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Choice and Imperfect Observation”

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Chaotic Asset Price Dynamics with Technology Choice and Imperfect Observation*

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Abstract

Asset price volatility, characterized by episodic booms and busts, is a prominent feature of real economies. To explain these phenomena, most existing literature has relied on bounded rationality or exogenous shocks. This paper studies land price dynamics within a rational agent general equilibrium model that incorporates endogenous technology choice. Entrepreneurs choose between a Leontief technology and a linear technology so as to maximize firm value, based on their relative productivity, but they can only observe the current land price with some noise. This informational friction results in probabilistic mixing of technology choices, leading to a continuous nonlinear land pricing map. We demonstrate that, under parameter restrictions that yield the Markov property, the resulting map exhibits observable chaos, and its invariant densities can be characterized analytically.

Keywords: Asset bubbles; Technology choice; Markov property; Chaotic dynamics

JEL: C61, G12, O33

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1 Introduction

Asset price volatility, characterized by episodic booms and busts, is widely observed in actual economies. However, standard deterministic asset pricing models typically exhibit either monotonically convergent or explosive paths, and fail to generate persistent fluctuations or complex dynamics without relying on exogenous shocks. Whereas nonlinear dynamics have been extensively explored in macroeconomic models of business cycles, their incorporation into formal asset pricing models with solid microfoundations remains limited.

There are two approaches to modeling the complex behavior of asset price fluctuations. One strand focuses on rational bubbles. Seminal contributions (Blanchard and Watson, 1982; Tirole, 1985) establish equilibrium conditions for the existence of bubbles in a partial equilibrium framework, whereas Santos and Woodford (1997) highlight the restrictiveness of these conditions in general equilibrium. Subsequent research examines the macroeconomic implications of bubbles (e.g., Kocherlakota, 2009; Martin and Ventura, 2012; Tirole and Farhi, 2012; Galí, 2014, 2021; Kunieda and Shibata, 2016; Hirano and Yanagawa, 2017; Brunnermeier et al., 2020). In these models, the primary focus is on the effects of bubbles on the real economy rather than on dynamic properties of the asset prices themselves.¹

A second strand emphasizes nonlinear dynamics and bounded rationality of investors. Day and Huang (1990) and Brock and Hommes (1997) show how switching behavior and heterogeneous expectations can generate complex dynamics, including chaos. While this literature effectively captures the “volatility clustering” observed in data, it is often criticized for lacking a rigorous general-equilibrium foundation with rational agents. As a result, the literature encounters a trade-off between the

¹Froot and Obstfeld (1991) and Ikeda and Shibata (1992, 1995) demonstrate that bubbles dependent on fundamentals can generate various asset price dynamics. However, their analyses are limited to partial equilibrium models. More recent advances in bubble economics are comprehensively reviewed by Hirano and Toda (2024). Among them, Hirano and Toda (2025) is a notable contribution. The paper demonstrates that under certain conditions—namely, when the counterfactual autarky interest rate is lower than the dividend growth rate, and the dividend growth rate is lower than the economic growth rate—asset price bubbles are not just possible, but inevitable. In other words, the paper shows that the equilibrium price of any asset satisfying these conditions incorporates a bubble component.

rigor of equilibrium analysis and the richness of dynamic behavior.

This paper bridges the gap between these two strands of literature by demonstrating that complex, chaotic price dynamics can arise without departing from the rational agents framework. Building on the rational bubble model presented in Section 6 of Hirano and Toda (2024), we introduce a nonlinear mechanism generated by the interaction between utility-maximizing households and wealth-maximizing entrepreneurs. Entrepreneurs choose among discrete production technologies to maximize firm value, while households make optimal intertemporal consumption-saving decisions. Although agents are fully rational and form expectations based on available information, imperfect observation of land price dynamics systematically distorts these decisions. We show that technology choice generates a piecewise linear, regime-dependent structure, while imperfect observation transforms the discrete choice into a continuous nonlinear map through probabilistic mixing. The interaction of these two elements leads to eventually expanding dynamics and, hence, observable chaos.

In the present model, endogenous technology choice generates a threshold structure, and observation errors smooth the threshold into an N-shaped map, which has the stretching-and-folding structure required for chaotic dynamics. The stretching phase is triggered by wealth-maximizing entrepreneurs who switch to superior production technologies, thereby accelerating price movements and pushing the economy toward a threshold. However, near this threshold, observation errors act as a stochastic perturbation, in a manner similar to the 'trembling hand' in game theory. This information friction causes misclassification around the threshold, smoothing the underlying discontinuity and forcing price trajectories back into a compact interval. The model thus generates recurrent but non-periodic asset-price movements even without exogenous shocks.

More generally, this mechanism reflects a class of discrete choice problems under imperfect information that can be modeled as piecewise linear Markov systems, as extensively analyzed by Zhang and Yokoo (2026a,b). Their analysis offers a mathematical foundation for "regime-switching" models that have been widely used empirically but lacked a unified theoretical basis.

This paper also links asset-pricing dynamics to the literature on endogenous busi-

ness cycles. A growing body of research integrates piecewise linear dynamic models with endogenous technology choice to generate complex dynamics (Aghion et al., 1999; Iwaisako, 2002; Matsuyama, 2007; Asano et al., 2012; Matsuyama et al., 2018; Asano and Yokoo, 2019; Asano et al., 2022, 2024).² We argue that the same force driving business cycles in the real economy—namely, threshold-based technology choice—is also a source of complex asset price dynamics. While these earlier studies, including the work by Zhang and Yokoo (2026a,b) on regime-switching Solow and OLG frameworks, have concentrated on capital accumulation, this paper demonstrates that the same threshold-based mechanism naturally extends to asset price dynamics.

