

On linear relations for special L -values over certain totally real number fields

Seiji Kuga

Graduate School of Mathematics, Kyushu University

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1 Introduction

Let F , $\mathfrak{o}_F$, and $\mathfrak{d}_F$ be a totally real number field over $\mathbb{Q}$ with degree m , the ring of integers of F , and the different of F over $\mathbb{Q}$. We denote the m embeddings of F to $\mathbb{R}$ by $\iota_1, \iota_2, \dots, \iota_m$. We define a congruence subgroup Γ by

$$\Gamma = \Gamma[\mathfrak{d}_F^{-1}, 4\mathfrak{d}_F] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(F) \mid a, d \in \mathfrak{o}_F, b \in \mathfrak{d}_F^{-1}, c \in 4\mathfrak{d}_F \right\}.$$

Let $\mathfrak{h} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$ be the upper-half plane. As usual, we embed $SL_2(F)$ into $SL_2(\mathbb{R})^m$ by $\gamma \mapsto (\iota_1(\gamma), \iota_2(\gamma), \dots, \iota_m(\gamma))$ and consider the Möbius transformation of $SL_2(F)$ on $\mathfrak{h}^m$ by this embedding.

For $\xi \in F$ and $\mathbf{z} = (z_1, z_2, \dots, z_m) \in \mathfrak{h}^m$, we set $q^\xi = e^{2\pi\sqrt{-1}\sum_{i=1}^m \iota_i(\xi)z_i}$. The standard theta series θ_F is given by

$$\theta_F(\mathbf{z}) = \sum_{\xi \in \mathfrak{o}_F} q^{\xi^2}.$$

We define the factor of automorphy $j_F(\gamma, \mathbf{z})$ by

$$j_F(\gamma, \mathbf{z}) = \frac{\theta_F(\gamma \mathbf{z})}{\theta_F(\mathbf{z})},$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $\mathbf{z} = (z_1, z_2, \dots, z_m) \in \mathfrak{h}^m$. When $F = \mathbb{Q}$, we write $\theta(z) = \theta_{\mathbb{Q}}(z)$ and $j(\gamma, z) = j_{\mathbb{Q}}(\gamma, z)$ briefly. It is known that

$$j(\gamma, z) = \varepsilon_d^{-1} \left(\frac{c}{d} \right) (cz + d)^{\frac{1}{2}}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4), \quad z \in \mathfrak{h}$$

where $(\cdot)$ is the Shimura's quadratic reciprocity symbol [5], and ε_d is 1 or $\sqrt{-1}$ according as $d \equiv 1 \pmod{4}$ or $d \equiv 3 \pmod{4}$.

Let k be in $\frac{1}{2}\mathbb{Z}$, a holomorphic function f on $\mathfrak{h}^m$ is a Hilbert modular form on Γ of parallel weight k if f satisfies

$$f(\gamma \mathbf{z}) = j_F(\gamma, \mathbf{z})^{2k} f(\mathbf{z}) \quad \text{for any } \gamma \in \Gamma, \mathbf{z} \in \mathfrak{h}^m$$

and that has the q -expansions of the forms

$$f(g\mathbf{z}) \prod_{i=1}^m (\iota_i(c) + \iota_i(d)z_i)^{-k} = c_g(0) + \sum_{\substack{\xi \in \mathfrak{o}_F \\ \xi \succ 0}} c_g(\xi) q^{\frac{\xi}{h_g}}$$

for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(F)$ where $h_g > 0$ is the constant which depends only on g and $\xi \succ 0$ means that $\iota_i(\xi) > 0$ for any $i = 1, 2, \dots, m$. We denote the space of Hilbert modular forms on Γ of parallel weight k by $M_k(\Gamma)$. We also denote the space of cusp forms by $S_k(\Gamma)$.

