

Bipartite Coherent Configurations

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Abstract

In this note, we establish several useful identities between the eigenvalues, dual eigenvalues, and intersection numbers of bipartite coherent configurations.

1 Introduction

Bipartite coherent configurations, introduced in [3], are a form of coherent configuration with an especially strong algebraic structure, similar to that of association schemes. In this note, we establish several useful identities relating the parameters of bipartite coherent configurations. This is in analogue to the basic properties of association schemes that can be found in [1] or [2]. Several of these properties were shown previously for bipartite coherent configurations [3] or a more general class of coherent configurations [4], but we collect them here for completeness.

In what follows, we will let I be the identity matrix and J be the matrix of all ones, with subscripts indicating the dimensions. We will denote Schur, or point-wise, multiplication by $\circ$. For a matrix A , let $\tau(A)$ denote the sum of the entries. We will be making frequent use of the identity that for matrices A, B of the same dimension,

$$\tau(A \circ B) = \text{tr}(AB^T) = \text{tr}(B^T A).$$

2 Definition

Let β and γ be sets and let t_β, t_γ , and $t_{\beta\gamma}$ be positive integers. Let $\{X_0, \dots, X_{t_\beta}\}$ be a set of $|\beta| \times |\beta|$ 01-matrices. Similarly, let $\{Y_0, \dots, Y_{t_\gamma}\}$ be a set of $|\gamma| \times |\gamma|$ 01-matrices, and let $\{N_1, \dots, N_{t_{\beta\gamma}}\}$ be a set of $|\beta| \times |\gamma|$ 01-matrices. Let

$$\mathcal{C} = \left\{ \begin{pmatrix} X_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \right\}_{i=0}^{t_\beta} \cup \left\{ \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & Y_j \end{pmatrix} \right\}_{j=0}^{t_\gamma} \cup \left\{ \begin{pmatrix} \mathbf{0} & N_h \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \right\}_{h=1}^{t_{\beta\gamma}} \cup \left\{ \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ N_h^T & \mathbf{0} \end{pmatrix} \right\}_{h=1}^{t_{\beta\gamma}}.$$

We say that $\mathcal{C}$ forms a *bipartite coherent configuration* if it satisfies the following:

(C1) The matrices $X_0 = I_\beta$ and $Y_0 = I_\gamma$;

(C2)

$$\sum_{M \in \mathcal{C}} M = J;$$

(C3) For all $M \in \mathcal{C}$, we have $M^T \in \mathcal{C}$.

(C4) For all $M_1, M_2 \in \mathcal{C}$, we have $M_1 M_2 \in \text{span}(\mathcal{C})$;

(C5) For all $1 \leq i, j \leq t_{\beta\gamma}$, we have

$$N_i N_j^T = N_j N_i^T$$

and

$$N_i^T N_j = N_j^T N_i;$$

(C6) The set

$$\{N_i N_j^T : 1 \leq i, j \leq t_{\beta\gamma}\} \cup \{I\}$$

spans $\{X_i : 0 \leq i \leq t_\beta\}$ and

$$\{N_i^T N_j : 1 \leq i, j \leq t_{\beta\gamma}\} \cup \{I\}$$

spans $\{Y_i : 0 \leq i \leq t_\gamma\}$.

Note that $\{X_0, \dots, X_{t_\beta}\}$ and $\{Y_0, \dots, Y_{t_\gamma}\}$ form association schemes. When restricted to association schemes, the results in this note are well known and can be found, for instance, in [1, 2].

Bipartite coherent configurations were defined in [3], which also established the following result.