As we mentioned earlier, introducing observation errors transforms the system into a continuous N-shaped map that exhibits observable chaos. Importantly, the resulting dynamics can satisfy the Markov property under suitable restrictions. While chaotic price fluctuations are inherently characterized by short-term unpredictability, our model allows for a certain degree of statistical predictability. Specifically, we analytically derive the invariant densities of the system. These densities provide an analytical characterization of the long-run statistical behavior and enable a probabilistic characterization of the economy’s recursive states, even when individual trajectories appear erratic.

The organization of this paper is as follows. Section 2 presents the model’s setting, while Section 3 discusses land price dynamics. Section 4 introduces observation errors into the model. To further investigate the model presented in the previous sections, Section 5 introduces the piecewise linear Markov model and derives chaotic dynamics and invariant densities. Finally, Section 6 concludes the paper, with some proofs relegated to the Appendix.

2 The settings of the model

The basic structure of our model follows that of Hirano and Toda (2024, Section 6), except for the introduction of two distinct technologies. Similar to Hirano and

²See also Horii et al. (2013) for the distributional implications of technology choice.

Toda (2024), we consider a two-sector economy consisting of a capital-intensive sector and a real estate sector. The economy consists of a single homogeneous good and a continuum of agents with a total measure of one.

Time is discrete, and the utility of a typical agent is given by

$$E_0 \sum_{t=0}^{\infty} \beta^t \log c_{it}, \quad (1)$$

where $E_t[\cdot]$ denotes the expectation conditional on information available at date t , $\beta \in (0, 1)$ is the discount factor, $i \in I = [0, 1]$ indexes the agents, and c_{it} represents the consumption of agent i at date t .

A unit of land (real estate) yields $D > 0$ units of consumption goods as dividends (land rents) every period. The aggregate supply of land is fixed at $S > 0$. Agents are unable to borrow and thus must self-finance. Let $k_{it}, s_{it} \geq 0$ denote the capital and land holdings of agent i and $P_t > 0$ be the land price at date t . At each date, an investment opportunity arrives to each agent with probability $\pi \in (0, 1)$ independently across time and agents. An agent with an investment opportunity is referred to as an entrepreneur; otherwise, the agent is a saver.

The budget constraint, subject to which each agent maximizes her utility, is given by:

$$c_{it} + k_{i,t+1} + P_t s_{it} = w_{it} = Z_{it} k_{it} + (P_t + D) s_{i,t-1} =: w_{it}, \quad (2)$$

where w_{it} represents individual wealth and Z_{it} denotes the marginal productivity of capital, which is constant and depends on the technology chosen by agent i at date t : Leontief or linear. Note that the latter technology is adopted in Hirano and Toda (2024).

We focus on the case where entrepreneurs behave symmetrically, such that $Z_{it} = Z_{jt} =: Z_t$ for $i \neq j$. By aggregating individual budget constraints (2) and using the land market condition $\int_I s_{it} di = S$, we can derive:

$$C_t + K_{t+1} + P_t S = W_t := Z_t K_t + (P_t + D) S, \quad (3)$$

where W_t denotes the aggregate wealth at date t and $C_t = \int_I c_{it} di$, $K_t = \int_I k_{it} di$, and $W_t = \int_I w_{it} di$.

We assume that the “Leontief” technology producing aggregate output takes the following form:

$$Y_t = A \min\{K_t, \xi\} + DS,$$

for some $\xi > 0$, while the linear technology is supposed to be of the form:

$$Y_t = BK_t + DS.^3$$

The marginal productivity of capital Z_t depends on the level of capital at date t , which we express as $Z_t = Z^i(K_t)$ when technology $i = 1, 2$ is chosen. Assuming equal depreciation rates across technologies, we obtain:

$$\begin{aligned} Z^1(K_t) &= \begin{cases} A + 1 - \delta, & \text{if } 0 < K_t < \xi, \\ 0, & \text{if } K_t > \xi, \end{cases} & (\text{Leontief technology}) \\ Z^2(K_t) &= B + 1 - \delta, & K_t > 0, \quad (\text{linear technology}) \end{aligned}$$

where we assume that $A > B$.

Upon entering the market, an entrepreneur is assumed to maximize her wealth by choosing one technology given the level of aggregate capital stock. Thus, in symmetric equilibrium, we can specify the wealth W_t as:⁴

$$W_t := \begin{cases} (A + 1 - \delta)K_t + (P_t + D)S, & \text{if } 0 < K_t \leq \xi, \\ (B + 1 - \delta)K_t + (P_t + D)S, & \text{if } \xi < K_t. \end{cases} \quad (4)$$

Since the utility function in (1) is logarithmic, the aggregate optimal consumption is known to be:

$$C_t = (1 - \beta)W_t. \quad (5)$$

³The two technologies may be interpreted as analytically convenient limiting cases of different degrees of input substitutability. The linear technology approximates a high-substitutability regime, such as IT- or automation-intensive production, where capital can more readily replace routine tasks. This interpretation aligns with Autor and Dorn (2013), who emphasize the falling cost of automating routine, codifiable tasks, and with Karabarbounis and Neiman (2014), who estimate an elasticity of substitution between capital and labor above unity and argue that the decline in the relative price of investment goods has induced firms to shift from labor toward capital. Conversely, the Leontief technology represents a low-substitutability limiting case in which inputs are strongly complementary, consistent with Chirinko’s (2008) survey evidence that many estimates of the capital-labor elasticity lie below unity. Another interpretation is developmental: Duffy and Papageorgiou (2000) find, using cross-country panel data, that physical capital and human-capital-adjusted labor are more substitutable in richer countries and less substitutable in poorer countries than under a Cobb-Douglas benchmark. Thus, the linear and Leontief technologies can be viewed as reduced-form approximations of high- and low-substitutability regimes rather than as literal descriptions of observed production functions.

⁴We assume that at $K_t = \xi$, the entrepreneurs choose technology 1 (Leontief technology).