In 1975, Cohen [1] constructed a special modular form $\mathcal{H}_r \in M_{r+\frac{1}{2}}(\Gamma_0(4))$ for all positive integers r , called the Cohen Eisenstein Series of weight $r + \frac{1}{2}$ which has the q -expansion of the form

$$\begin{aligned} \mathcal{H}_r(z) = & \zeta(1-2r) + \sum_{\substack{N \geq 1 \\ (-1)^r N \equiv 0,1 \pmod{4}}} L(1-r, \mathfrak{X}_{(-1)^r N}) \\ & \times \sum_{d|f_{(-1)^r N}} \mu(d) \mathfrak{X}_{(-1)^r N}(d) d^{r-1} \sigma_{2r-1} \left(\frac{f_{(-1)^r N}}{d} \right) q^N \end{aligned}$$

where $\mathfrak{X}_N$ is the quadratic character corresponding to $\mathbb{Q}(\sqrt{N})/\mathbb{Q}$, f_N is the natural number such that $N = D_N f_N^2$, and D_N is the discriminant of $\mathbb{Q}(\sqrt{N})/\mathbb{Q}$. The space of modular forms on $\Gamma_0(4)$ of weight $r + \frac{1}{2}$ whose n th Fourier coefficient vanishes unless $(-1)^r n$ is congruent to 0 or 1 modulo 4 is called the Kohnen plus space introduced and investigated by Kohnen in 1980 [3].

In 2013, Hiraga and Ikeda gave a generalization of the Kohnen plus space for Hilbert modular forms of half-integral weight [2].

Let κ be a positive integer, the Kohnen plus space $M_{\kappa+\frac{1}{2}}^+(\Gamma)$ with respect to $M_{\kappa+\frac{1}{2}}(\Gamma)$ is defined by the subspace of $M_{\kappa+\frac{1}{2}}(\Gamma)$ which consists of all $h \in M_{\kappa+\frac{1}{2}}(\Gamma)$ with Fourier coefficient of the form

$$h(z) = c(0) + \sum_{\substack{\xi \in \mathfrak{o}_F, \xi \succ 0 \\ (-1)^\kappa \xi \equiv \square \pmod{4}}} c(\xi) q^\xi.$$

Here, we define $\xi \equiv \square(4)$ if there exists $x \in \mathfrak{o}_F$ such that $\xi - x^2 \in 4\mathfrak{o}_F$. We also define $S_{\kappa+\frac{1}{2}}^+(\Gamma) = M_{\kappa+\frac{1}{2}}^+(\Gamma) \cap S_{\kappa+\frac{1}{2}}(\Gamma)$.

In 2016, Su constructed the Eisenstein series $G_{\kappa+\frac{1}{2}, \chi'} \in M_{\kappa+\frac{1}{2}}^+(\Gamma)$ which is a generalization of the Cohen Eisenstein series [6]. Let χ' be a character of the class group of F , then $G_{\kappa+\frac{1}{2}, \chi'}$ is defined by

$$G_{\kappa+\frac{1}{2}, \chi'}(\mathbf{z}) = L_F(1 - 2\kappa, \overline{\chi'}^2) + \sum_{\substack{(-1)^\kappa \xi \equiv \square(4) \\ \xi > 0}} \mathcal{H}_\kappa(\xi, \chi') q^\xi$$

where $L_F(s, \chi)$ is the L -function over F with respect to the character χ defined by

$$L_F(s, \chi) = \sum_{\substack{0 \neq \mathfrak{a} \subset \mathfrak{o}_F \\ \text{ideal}}} \frac{\chi(\mathfrak{a})}{N_{F/\mathbb{Q}}(\mathfrak{a})^s}$$

for $\text{Re}(s) > 1$ and

$$\begin{aligned} \mathcal{H}_\kappa(\xi, \chi') &= \chi'(\mathfrak{D}_{(-1)^\kappa \xi}) L_F(1 - \kappa, \overline{\mathfrak{X}_{(-1)^\kappa \xi} \chi'}) \\ &\quad \times \sum_{\mathfrak{a} | \mathfrak{F}_\xi} \mu(\mathfrak{a}) \mathfrak{X}_\xi(\mathfrak{a}) \chi'(\mathfrak{a}) N_{F/\mathbb{Q}}(\mathfrak{a})^{\kappa-1} \sigma_{F, 2\kappa-1, \chi'^2}(\mathfrak{F}_\xi \mathfrak{a}^{-1}). \end{aligned} \quad (1)$$