2.1 Theorem. *Let $\mathcal{C}$ be a bipartite coherent configuration. Then $\text{span}(\mathcal{C})$ has a second basis*

$$\mathcal{B} = \left\{ \begin{pmatrix} L_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \right\}_{i=0}^{t_\beta} \cup \left\{ \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & R_j \end{pmatrix} \right\}_{j=0}^{t_\gamma} \cup \left\{ \begin{pmatrix} \mathbf{0} & D_h \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \right\}_{h=0}^{t_{\beta\gamma}-1} \cup \left\{ \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ D_h^T & \mathbf{0} \end{pmatrix} \right\}_{h=0}^{t_{\beta\gamma}-1}$$

where for all $0 \leq r \leq t_\beta$, we have $L_r^T = L_r$ and for all $0 \leq r \leq t_\gamma$, we have $R_r^T = R_r$ further satisfying:

(D1) For all $0 \leq r, s \leq t_\beta$, we have

$$L_r L_s = \delta_{rs} L_r$$

and for all $0 \leq r, s \leq t_\gamma$, we have

$$R_r R_s = \delta_{rs} R_r;$$

(D2) $L_0 = \frac{1}{|\beta|} J_\beta$, $D_0 = \frac{1}{\sqrt{|\beta||\gamma|}} J_{\beta,\gamma}$, and $R_0 = \frac{1}{|\gamma|} J_\gamma$;

(D3)

$$\sum_{r=0}^{t_\beta} L_r = I_\beta$$

and

$$\sum_{r=0}^{t_\gamma} R_r = I_\gamma;$$

(D4) For all $0 \leq r \leq t_{\beta\gamma} - 1$, we have $L_r = D_r D_r^T$ and $R_r = D_r^T D_r$; and

(D5) For all $M_1, M_2 \in \mathcal{B}$, we have $M_1 \circ M_2 \in \mathcal{B}$.

We can think of a bipartite coherent configuration as a collection of two-by-two block matrices, and is convenient to work with the bipartite coherent configuration block-wise. With some abuse of notation, we will refer to the matrices N_i as being “in” $\mathcal{C}$, even though more accurately it’s the matrix $\begin{pmatrix} \mathbf{0} & N_i \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \in \mathcal{C}$, and similarly for the other blocks and for the dual basis $\mathcal{D}$.

When we give identities using the Schur product, we will only give the proof in terms of the N_i and D_r , with the understanding that not only do the same proofs hold for the other blocks, but that they are also standard results on association schemes.

Ordinary matrix multiplication is slightly more involved, and splits into four cases, which we will differentiate using matrices in $\mathcal{C}$.

- (i) $X_i X_j$ or equivalently $Y_i Y_j$
- (ii) $X_i N_j$ or equivalently $Y_i N_j^T$
- (iii) $N_i Y_j$ or equivalently $N_i^T X_j$
- (iv) $N_i N_j^T$ or equivalently $N_i^T N_j$

Case (i) again reduces to known results on association schemes. Cases (ii) and (iii) are very similar, though there will occasionally be subtle differences. The emphasis for this note is to establish the results on cases (ii) or (iii) and (iv), with the understanding that the remaining cases follow easily.

3 Orthogonality

Under Schur multiplication, the matrices in $\mathcal{C}$ are pairwise orthogonal idempotents. Under ordinary matrix multiplication, (D1) tells us that L_r and R_r are each sets of pairwise orthogonal idempotents and (D4) tells us that there is a close relationship between the D_r and the L_r or R_r . This relationship is sufficient to establish the remaining cases for orthogonality.

3.1 Lemma. For all $0 \leq r \leq t_\beta$ and $0 \leq s \leq t_{\beta\gamma} - 1$, we have

$$L_r D_s = \delta_{r,s} D_s.$$

Proof. By (D4) and (D1), we have

$$L_r D_s D_s^T L_r = L_r L_s L_r = \delta_{r,s} L_s.$$

If $r \neq s$, then since $(L_r D_s)(L_r D_s)^T = \mathbf{0}$, it follows that $L_r D_s = \mathbf{0}$. Then by (D3), we see that

$$L_r D_r = \sum_{s=0}^{t_\beta} L_s D_r = I_\beta D_r = D_r. \quad \square$$

3.2 Lemma. For all $0 \leq r, s \leq t_{\beta\gamma} - 1$ we have

$$D_r D_s^T = \delta_{r,s} L_r.$$

Proof. By (D4), we know that $D_r D_r^T = L_r$, so suppose $r \neq s$. Then we have

$$(D_r D_s^T)(D_r D_s^T)^T = D_r D_s^T D_s D_r^T = D_r R_s D_r^T = \mathbf{0}$$

by Lemma 3.1, which implies $D_r D_s^T = \mathbf{0}$. $\square$

Suda [4] worked with a class of coherent configurations where there is a second distinguished basis with special properties. For bipartite coherent configurations, two of these properties are implicit in the set-up of the dual basis $\mathcal{D}$ and one of them is equivalent to (D2). The last condition, related to orthogonality, is equivalent to (D1) combined with Lemma 3.1 and 3.2.