By combining (3) and (5), we can derive aggregate savings as:

$$K_{t+1} + P_t S = \beta W_t.$$

Let ϕ_t represent the fraction of aggregate savings into capital. Aggregate savings are then allocated to capital and land according to:

$$(K_{t+1}, P_t S) = (\beta \phi_t W_t, \beta(1 - \phi_t) W_t). \quad (6)$$

Note that $\phi_t \in [0, \pi]$. Because entrepreneurs can choose between capital and land investments, an arbitrage argument implies that if

$$\min\{A + 1 - \delta, B + 1 - \delta\} = B + 1 - \delta > R_t := \frac{P_{t+1} + D}{P_t},$$

then $\phi_t = \pi$.

Substituting $\phi_t = \pi$ into (6), we obtain:

$$(K_{t+1}, P_t S) = (\beta \pi W_t, \beta(1 - \pi) W_t). \quad (7)$$

From (7) and (4), it follows that

$$W_t = \begin{cases} \frac{(A+1-\delta)}{1-\beta(1-\pi)} K_t + \frac{DS}{1-\beta(1-\pi)}, & \text{if } 0 < K_t \leq \xi, \\ \frac{(B+1-\delta)}{1-\beta(1-\pi)} K_t + \frac{DS}{1-\beta(1-\pi)}, & \text{if } \xi < K_t. \end{cases} \quad (8)$$

Substituting (8) into (7), we obtain

$$K_{t+1} = H(K_t) := \begin{cases} aK_t + c =: H_L(K_t) & \text{if } 0 < K_t \leq \xi, \\ bK_t + c =: H_R(K_t) & \text{if } \xi < K_t, \end{cases} \quad (9)$$

where

$$a = \frac{(A + 1 - \delta)\beta\pi}{1 - \beta(1 - \pi)}, \quad b = \frac{(B + 1 - \delta)\beta\pi}{1 - \beta(1 - \pi)}, \quad \text{and} \quad c = \frac{\beta\pi DS}{1 - \beta(1 - \pi)}.$$

Note that $a > b > 0$ and $c > 0$ by assumption.

We define the thresholds for productivity, as in Proposition 5 of Hirano and Toda (2024), as:

$$\underline{A} := \frac{1 - \beta}{\beta} + \delta < \frac{1 - \beta}{\beta\pi} + \delta =: \bar{A}.$$

We assume that:

$$\underline{A} < B < \bar{A} \quad \text{and} \quad \bar{A} < A, \quad (10)$$

which implies that:

$$a > 1 > b.$$

Furthermore, H_L has no steady state and H_R has a pseudo-steady state defined by the solution to the equation:

$$H_R(\bar{K}) = \bar{K},$$

which is given by:

$$\bar{K} = \frac{c}{1-b} = \frac{\beta\pi DS}{1-\beta-(B-\delta)\beta\pi}. \quad (11)$$

Thus, in order for H_R to have no actual steady state, it suffices to assume that:

$$\bar{K} < \xi \quad (12)$$

or, equivalently,

$$\frac{\xi}{S} > \frac{\beta\pi D}{1-\beta-(B-\delta)\beta\pi} =: \tilde{K} = \tilde{K}(D).$$

Note that the value of $\tilde{K}$ can be made arbitrarily small by taking D sufficiently small.

See Figure 1.

3 Land price dynamics

By Lemma 1 in Appendix, the long-run wealth dynamics are described by the map $\tilde{H} : I_T \rightarrow I_T$ in (23) when $\bar{K} < \xi$. Using (7), we can transform the long-run dynamics of wealth into that of the land price. Specifically, there exists a closed (trapping) interval

$$M = [b\xi' + d, a\xi' + d]$$

such that the long-run price dynamics is governed by the map $\tilde{L} : M \rightarrow M$ given by

$$P_{t+1} = \tilde{L}(P_t) := \begin{cases} aP_t + d =: \tilde{L}_L(P_t) & \text{if } b\xi' + d \leq P_t \leq \xi', \\ bP_t + d =: \tilde{L}_R(P_t) & \text{if } \xi' < P_t \leq a\xi' + d, \end{cases} \quad (13)$$

where

$$\xi' = \frac{(1-\pi)\xi}{\pi S} > \frac{(1-\pi)\tilde{K}}{\pi},$$

$$a = \frac{(A+1-\delta)\beta\pi}{1-\beta(1-\pi)}, \quad b = \frac{(B+1-\delta)\beta\pi}{1-\beta(1-\pi)}, \quad \text{and} \quad d = \frac{\beta(1-\pi)D}{1-\beta(1-\pi)},$$

with $a > 1 > b > 0$ and $d > 0$. Note that the threshold ξ' must be greater than a certain positive value, but it is possible if S is taken sufficiently small.

For convenience, we introduce the following variable change:

$$x_t = \tilde{h}(P_t) = \frac{P_t - (b\xi' + d)}{(a-b)\xi'}$$

to obtain a map from the unit interval onto itself, equivalent to (13). Thus, we obtain $L : I = [0, 1] \rightarrow I$ as follows:

$$x_{t+1} = L(x_t) := \begin{cases} 1 + a(x_t - \theta) =: L_L(x_t) & \text{if } 0 \leq x_t \leq \theta, \\ b(x_t - \theta) =: L_R(x_t) & \text{if } \theta < x_t \leq 1, \end{cases} \quad (14)$$

where

$$\theta = \tilde{h}(\xi') = \frac{(1-b)\xi' - d}{(a-b)\xi'} = \frac{1-b}{a-b} - \frac{d\pi S}{(1-\pi)(a-b)\xi} < \frac{1-b}{a-b},$$

$$a = \frac{(A+1-\delta)\beta\pi}{1-\beta(1-\pi)} \quad \text{and} \quad b = \frac{(B+1-\delta)\beta\pi}{1-\beta(1-\pi)}$$

with $a > 1 > b > 0$. Additionally, note that

$$1 - b > (a - b)\theta, \quad (15)$$

which is used in the proof of Proposition 1.

In what follows, we treat θ as a constant independent of a and b , by appropriately tuning other parameters such as ξ and S .