Here, $\mathfrak{D}_\xi$ and $\mathfrak{X}_\xi$ are the relative discriminant and the quadratic character corresponding to $F(\sqrt{\xi})/F$ respectively, $\mathfrak{F}_\xi$ is the integral ideal such that $\mathfrak{F}_\xi^2 \mathfrak{D}_\xi = (\xi)$, μ is the Möbius function for ideals, and $\sigma_{F, k, \chi}$ is defined by

$$\sigma_{F, k, \chi}(\mathfrak{a}) = \sum_{\mathfrak{b} | \mathfrak{a}} N_{F/\mathbb{Q}}(\mathfrak{b})^k \chi(\mathfrak{b}).$$

When F is a real quadratic field such that $\mathfrak{d}_F = (\delta)$ with a totally real positive element δ , Su gave linear relations between special L -values over F and some arithmetic functions [7].

In this paper, we give generalization of these linear relations.

We define arithmetic functions $\alpha_k(n)$ and $\beta_k(n)$ by

$$\alpha_k(n) := \begin{cases} -\frac{2k}{(2^k-1)B_k} \tilde{\sigma}_{k-1}\left(\frac{n}{2}\right) & \text{if } k \in 2\mathbb{Z}, \\ \frac{2}{L(-k+1, \chi_{-4})} \sigma_{k-1, \chi_{-4}}(n) & \text{if } k \in \mathbb{Z} \setminus 2\mathbb{Z}, \\ -\frac{2k-1}{(2^{k-\frac{1}{2}}-1)B_{k-\frac{1}{2}}} \sum_{s^2 < n} \tilde{\sigma}_{k-\frac{3}{2}}\left(\frac{n-s^2}{2}\right) + 2\lambda(n) & \text{if } k \notin \mathbb{Z}, k - \frac{1}{2} \in 2\mathbb{Z}, \\ \frac{2}{L(-k+\frac{3}{2}, \chi_{-4})} \sum_{s^2 < n} \sigma_{k-\frac{3}{2}, \chi_{-4}}(n - s^2) + 2\lambda(n) & \text{if } k \notin \mathbb{Z}, k - \frac{1}{2} \in \mathbb{Z} \setminus 2\mathbb{Z}, \end{cases}$$

$$\beta_k(n) := \begin{cases} \frac{(-1)^{\frac{k}{2}+1} 2k}{(2^k-1)B_k} \sigma'_{k-1, \chi_4}(n) & \text{if } k \in 2\mathbb{Z}, \\ \frac{2^k(-1)^{\frac{k-1}{2}}}{L(-k+1, \chi_{-4})} \sigma'_{k-1, \chi_{-4}}(n) & \text{if } k \in \mathbb{Z} \setminus 2\mathbb{Z}, \\ \frac{(-1)^{\frac{k}{2}+\frac{3}{4}} (2k-1)}{(2^{k-\frac{1}{2}}-1)B_{k-\frac{1}{2}}} \sum_{s^2 < n} \sigma'_{k-\frac{3}{2}, \chi_4}(n-s^2) & \text{if } k \notin \mathbb{Z}, k-\frac{1}{2} \in 2\mathbb{Z}, \\ \frac{2^{k-\frac{1}{2}}(-1)^{\frac{k}{2}-\frac{3}{4}}}{L(-k+\frac{3}{2}, \chi_{-4})} \sum_{s^2 < n} \sigma'_{k-\frac{3}{2}, \chi_{-4}}(n-s^2) & \text{if } k \notin \mathbb{Z}, k-\frac{1}{2} \in \mathbb{Z} \setminus 2\mathbb{Z}, \end{cases}$$

where B_k is the k -th Bernoulli number,

$$\tilde{\sigma}_k(n) = \sum_{d|n} d^k (-1)^d, \quad \lambda(n) = \begin{cases} 1 & \text{if } n \text{ is a square} \\ 0 & \text{otherwise} \end{cases},$$

and we set $a(x) = 0$ for an arithmetic function $a(n)$ and $x \notin \mathbb{N} \cup \{0\}$.