4 Dual Eigenvalues

Schur multiplication preserves the block structure of a bipartite coherent configuration in a more straightforward way than ordinary matrix multiplication. Thus, in defining how to switch from one basis to another, it is convenient to begin by expressing the elements of the spectral basis $\mathcal{D}$ in terms of the Schur basis $\mathcal{C}$.

Since $\{N_1, \dots, N_{t_{\beta\gamma}}\}$ spans $\{D_0, \dots, D_{t_{\beta\gamma}-1}\}$, for any $0 \leq i \leq t_{\beta\gamma} - 1$, we know there exist coefficients $Q_{1,i}^{\beta\gamma}, \dots, Q_{t_{\beta\gamma},i}^{\beta\gamma}$ such that

$$D_r = \frac{1}{\sqrt{|\beta||\gamma|}} \sum_{i=1}^{t_{\beta\gamma}} Q_{i,r}^{\beta\gamma} N_i.$$

The coefficients $Q_{j,r}^{\beta\gamma}$ are called the *dual eigenvalues associated to $\beta\gamma$* . By taking transposes, we see that the eigenvalues associated to $\gamma\beta$ are equivalent to the eigenvalues associated to $\beta\gamma$.

In the same way, the eigenvalues associated to β and γ respectively are the coefficients such that

$$L_r = \frac{1}{|\beta|} \sum_{i=0}^{t_\beta} Q_{i,r}^\beta X_i$$

and

$$R_r = \frac{1}{|\gamma|} \sum_{i=0}^{t_\gamma} Q_{i,r}^\gamma Y_i.$$

Since the N_i are pairwise orthogonal idempotents under Schur multiplication, it follows that for any $0 \leq r \leq t_{\beta\gamma} - 1$ and $1 \leq i \leq t_{\beta\gamma}$ we must have

$$D_r \circ N_i = \frac{1}{\sqrt{|\beta||\gamma|}} \sum_{j=1}^{t_{\beta\gamma}} Q_{j,r}^{\beta\gamma} N_j \circ N_i = \frac{1}{\sqrt{|\beta||\gamma|}} Q_{i,r}^{\beta\gamma} N_i. \quad (4.1)$$

Analogously,

$$L_r \circ X_i = \frac{1}{|\beta|} Q_{i,r}^\beta X_i \quad (4.2)$$

and

$$R_r \circ Y_i = \frac{1}{|\gamma|} Q_{i,r}^\gamma Y_i.$$

We define the *multiplicity relative to β* by $m_r^\beta = \text{tr}(L_r)$. As a special consequence of Equation 4.2, we have

$$m_r^\beta = \tau(L_i \circ X_0) = \tau\left(\frac{1}{|\beta|} Q_{0,i}^\beta X_0\right) = Q_{0,i}^\beta.$$

Similarly, the multiplicity relative to γ is

$$m_r^\gamma = \text{tr}(R_r) = Q_{0,i}^\gamma.$$

A useful fact in linear algebra is that for any matrix B , the matrices BB^T and B^TB share nonzero eigenvalues with multiplicity. Thus (D4) tells us that for $0 \leq r \leq t_{\beta\gamma} - 1$, the multiplicities $m_r^\beta = m_r^\gamma$. It will be helpful to let $m_r^{\beta\gamma}$ to be this common quantity to emphasize that it independent of the specific choice of β or γ .