The normalized dynamic equation in (14) is formally equivalent to that studied in Umezaki and Yokoo (2019) in a quite different context, except that the slope of one of the two branches in the graph of L is greater than 1 while both slopes in the counterpart in Umezaki and Yokoo (2019) are between 0 and 1. As this difference does not affect the dynamic implications due to the result by Keener (1980) or Hata (2014), we can directly apply the results of Umezaki and Yokoo (2019) to our present model.

Proposition 1. *The long-run price dynamics given by (14) virtually exhibits a periodic cycle that is globally attracting. Specifically, for any irreducible fraction $p/q \in (0, (1-b)/(a-b)) \subset (0, 1)$, where p and q are positive integers, there exists an open interval $\Delta(p/q)$ such that if $\theta \in \Delta(p/q)$ then for any initial condition x_0 , x_t converges to a period- q cycle as t goes to infinity.*

Proof. First note that $L : I \rightarrow I$ has no actual steady state. Furthermore, it holds that

$$L_L(0) = 1 - a\theta > b(1 - \theta) = L_R(1)$$

due to (15). This implies that L restricted on I corresponds to the “non-overlapping case” coined by Keener (1980), who shows that this type of map exhibits globally attracting periodic motions, except for a measure-zero set of parameter values. For more details, see Umezaki and Yokoo (2019). $\square$

4 Introduction of observation errors

So far, we have considered the case in which all firms act symmetrically. In the following, we will consider the case where there is some heterogeneity among the firms in observing the state of the economy. It is a natural assumption that firms can only observe the world with finite precision. Specifically, we assume that firms observe the normalized land price at time t with some noise:

$$\hat{x}_{i,t} = x_t + \sigma \varepsilon_{i,t},$$

where $\hat{x}_{i,t}$ denotes the land price observed by firm i at time t and $\varepsilon_{i,t}$ is the disturbance term drawn from a zero-mean distribution G , independent of i and t . The constant $\sigma \geq 0$ represents the degree of imprecision. In particular, if $\sigma = 0$, that is, if there are no observation errors, then the model reduces to (14). The probability of technology 1 (Leontief) being chosen is given by

$$\text{Prob}(\hat{x}_{i,t} \leq \theta) = \text{Prob}\left(\varepsilon_{i,t} \leq \frac{\theta - x_t}{\sigma}\right) = G\left(\frac{\theta - x_t}{\sigma}\right).$$

Thus, the land price dynamics (14) now become

$$x_{t+1} = G\left(\frac{\theta - x_t}{\sigma}\right) L_L(x_t) + \left(1 - G\left(\frac{\theta - x_t}{\sigma}\right)\right) L_R(x_t) \quad (16)$$

$$= (1 + (a - b)(x_t - \theta))G\left(\frac{\theta - x_t}{\sigma}\right) + b(x_t - \theta).$$

In what follows, we will examine the case in which the model is piecewise linear. That is, the distribution is assumed to be such that the resulting map exhibits piecewise linearity with the Markov property.

5 Example: The piecewise linear Markov model

In this section, we develop a piecewise linear Markov model as a simplified version of the smooth model.⁵ Specifically, we constrain the parameters to maintain the Markov property, which allows us to investigate the chaotic dynamics of the model in greater depth.

To obtain a simple explicit model, we assume that

$$G\left(\frac{\theta - x_t}{\sigma}\right) = \begin{cases} 0 & \text{if } \theta + \sigma \leq x_t, \\ G_M\left(\frac{\theta - x_t}{\sigma}\right) & \text{if } \theta - \sigma \leq x_t < \theta + \sigma, \\ 1 & \text{if } x_t < \theta - \sigma, \end{cases}$$

where G_M is a nondecreasing function. Let $\theta_L = \theta - \sigma$ and $\theta_R = \theta + \sigma$. If σ is not too large and if G_M is specified properly, we can apply a scale transformation $h : [\sigma b, 1 - \sigma a] \rightarrow I = [0, 1]$ such that

$$x'_t = h(x_t) = \frac{x_t - \sigma b}{1 - \sigma a - \sigma b}.$$

We also have $\theta'_j = h(\theta_j)$ ($j = L, R$). Abusing notation and omitting the apostrophes for simplicity, we can transform (16) into:

$$x_{t+1} = \begin{cases} 1 + a(x_t - \theta_L), & \text{if } 0 \leq x_t \leq \theta_L, \\ \varphi_M(x_t), & \text{if } \theta_L < x_t \leq \theta_R, \\ b(x_t - \theta_R), & \text{if } \theta_R < x_t \leq 1, \end{cases}$$

where φ_M is a function determined by G_M . To simplify the model, we define $\varphi_M(x_t)$ in the above expression as

$$\varphi_M(x_t) = \frac{\theta_R - x_t}{\theta_R - \theta_L}.$$

⁵A smooth formulation can be obtained by assuming that the disturbance term follows a logistic distribution. While this formulation can exhibit chaotic dynamics over a wide range of parameter values, it is analytically intractable; hence, we focus on the piecewise linear model.

Let $y_t = (\theta - x_t)/\sigma$ or $x_t = \theta - \sigma y_t$. Then, $G_M(y_t)$ must satisfy

$$[1 - \sigma(a - b)y_t]G_M(y_t) - \sigma b y_t = h^{-1}(\varphi_M[h(\theta - \sigma y_t)]).$$

Solving this equation for $G_M(y_t)$ yields

$$\begin{aligned} G_M(y_t) &= \frac{h^{-1}(\varphi_M[h(\theta - \sigma y_t)]) + \sigma b y_t}{1 - \sigma(a - b)y_t} \\ &= \frac{[1 - \sigma(a - b)](1 + y_t)}{2[1 - \sigma(a - b)y_t]}. \end{aligned}$$

For a more detailed derivation of G_M , see the Appendix. Note that $\lim_{y \rightarrow -1} G_M(y) = 0$ and $\lim_{y \rightarrow 1} G_M(y) = 1$.

See Figure 2.