Our main result is the following.

Theorem 1.1. *Let F be a totally real number field such that $8 \nmid m$ and $\mathfrak{d}_F = (\delta)$ with totally real positive element δ , we have*

$$\mathcal{R}G_{\kappa+\frac{1}{2}, \chi'} = L_F(1-2\kappa, \overline{\chi'}^2) \left\{ E_{m(\kappa+\frac{1}{2})} + 2^{-m\kappa} (-1)^{\frac{m\kappa(\kappa+1)}{2}} E_{m(\kappa+\frac{1}{2})}^* \right\} + Q$$

where Q is some cusp form in $S_{m(\kappa+\frac{1}{2})}(\Gamma_0(4))$.

By comparing the Fourier coefficients of both sides, we deduce the following corollary.

Corollary 1.1. *With the above notation, if $\mathcal{H}_\kappa(\xi, \chi')$ is as in (1), we have*

$$\begin{aligned} \sum_{\substack{\xi \in \mathfrak{o}_F, \xi \succ 0 \\ (-1)^\kappa \xi \equiv \square \pmod{4} \\ \text{Tr}(\frac{\xi}{\delta}) = n}} \mathcal{H}_\kappa(\xi, \chi') &= L_F(1-2\kappa, \overline{\chi'}^2) \\ &\times \left\{ \alpha_{m(\kappa+\frac{1}{2})}(n) + 2^{-m\kappa} (-1)^{\frac{m\kappa(\kappa+1)}{2}} \beta_{m(\kappa+\frac{1}{2})}(n) \right\} + c(n) \end{aligned}$$

where $c(n)$ is the q -coefficient of some cusp form in $S_{m(\kappa+\frac{1}{2})}(\Gamma_0(4))$.

2 Outline of the proof

From here until the end, we assume that F is a totally real number field such that $\mathfrak{d}_F = (\delta)$ with totally real positive element δ .

Let $\mathbb{A}_F$, $\psi = \prod \psi_v$ be the adele ring of F , the additive character on $\mathbb{A}_F/F$ with

$\psi_v(x) = e^{(-1)^\kappa 2\pi\sqrt{-1}x}$ for archimedean places v . Let f be a complex valued function on $\mathfrak{h}^m$, we define a complex valued function $\mathcal{R}f$ on $\mathfrak{h}$ as follows.

$$(\mathcal{R}f)(z) = f\left(\frac{z}{\iota_1(\delta)}, \frac{z}{\iota_2(\delta)}, \dots, \frac{z}{\iota_m(\delta)}\right).$$

Lemma 2.1. [7, Theorem 2.1] *For $f \in M_{k+\frac{1}{2}}(\Gamma)$, we have*

$$\mathcal{R}f \in M_{m(\kappa+\frac{1}{2})}(\Gamma_0(4)).$$

Moreover, if we write $f(z) = \sum_{\xi \in \mathfrak{o}_F} c(\xi)q^\xi$, $\mathcal{R}f(z)$ has the q -expansion of the form

$$(\mathcal{R}f)(z) = \sum_{n=0}^{\infty} \left(\sum_{\text{Tr}_{F/\mathbb{Q}}(\frac{\xi}{\delta})=n} c(\xi) \right) q^n. \quad (2)$$

The most important part of the proof of Theorem 1.1 is the calculations of the constant terms of $\mathcal{R}G_{\kappa+\frac{1}{2}, \chi'}$ at each cusp. For a complex valued function $f : \mathfrak{h}^m \rightarrow \mathbb{C}$, we put

$$(\mathcal{W}_F f)(z) = \prod_{i=1}^m (-2\sqrt{-1}\iota_i(\delta)z_i)^{-\kappa-\frac{1}{2}} f\left(-(4\iota_1(\delta)^2 z_1)^{-1}, \dots, -(4\iota_m(\delta)^2 z_m)^{-1}\right)$$

and

$$(\mathcal{U}_F f)(z) = \prod_{i=1}^m (2\iota_i(\delta)z_i + 1)^{-\kappa-\frac{1}{2}} f\left(\frac{z_1}{2\iota_1(\delta)z_1 + 1}, \dots, \frac{z_m}{2\iota_m(\delta)z_m + 1}\right).$$

The following lemmas give the constant terms of $\mathcal{W}_F G_{\kappa+\frac{1}{2}, \chi'}$ and $\mathcal{U}_F G_{\kappa+\frac{1}{2}, \chi'}$.