5 Eigenvalues

The *eigenvalues associated to $\beta\gamma$, β , or γ* are given by, respectively

$$N_i = \sum_{r=0}^{t_{\beta\gamma}-1} P_{r,i}^{\beta\gamma} D_r,$$

$$X_i = \sum_{r=0}^{t_\beta} P_{r,i}^\beta L_r,$$

or

$$Y_i = \sum_{r=0}^{t_\gamma} P_{r,i}^\gamma R_r.$$

Let $0 \leq i \leq t_\beta$ and $0 \leq r \leq t_{\beta\gamma} - 1$. Then by Lemma 3.1 we have

$$X_i D_r = \sum_{s=0}^{t_\beta} P_{s,i}^\beta L_s D_r = P_{r,i}^\beta D_r. \quad (5.1)$$

The same argument gives us that for $0 \leq i \leq t_{\beta\gamma}$ and $0 \leq r \leq t_{\beta\gamma} - 1$, we have

$$N_i R_r = P_{r,i}^{\beta\gamma} D_r. \quad (5.2)$$

There is a nuance here that if $r \geq t_{\beta\gamma}$, the matrix D_r and eigenvalues $P_{r,i}^{\beta\gamma}$ are not defined. It can be notationally convenient to set $D_s = \mathbf{0}$ and $P_{s,i}^{\beta\gamma} = 0$ for all $s \geq t_{\beta\gamma}$ so we can extend Equation 5.2 to all $0 \leq r \leq t_\beta$.

Let $0 \leq i \leq t_{\beta\gamma}$ and $0 \leq r \leq t_{\beta\gamma} - 1$. Then Lemma 3.2 tells us that

$$N_i D_r^T = \sum_{s=0}^{t_{\beta\gamma}-1} P_{s,i}^{\beta\gamma} D_s D_r^T = P_{r,i}^{\beta\gamma} L_r. \quad (5.3)$$

Combining Equation 5.3 and (D2), we see that

$$\frac{1}{\sqrt{|\beta||\gamma|}} N_i J_\gamma = N_i D_0^T = P_{0,i}^{\beta\gamma} L_0 = \frac{1}{|\beta|} P_{0,i}^{\beta\gamma} J_\beta,$$

so N_i has constant row sum

$$\sqrt{\frac{|\gamma|}{|\beta|}} P_{0,i}^{\beta\gamma}. \quad (5.4)$$

For $1 \leq i \leq t_{\beta\gamma}$, the *valency associated to $\beta\gamma$* , denoted $k_i^{\beta\gamma}$, is the row sum of N_i .

The valency associated to $\gamma\beta$ is the column sum of N_i , and is given by

$$k_i^{\gamma\beta} = \sqrt{\frac{|\beta|}{|\gamma|}} P_{0,i}^{\beta\gamma}.$$

6 Useful Identities

The following result gives an analogue of a subset of conditions of Lemma 2. 2. 1 of [2].

6.1 Proposition. *The eigenvalues and dual eigenvalues of a bipartite coherent configuration associated to $\beta\gamma$ satisfy:*

(i) For $1 \leq i \leq t_{\beta\gamma}$, we have

$$Q_{i,0}^\beta = 1;$$

(ii) For $1 \leq i \leq t_{\beta\gamma}$, we have

$$P_{0,i}^{\beta\gamma} = \sqrt{\frac{|\beta|}{|\gamma|}} k_i^{\beta\gamma};$$

(iii) For $0 \leq r \leq t_{\beta\gamma} - 1$ and $1 \leq i \leq t_{\beta\gamma}$, we have

$$m_r^{\beta\gamma} P_{r,i}^{\beta\gamma} = P_{0,i}^{\beta\gamma} Q_{i,r}^{\beta\gamma};$$

(iv) For $1 \leq i, j \leq t_{\beta\gamma}$, we have

$$\sum_{r=0}^{t_{\beta\gamma}-1} m_r^{\beta\gamma} P_{r,i}^{\beta\gamma} P_{r,j}^{\beta\gamma} = \delta_{i,j} |\beta| k_i^{\beta\gamma};$$