Given the functional form of G_M above, we have obtained a piecewise linear map of the unit interval onto itself:

$$\varphi : I \rightarrow I,$$

$$x_{t+1} = \varphi(x_t) = \begin{cases} 1 + a(x_t - \theta_L) =: \varphi_L(x_t), & \text{if } 0 \leq x_t \leq \theta_L, \\ (\theta_R - x_t)/(\theta_R - \theta_L) =: \varphi_M(x_t), & \text{if } \theta_L < x_t \leq \theta_R, \\ b(x_t - \theta_R) =: \varphi_R(x_t), & \text{if } \theta_R < x_t \leq 1. \end{cases} \quad (17)$$

In what follows, we focus on a specific set of parameter values under which the model satisfies the Markov property. With this property, the model exhibits chaotic behavior and a degree of predictability in the sense that the stationary distribution of the chaotic trajectories generated by the model can be calculated explicitly.

5.1 Some definitions related to the Markov property

We provide mathematical definitions related to the Markov property. The term “Markov property” is often associated with stochastic processes; however, it refers to a slightly different concept in the context of deterministic processes. For more details, see Boyarsky and Góra (1997). Let $I = [0, 1]$ and $\psi : I \rightarrow I$ be a map on I onto itself. Let $\mathcal{P}$ be a finite partition of I and I_k ($k = 1, 2, \dots, n$) be subintervals of the partition. Moreover, let ψ_k be the restriction of ψ to I_k . If each ψ_k is a homeomorphism (that is, a continuous bijection with a continuous inverse) from I_k

onto some connected union of subintervals I_m , then ψ is said to be Markov. Partition $\mathcal{P}$ is said to be a Markov partition with respect to ψ . To examine complex dynamics in piecewise linear maps or, more generally, in piecewise smooth maps, we also need to discuss the following property. Let ψ be a piecewise smooth map that is not necessarily Markov, and let ψ^n be an n -th iterate of map ψ . If there exists an integer $n \geq 1$ such that $\inf \left| \frac{d\psi^n(x)}{dx} \right| > 1$ on each I_k whenever the derivative exists, then ψ is said to be eventually expanding.

It is evident that the eventually expanding property implies that the map has no attracting periodic cycles. As a result, it is expected that the map exhibits complex dynamics with long-run observability. This contrasts with so-called topological chaos, which does not necessarily involve the observability of such long-term complexity. In economics, this long-run observability is particularly important. In this paper, we say that a map on the interval exhibits *observable chaos* if it is eventually expanding.

5.2 Chaotic dynamics

In this subsection, we first show that the map given by (17) can be made Markov by appropriately selecting parameters. Consider a Markov partition defined by the following periodic orbit of period 5:

$$0 = \varphi^5(0) < \theta_{5,L} = \varphi^2(0) < \theta_{5,R} = \varphi^4(0) < \varphi(0) < \varphi^3(0) = 1,$$

and

$$\mathcal{P} = \{I_1 = [0, \theta_{5,L}), I_2 = [\theta_{5,L}, \theta_{5,R}), I_3 = [\theta_{5,R}, \varphi(0)), I_4 = [\varphi(0), 1]\}.$$

This situation is illustrated in Fig. 3. The following proposition states that such a period-5 Markov partition is indeed possible, and consequently, that the map exhibits observable chaos.

See Figure 3.

By applying the mathematical results established by Zhang and Yokoo (2026a, b) in the context of macroeconomic growth, the following propositions characterize the long-run properties of asset price dynamics within our framework. Specifically, we identify the conditions for the emergence of observable chaos as follows.

Proposition 2. (*Observable chaos on a period-5 Markov partition*) If $\theta_L = \theta_{5,L}$, $\theta_R = \theta_{5,R}$, $a = \hat{a} = a(\theta_{5,L}, \theta_{5,R})$, and $\sigma = \hat{\sigma} = \sigma(\hat{a})$, then φ defined by (17) has a period-5 Markov partition of the unit interval and exhibits observable chaos.

Proof. The induced map is the same as that in Proposition 2 of Zhang and Yokoo (2026b). Hence, the same argument applies, implying that φ exhibits observable chaos. It remains only to derive the parameter values that generate the corresponding period-5 Markov partition. We briefly outline the derivation of these parameter values below.

First, we solve

$$\varphi_R(\varphi_L(0)) = \theta_L \quad \text{and} \quad \varphi_R(1) = \theta_R$$

for θ_L and θ_R to obtain

$$\theta_L = \frac{b}{(1+b)(1+ab)} := \theta_{5,L} \quad \text{and} \quad \theta_R = \frac{b}{1+b} := \theta_{5,R}.$$

Next, we solve the simultaneous equations:

$$h(\theta_L) = \frac{\theta - \sigma(1+b)}{1 - \sigma(a+b)} = \theta_{5,L}, \tag{18}$$

$$h(\theta_R) = \frac{\theta + \sigma(1-b)}{1 - \sigma(a+b)} = \theta_{5,R} \tag{19}$$

for σ and a . Solving (19) for σ gives

$$\sigma = \frac{b - (1+b)\theta}{1+ab}.$$

Substituting this into (18) yields

$$\frac{(1+ab)\theta - (1+b)[b - (1+b)\theta]}{(1+ab) - (a+b)[b - (1+b)\theta]} = \frac{b}{(1+b)(1+ab)}. \tag{20}$$

Since (20) is a quadratic equation in a , one can obtain an explicit solution, denoted $\hat{a}$. See Appendix for the explicit formula for $\hat{a}$. Using (18) and (19) with $\hat{a}$ substituted, $\hat{\sigma}$ is given by

$$\hat{\sigma} = \frac{\hat{a}b\theta}{2 + \hat{a}b(1+b)}.$$

□

Proposition 3. (*Observable chaos on a period- $(2n+3)$ Markov partition*) For any integer $n \geq 1$, if $\theta_L = \theta_{2n+3,L}$, $\theta_R = \theta_{2n+3,R}$, $a = \hat{a} = a(\theta_{2n+3,L}, \theta_{2n+3,R})$ and $\sigma = \hat{\sigma} = \sigma(\hat{a})$, then φ defined by (17) has a period- $(2n+3)$ Markov partition of the unit interval and exhibits observable chaos.