Lemma 2.2. [7] *The constant term of $\mathcal{W}_F G_{\kappa+\frac{1}{2}, \chi'}$ is equal to*

$$2^{-m\kappa}(-1)^{\frac{m\kappa(\kappa+1)}{2}} L_F(1 - 2\kappa, \overline{\chi'}^2).$$

Lemma 2.3 (Kuga). *The constant term of $\mathcal{U}_F G_{\kappa+\frac{1}{2}, \chi'}$ is equal to*

$$2^{-m\kappa} L_F(1 - 2\kappa, \overline{\chi'}^2) \prod_{v|2} \int_{\mathfrak{o}_v} \psi_v\left(\frac{x^2}{2\delta}\right) dx.$$

Especially when $8 \nmid m$, this value is equal to 0.

Sketch of proof of the Theorem 1.1.

When $4 \nmid m$, the proof is similar to that of [7].

When $4 \mid m$ and $8 \nmid m$, we define operators $\mathcal{U}$ and $\mathcal{W}$ on $M_{m(\kappa+\frac{1}{2})}(\Gamma_0(4))$ as follows.

For $h \in M_{m(\kappa+\frac{1}{2})}(\Gamma_0(4))$,

$$(\mathcal{W}h)(z) = \left(\frac{2z}{\sqrt{-1}} \right)^{-m(\kappa+\frac{1}{2})} h \left(-\frac{1}{4z} \right).$$

$$(\mathcal{U}h)(z) = (2z+1)^{-m(\kappa+\frac{1}{2})} h \left(\frac{z}{2z+1} \right).$$

Then, by a simple calculation, we can check that

$$\mathcal{W}\mathcal{R}G_{\kappa+\frac{1}{2},\chi'} = \mathcal{R}\mathcal{W}_F G_{\kappa+\frac{1}{2},\chi'}$$

and

$$\mathcal{U}\mathcal{R}G_{\kappa+\frac{1}{2},\chi'} = \mathcal{R}\mathcal{U}_F G_{\kappa+\frac{1}{2},\chi'}.$$

By Lemmas 2.1, 2.2, and 2.3, we have

$$\begin{aligned} \mathcal{R}G_{\kappa+\frac{1}{2},\chi'} = & L_F(1-2\kappa, \overline{\chi'}^2) E_{m(\kappa+\frac{1}{2})} \\ & + 2^{-m\kappa} (-1)^{\frac{m\kappa(\kappa+1)}{2}} L_F(1-2\kappa, \overline{\chi'}^2) E_{m(\kappa+\frac{1}{2})}^* + Q \end{aligned}$$

where

$$E_k(z) = \begin{cases} \sum_{\gamma \in \Gamma_0(4)_\infty \setminus \Gamma_0(4)} j(\gamma, z)^{-2k} & \text{if } k \in \mathbb{Z} \\ \theta(z) \sum_{\gamma \in \Gamma_0(4)_\infty \setminus \Gamma_0(4)} j(\gamma, z)^{-2k+1} & \text{if } k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}, \end{cases}$$

$$E_k^*(z) = \left(\frac{2z}{\sqrt{-1}} \right)^{-k} E_k \left(-\frac{1}{4z} \right).$$

and $Q \in S_{m(\kappa+\frac{1}{2})}(\Gamma_0(4))$. By comparing the Fourier coefficients of both side, we complete the proof. Indeed, $\alpha_k(n)$ and $\beta_k(n)$ represent the n th Fourier coefficients of E_k and E_k^* respectively. $\square$

References

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