(v) For $0 \leq r, s \leq t_{\beta\gamma} - 1$, we have

$$\sum_{i=1}^{t_{\beta\gamma}} k_i^{\beta\gamma} Q_{i,r}^{\beta\gamma} Q_{i,s}^{\beta\gamma} = \delta_{r,s} |\gamma| m_r^{\beta\gamma};$$

(vi) For $0 \leq r \leq t_{\beta\gamma} - 1$ and $1 \leq i \leq t_{\beta\gamma}$, we have

$$|P_{r,i}^{\beta\gamma}| \leq P_{0,i}^{\beta\gamma};$$

(vii) For $0 \leq r \leq t_{\beta\gamma}$ and $1 \leq i \leq t_{\beta\gamma}$, we have

$$|Q_{i,r}^{\beta\gamma}| \leq m_r^{\beta\gamma}.$$

Proof. We already saw (ii) in Equation 5.4.

For (i), Equation 4.1 and (D2) gives us

$$\frac{1}{\sqrt{|\beta||\gamma|}} Q_{i,0}^{\beta\gamma} N_i = D_0 \circ N_i = \frac{1}{\sqrt{|\beta||\gamma|}} N_i,$$

so we must have $Q_{i,0}^{\beta\gamma} = 1$.

To show (iii), we combine Equation 4.1 and Equation 5.3 to see that

$$m_r^{\beta\gamma} P_{r,i}^{\beta\gamma} = \text{tr}(N_i D_r^T) = \tau(D_r \circ N_i) = \frac{1}{\sqrt{|\beta||\gamma|}} Q_{i,r}^{\beta\gamma} \tau(N_i) = \sqrt{\frac{|\beta|}{|\gamma|}} k_i^{\beta\gamma} Q_{i,r}^{\beta\gamma},$$

and so (iii) follows from (ii).

For (iv), we have that

$$\text{tr}(N_i N_j^T) = \tau(N_i \circ N_j) = |\beta| k_i^{\beta\gamma} \delta_{i,j}. \quad (6.1)$$

We can also write $N_i N_j^T$ in the other basis to see that

$$\text{tr}(N_i N_j^T) = \text{tr} \left(\sum_{r=0}^{t_{\beta\gamma}-1} \sum_{s=0}^{t_{\beta\gamma}-1} P_{r,i}^{\beta\gamma} P_{s,j}^{\beta\gamma} D_r D_s^T \right) = \sum_{r=0}^{t_{\beta\gamma}-1} m_r^{\beta\gamma} P_{r,i}^{\beta\gamma} P_{r,j}^{\beta\gamma} \quad (6.2)$$

by Lemma 3.2. Then combining Equation 6.1 with Equation 6.2 gives us (iv).

We can show (v) in a similar way, since

$$\delta_{r,s} m_r^{\beta\gamma} = \tau(D_r \circ D_s) = \frac{1}{|\beta||\gamma|} \sum_{i=1}^{t_{\beta\gamma}} Q_{i,r}^{\beta\gamma} Q_{i,s}^{\beta\gamma} \tau(N_i) = \frac{1}{|\gamma|} \sum_{i=1}^{t_{\beta\gamma}} Q_{i,r}^{\beta\gamma} Q_{i,s}^{\beta\gamma} k_i^{\beta\gamma}.$$

To see (vi), we observe that $P_{r,i}^{\beta}$ is indeed an eigenvalue of the matrix

$$\begin{pmatrix} \mathbf{0} & N_i \\ N_i^T & \mathbf{0} \end{pmatrix}.$$

Then (vii) follows from (vi) and (iii). $\square$

Identities (i) and (ii), and (v) were previously shown in [3]. Identities (i), (ii), (iii), and (iv) were also proved in a more general setting in Propositions A. 4 and A. 4 of [4].

The following result will also be useful.