Proof. See Appendix. □

5.3 Stationary distribution

In the previous subsection, we demonstrated that our model can exhibit Markov properties and observable chaos. In this subsection, we show that, when such a case occurs, the chaotic trajectories have a stationary distribution in the sense that the map admits an ergodic invariant measure that is absolutely continuous with respect to Lebesgue measure. Furthermore, the corresponding density function turns out to be a piecewise constant function with an explicit representation. This implies that *the economy fluctuates seemingly randomly but is statistically predictable* to some extent.

To illustrate this, we first introduce a matrix version of the Frobenius–Perron operator, $M_\varphi = (m_{ij})_{1 \leq i, j \leq n}$, where the entries are given as follows:

$$m_{ij} = \frac{q_{ij}}{|\varphi'_i|}, \quad 1 \leq i, j \leq n,$$

where $Q_\varphi = (q_{ij})_{1 \leq i, j \leq n}$ is the incidence matrix induced by the piecewise linear transformation φ and the partition $\mathcal{P}$ of I . More specifically,

$$q_{ij} = \begin{cases} 1, & \text{if } I_j \subset \varphi(I_i), \\ 0, & \text{otherwise.} \end{cases}$$

The invariant density ω is then given by the left eigenvector with respect to the eigenvalue 1 of M_φ , that is,

$$\omega M_\varphi = \omega.$$

For more details, see Boyarsky and Góra (1997).

Building on the analytical techniques for piecewise linear Markov systems developed by Zhang and Yokoo (2026a,b), we explicitly characterize the long-run average behavior of asset price fluctuations. In particular, we derive the invariant density associated with the period-5 Markov partition established in their work.

Proposition 4. (*Invariant density on a period-5 Markov partition*) If φ is Markov on $\{I_i\}_{i=1}^4$, $\inf |(\varphi^4)'| > 1$, and matrix $M_\varphi = M_{\varphi_5}$, then φ admits an invariant density function $\omega = \omega_5^*(x)$, and the corresponding probability density is $p(x) = \omega_5^*(x) / \sum_{i=1}^4 \omega_i |I_i|$.

Proof. We derive the invariant density for the period-5 Markov case. Noting that

$$I_4 \subset \varphi(I_1), \quad \cup_{i=1}^4 I_i \subset \varphi(I_2), \quad I_1 \subset \varphi(I_3), \quad \text{and} \quad I_2 \subset \varphi(I_4),$$

yields the incidence matrix:

$$Q_{\varphi_5} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Also note that

$$|\varphi'_L| = \hat{a}, \quad |\varphi'_M| = \frac{1}{\theta_R - \theta_L} = \frac{(1+b)(1+\hat{a}b)}{\hat{a}b^2}, \quad \text{and} \quad |\varphi'_R| = b,$$

then the Frobenius–Perron operator matrix M_{φ_5} is given by

$$M_{\varphi_5} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{\hat{a}} \\ \lambda & \lambda & \lambda & \lambda \\ \frac{1}{b} & 0 & 0 & 0 \\ 0 & \frac{1}{b} & 0 & 0 \end{pmatrix},$$

where $\lambda = \hat{a}b^2 / (1+b)(1+\hat{a}b)$. Solving $\omega M_{\varphi_5} = \omega$ yields a piecewise constant density function:

$$\omega_5^*(x) = \begin{cases} \hat{a}b + \hat{a}b^2 \equiv \omega^1, & x \in I_1 \\ \hat{a}b^2 + \hat{a}b + 1 + b \equiv \omega^2, & x \in I_2 \\ \hat{a}b^2 \equiv \omega^3, & x \in I_3 \\ \hat{a}b^2 + b + b^2 \equiv \omega^4, & x \in I_4 \end{cases}$$

□

The corresponding probability density is illustrated in Fig. 4. The area of the rectangle on I_i indicates the frequency with which the chaotic trajectory visits I_i .

See Figure 4.

For a generalized version, we can show the following proposition, which is proved by Zhang and Yokoo (2026a, b) in different contexts.

Proposition 5. (*Invariant density on a period-(2n+3) Markov partition*) If φ is Markov on $\{I_i\}_{i=1}^{2n+2}$, $\inf |(\varphi^{2n+2})'| > 1$, and matrix $M_\varphi = M_{\varphi^{2n+3}}$, then φ admits an invariant density function $\omega = \omega_{2n+3}^*(x)$ and its corresponding probability density is $p(x) = \omega_{2n+3}^*(x) / \sum_{i=1}^{2n+2} \omega_i |I_i|$.

Proof. See Appendix. □

Figure 5 shows a typical chaotic trajectory for a period-11 Markov property. Endogenous boom-and-bust fluctuations can be observed.

See Figure 5.

5.4 Stability of invariant densities

Since the invariant densities derived previously depend on a specific shape of the distribution function for observation errors, one might argue that the analysis is not general. However, in this subsection, we demonstrate that the invariant densities remain stable under perturbations associated with observation errors. To this end, we introduce the following definition and theorem.

