 228 (1971).
- [18] Sebastiani, M., Thom, R., Un resultat sur la monodromie, Inventiones Math. 13 (1971)
- [19] Varchenko, A.N., Zeta- Function of Monodromy and Newton's Diagram. Inventiones Math. 37, 253 - 262 (1976).
- [20] Wagreich, P., Singularities of complex surfaces with solvable local fundamental group. Topology 11, 51 - 72 (1972).
- [21] Oka, M., On the resolution of three dimensional hypersurface singularity, in preparation.