6.2 Proposition. *The eigenvalues and dual eigenvalues associated to a bipartite coherent configuration satisfy:*

(a) For all $1 \leq i, j \leq t_{\beta\gamma}$, we have

$$\sqrt{|\beta||\gamma|} \delta_{ij} = \sum_{r=0}^{t_{\beta\gamma}-1} Q_{j,r}^{\beta\gamma} P_{r,i}^{\beta\gamma},$$

(b) For all $0 \leq r, s \leq t_{\beta\gamma} - 1$, we have

$$\sqrt{|\beta||\gamma|}\delta_{rs} = \sum_{i=1}^{t_{\beta\gamma}} P_{r,i}^{\beta\gamma} Q_{i,s}^{\beta\gamma}.$$

Proof. We have

$$N_i = \sum_{r=0}^{t_{\beta\gamma}-1} P_{r,i}^{\beta\gamma} D_r = \frac{1}{\sqrt{|\beta||\gamma|}} \sum_{r=0}^{t_{\beta\gamma}-1} \sum_{j=1}^{t_{\beta\gamma}} P_{r,i}^{\beta\gamma} Q_{j,r}^{\beta\gamma} N_j,$$

from which (a) follows. Similarly,

$$D_r = \frac{1}{\sqrt{|\beta||\gamma|}} \sum_{i=1}^{t_{\beta\gamma}} Q_{i,r}^{\beta\gamma} N_i = \frac{1}{\sqrt{|\beta||\gamma|}} \sum_{i=1}^{t_{\beta\gamma}} \sum_{s=0}^{t_{\beta\gamma}-1} Q_{i,r}^{\beta\gamma} P_{s,i}^{\beta\gamma},$$

giving us (b). $\square$

This can also be shown in terms of eigenmatrices and dual eigenmatrices as done in [3] and in a more general setting in [4].

7 Intersection Numbers

By (C4), we know there exist coefficients, called intersection numbers, such that for $0 \leq i \leq t_\beta$ and $1 \leq j \leq t_{\beta\gamma}$ we have

$$X_i N_j = \sum_{h=1}^{t_{\beta\gamma}} \xi_{2i,2j-1}^{2h-1} N_h.$$

Similarly, for $1 \leq i \leq t_{\beta\gamma}$ and $0 \leq j \leq t_\gamma$ we have

$$N_i Y_j = \sum_{h=1}^{t_{\beta\gamma}} \xi_{2i-1,2j}^{2h-1} N_h;$$

for $0 \leq i, j \leq t_\beta$ we have

$$X_i X_j = \sum_{h=0}^{t_\beta} \xi_{2i,2j}^{2h} X_h;$$

and for $1 \leq i, j \leq t_{\beta\gamma}$ we have

$$N_i N_j^T = \sum_{h=0}^{t_\beta} \xi_{2i-1,2j-1}^{2h} X_h.$$

Note $\xi_{i,j}^h = 0$ for all $h > t_\beta$ if h is even and $h > t_{\beta\gamma}$ if h is odd.

The corresponding intersection numbers for γ are defined analogously by $\sigma_{i,j}^h$.

The notation used is heavily inspired by distance-biregular graphs, as discussed in Section 4 of [3]. In the more general context of bipartite coherent configurations, representing

the various intersection numbers using parity loses its combinatorial interpretation, but retains the advantage of reducing the number of parameters that are written.

We now relate the intersection numbers to eigenvalues by extending more of Lemma 2. 2. 1 of [2] to bipartite coherent configurations.

7.1 Proposition. *The intersection numbers, eigenvalues, and dual eigenvalues of a bipartite coherent configuration associated to $\beta\gamma$ satisfy:*

(i) For $0 \leq r \leq t_{\beta\gamma} - 1$, $1 \leq i \leq t_{\beta\gamma}$, and $0 \leq j \leq t_\beta$, we have

$$P_{r,i}^{\beta\gamma} P_{r,j}^\gamma = \sum_{h=1}^{t_{\beta\gamma}} \xi_{2i-1,2j}^{2h-1} P_{r,h}^{\beta\gamma},$$

(ii) For $0 \leq r \leq t_{\beta\gamma} - 1$ and $1 \leq i, j \leq t_{\beta\gamma}$, we have

$$P_{r,i}^{\beta\gamma} P_{r,j}^{\beta\gamma} = \sum_{h=0}^{t_\beta} \xi_{2i-1,2j-1}^{2h} P_{r,h}^{\beta\gamma},$$