Let $I = [0, 1]$ with normalized Lebesgue measure λ . Let $\mathcal{T}(I)$ denote the class of all piecewise expanding transformations. The Skorokhod metric on $\mathcal{T}(I)$ is defined as follows: for $\tau_1, \tau_2 \in \mathcal{T}$,

$$d_S(\tau_1, \tau_2) = \inf \{ \varepsilon > 0 : \exists A \subseteq I \text{ and } \exists \xi : I \rightarrow I \text{ such that}$$

$$\lambda(A) > 1 - \varepsilon, \xi \text{ is a diffeomorphism,}$$

$$\tau_1|_A = \tau_2 \circ \tau|_A \text{ and } \forall x \in A, |\xi(x) - x| < \varepsilon, \left| \frac{1}{\xi'(x)} - 1 \right| < \varepsilon \}.$$

Theorem 1 (Boyarsky & Góra, 1997, Theorem 11.2.2). *Let $\tau, \tau_n \in \mathcal{T}(I)$. Suppose that there exist $\delta > 0$ and $0 < r < 1$ such that for any $\eta = \tau$ or $\tau_n, n = 1, 2, \dots$,*

$$(a) \min_{1 \leq i \leq q} \lambda \left(I_i^{(\eta)} \right) \geq \delta,$$

$$(b) 2 \left(\inf_{x \in I} |\eta'(x)| \right)^{-1} + \max_{1 \leq i \leq q} V_{I_i^{(\eta)}} \left(|\eta'(x)|^{-1} \right) \leq r,$$

where $V_K f$ denotes the total variation of function f on interval K . If τ is ergodic and $d_S(\tau, \tau_n) \rightarrow 0$ as $n \rightarrow \infty$, then τ_n is ergodic for n large enough and $\pi_n \rightarrow \pi$ in $\mathcal{L}^1$ as $n \rightarrow \infty$, where π_n and π are the invariant densities of τ_n and τ , respectively.

Let ε be sufficiently small, and suppose that φ_ε is the perturbation of φ , and let $\{\varphi_n\}$ be a sequence corresponding to $\{\varepsilon_n : \varepsilon_1 > \varepsilon_2 > \dots > 0\}$. The stability of invariant densities can be established by showing the convergence of the sequence $\{\varphi_n\}$ in the Skorokhod metric. The piecewise linearity ensures that φ and φ_n satisfy the conditions (a) and (b) in Theorem 11.2.2. We consider two maps defined on the interval $[\theta_L, \theta_R]$.⁶

$$\varphi_\varepsilon(x) = \begin{cases} \frac{(1 - \theta_L)\bar{x} - (1 + \mu\varepsilon - \bar{x})x}{\bar{x} - \theta_L(1 + \mu\varepsilon)}, & \text{if } \theta_L \leq x < \frac{\bar{x}}{1 + \mu\varepsilon}, \\ \frac{-\bar{x}(x - \theta_R)}{(1 + \mu\varepsilon)\theta_R - \bar{x}}, & \text{if } \frac{\bar{x}}{1 + \mu\varepsilon} \leq x \leq \theta_R. \end{cases}$$

$$\varphi_M(x) = \frac{\theta_R - x}{\theta_R - \theta_L} = \begin{cases} \frac{(1 - \theta_L)\bar{x} - (1 - \bar{x})x}{\bar{x} - \theta_L}, & \text{if } \theta_L \leq x < \bar{x}, \\ \frac{-\bar{x}(x - \theta_R)}{\theta_R - \bar{x}}, & \text{if } \bar{x} \leq x \leq \theta_R. \end{cases}$$

Note that μ is a sufficiently small positive number introduced for tuning, and that φ_M can be expressed in this form, with φ_ε defined accordingly. As ε increases, $\bar{x}/(1 + \mu\varepsilon)$ deviates from $\bar{x}$, and the straight line structure is transformed into two intersecting branches, as illustrated in Fig. 6.

See Figure 6.

Using the mathematical framework developed by Zhang and Yokoo (2026b), we show that the long-run statistical properties of our asset pricing model are robust to small observation errors. The following proposition establishes the stability of the invariant densities under such perturbations.

Proposition 6. (*Robustness of invariant densities*) *Let the assumptions stated in proposition 3 be satisfied, so that φ satisfies the Markov property. If the perturbation φ_ε is defined as above, then the unique invariant densities of $\{\varphi_n\}$ converge in $\mathfrak{L}^1$ to the unique invariant density of φ .*

Proof. To estimate the Skorokhod metric $d_S(\varphi_\varepsilon, \varphi_M)$, let $\bar{x} > 0$ be such that $\varphi_M(\bar{x}) = \bar{x}$ and $A = A_1 \cup A_2$, where $A_1 = [\theta_L, \bar{x}/(1 + \mu\varepsilon)]$ and $A_2 = [\bar{x}, \theta_R]$. We first construct a diffeomorphism $\xi(x)$ such that $\varphi_\varepsilon = \varphi_M \circ \xi$ on A .

⁶Although the full interval I could also be considered, focusing on the interval $[\theta_L, \theta_R]$ is sufficient for our analysis.

On A_1 , we have

$$\xi_{|A_1}(x) = \xi_1(x) = \frac{(1 + \mu\varepsilon - \bar{x})(\bar{x} - \theta_L) x - \mu\varepsilon\theta_L(1 - \theta_L)\bar{x}}{[\bar{x} - (1 + \mu\varepsilon)\theta_L](1 - \bar{x})}.$$

On A_2 , we similarly have

$$\xi_{|A_2}(x) = \xi_2(x) = \frac{(\theta_R - \bar{x}) x + \mu\varepsilon\theta_R^2}{\theta_R(1 + \mu\varepsilon) - \bar{x}}.$$

To construct ξ on $(\bar{x}/(1 + \mu\varepsilon), \bar{x})$, we define ξ' to be piecewise linear and then define ξ as the integral of ξ' . We then have $\xi(\bar{x}/(1 + \mu\varepsilon))$ and $\xi(\bar{x})$. Thus, the average value of ξ' on $(\bar{x}/(1 + \mu\varepsilon), \bar{x})$ is given by

$$\xi'_{\text{average}} = \frac{|\xi(\bar{x}) - \xi(\bar{x}/(1 + \mu\varepsilon))|}{\mu\varepsilon\bar{x}/(1 + \mu\varepsilon)}.$$

We also have

$$\xi'_L = \xi' \left(\frac{\bar{x}}{1 + \mu\varepsilon} \right) = \frac{(1 + \mu\varepsilon - \bar{x})(\bar{x} - \theta_L)}{[\bar{x} - (1 + \mu\varepsilon)\theta_L](1 - \bar{x})} \quad \text{and}$$

$$\xi'_R = \xi'(\bar{x}) = \frac{\theta_R - \bar{x}}{\theta_R(1 + \mu\varepsilon) - \bar{x}}.$$

Let us construct the graph of ξ' on $[\bar{x}/(1 + \mu\varepsilon), \bar{x}]$ as follows: Draw it as a continuous piecewise linear curve, whose endpoints are given by ξ'_L and ξ'_R , such that the two regions enclosed by this curve and the line given by ξ'_{average} , one region above this line and the other below it have equal areas. See Boyarsky and Góra (1997, p.233) for details.