(iii) For $1 \leq i, j \leq t_{\beta\gamma}$ and $0 \leq r \leq t_{\beta\gamma} - 1$ we have

$$P_{r,i}^{\beta\gamma} Q_{j,r}^{\beta\gamma} = \sqrt{\frac{|\gamma|}{|\beta|}} \sum_{h=0}^{t_\gamma} \xi_{2i-1,2h}^{2j-1} Q_{h,r}^\gamma;$$

(iv) For $1 \leq i \leq t_{\beta\gamma}$, $0 \leq j \leq t_\beta$, and $0 \leq r \leq t_{\beta\gamma} - 1$ we have

$$P_{r,i}^{\beta\gamma} Q_{j,r}^\beta = \sqrt{\frac{|\beta|}{|\gamma|}} \sum_{h=1}^{t_{\beta\gamma}} \xi_{2i-1,2h-1}^{2j} Q_{h,r}^{\beta\gamma};$$

Proof. For (i), we consider $N_i Y_j$. By Equation 5.2 and the corresponding analogue of Equation 5.1, we have

$$N_i Y_j D_r^T = P_{r,j}^\gamma N_i D_r^T = P_{r,i}^{\beta\gamma} P_{r,j}^\gamma D_r^T. \quad (7.1)$$

By the definition of intersection numbers and Equation 5.2 we have

$$N_i Y_j D_r^T = \sum_{h=1}^{t_{\beta\gamma}} \xi_{2i-1,2j}^{2h-1} P_{r,h}^{\beta\gamma} D_r^T. \quad (7.2)$$

Then (i) comes from combining Equation 7.1 and Equation 7.2.

In a similar way, we have

$$P_{r,i}^{\beta\gamma} P_{r,j}^{\beta\gamma} L_r = N_i N_j^T L_r = \sum_{h=0}^{t_\beta} \xi_{2i-1,2j-1}^{2h} P_{r,h}^\beta L_r,$$

establishing (ii).

To prove (iii), consider $N_i R_r \circ N_j$. From Equation 5.2 and Equation 4.1 we have

$$N_i R_r \circ N_j = P_{r,i}^{\beta\gamma} D_r \circ N_j = \frac{1}{\sqrt{|\beta| |\gamma|}} P_{r,i}^{\beta\gamma} Q_{j,r}^{\beta\gamma} N_j. \quad (7.3)$$

On the other hand, writing R_r in the basis of Schur idempotents and using the definition of intersection numbers gives us

$$N_i R_r \circ N_j = \frac{1}{|\gamma|} \sum_{h=0}^{t_\gamma} Q_{h,r}^\gamma \sum_{\ell=1}^{t_{\beta\gamma}} \xi_{2i-1,2h}^{2\ell-1} N_\ell \circ N_j = \frac{1}{|\gamma|} \sum_{h=0}^{t_\gamma} Q_{h,r}^\gamma \xi_{2i-1,2h}^{2j-1} N_j. \quad (7.4)$$

Combining Equations 7.3 and 7.4 gives us (iii).

In a similar way, we have

$$\frac{1}{|\beta|} P_{r,i}^{\beta\gamma} Q_{j,r}^\beta X_j = N_i D_r^T \circ X_j = \frac{1}{\sqrt{|\beta| |\gamma|}} \sum_{h=1}^{t_{\beta\gamma}} Q_{h,r}^{\beta\gamma} \xi_{2i-1,2h-1}^{2j} X_j,$$

establishing (iv). □

The subtle difference between considering $N_i R_r \circ N_j$ and $X_i D_r \circ N_j$ leads to a slightly different version of (iii).

7.2 Corollary. For $1 \leq i \leq t_\beta$, $j \leq t_{\beta\gamma}$, and $0 \leq r \leq t_{\beta\gamma} - 1$ we have

$$P_{r,i}^\beta Q_{j,r}^{\beta\gamma} = \sum_{h=1}^{t_{\beta\gamma}} \xi_{2i,2h-1}^{2j-1} Q_{h,r}^{\beta\gamma}.$$

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