Next, we estimate $d_S(\varphi_1, \varphi_2)$ for sufficiently small $\mu > 0$.⁷

On $A_1 = [\theta_L, \bar{x}/(1 + \mu\varepsilon)]$, we have

$$|\xi_1(x) - x| = \left| \frac{\mu\varepsilon(1 - \theta_L)\bar{x}(x - \theta_L)}{[\bar{x} - (1 + \mu\varepsilon)\theta_L](1 - \bar{x})} \right| \leq \frac{\mu(1 - \theta_L)\bar{x}\varepsilon}{1 - \bar{x}} \leq \varepsilon.$$

$$\left| \frac{1}{\xi'_1(x)} - 1 \right| = \left| \frac{\mu\varepsilon\bar{x}(\theta_L - 1)}{(1 + \mu\varepsilon - \bar{x})(\bar{x} - \theta_L)} \right| \leq \frac{\mu\bar{x}(1 - \theta_L)\varepsilon}{(1 - \bar{x})(\bar{x} - \theta_L)} \leq \varepsilon.$$

⁷To be more specific, it suffices to set

$$\mu = \min \left\{ \frac{1 - \bar{x}}{(1 - \theta_L)\bar{x}}, \frac{(1 - \bar{x})(\bar{x} - \theta_L)}{\bar{x}(1 - \theta_L)}, \frac{1}{\theta_R}, \frac{\theta_R - \bar{x}}{\theta_R} \right\}.$$

On $A_2 = [\bar{x}, \theta_R]$, we have

$$|\xi_2(x) - x| = \left| \frac{\mu\varepsilon\theta_R(\theta_R - x)}{\theta_R(1 + \mu\varepsilon) - \bar{x}} \right| < \mu\theta_R\varepsilon \leq \varepsilon.$$

$$\left| \frac{1}{\xi_2'(x)} - 1 \right| = \frac{\mu\theta_R\varepsilon}{\theta_R - \bar{x}} \leq \varepsilon.$$

Thus, it follows that $d_S(\varphi_\varepsilon, \varphi_M) \leq \varepsilon$ for ε sufficiently small on $A = A_1 \cup A_2$, which implies that $d_S(\varphi_n, \varphi) \rightarrow 0$ as $n \rightarrow +\infty$. As φ is ergodic, so is φ_n for n sufficiently large. Let $\{\omega_n\}$ be a sequence of invariant densities corresponding to $\{\varepsilon_n : \varepsilon_1 > \varepsilon_2 > \dots > 0\}$. By Theorem 11.2.2 of Boyarsky and Góra (1997), the unique invariant densities of $\{\varphi_n\}$ converge in $\mathcal{L}^1$ to the unique invariant density of φ . $\square$

Fig. 7 illustrates the bifurcation diagram with respect to ε , showing that observable chaos persists for $\varepsilon > 0$ near zero. Moreover, the invariant densities remain stable under such perturbations, indicating that the stationary distribution of chaotic dynamics is robust.

See Figure 7.

6 Concluding remarks

In this paper, we have developed a general equilibrium model based on explicit optimizing behavior and shown that the interaction between endogenous technology choice and imperfect observation can generate complex asset price dynamics without exogenous shocks or departures from rational behavior.

The model connects rational-bubble theory with endogenous-fluctuation models and piecewise dynamical systems. Our findings suggest that asset price instability can arise from forward-looking rational behavior when the underlying economic environment is subject to threshold effects and informational frictions. It also shows that, under suitable conditions, the long-run statistical properties of the system remain tractable through invariant densities.

Our analysis has clear limitations. To obtain explicit analytical results, we restricted our attention to a class of piecewise linear maps that satisfy the Markov

property. While this restriction allows us to characterize invariant densities in closed form, it raises questions about the generality of our results.

One remaining issue is whether the stationary distributions obtained in this paper have an empirical counterpart in observed asset price behavior. Another issue is how the dynamic patterns of asset prices change when the model is extended to richer environments with production, credit frictions, or heterogeneous beliefs. Such extensions would clarify the extent to which the mechanism can account for the persistence and irregularity of fluctuations in actual asset markets.

Appendix

First, we impose an assumption.

Assumption 1.

$$A < \frac{(1 - \beta + \beta\pi)B + (1 - \beta)(1 - \delta)}{\beta\pi}. \quad (21)$$

Next, we provide some lemmas.

Lemma 1. *Let the productivity parameters A and B satisfy (10) and (21). Let the initial aggregate wealth satisfy*

$$W_0 \geq \bar{W}_0,$$

where $\bar{W}_0$ is given by (22). Furthermore, let the threshold $\xi > \max\{\bar{K}, \bar{\xi}\}$, where $\bar{K}$ and $\bar{\xi}$ are given by (11) and (24), respectively. Then $\phi_t = \pi$ for all t is an equilibrium path.

Proof. In order for $\phi_t = \pi$ to hold for all t , it is sufficient to impose that

$$R_t \leq \min\{A + 1 - \delta, B + 1 - \delta\} = B + 1 - \delta.$$

Setting $\phi_t = \pi$ for all t and using (7), we have

$$\begin{aligned} R_t = \frac{P_{t+1} + D}{P_t} &= \frac{\beta Z_t \pi}{1 - \beta + \beta\pi} + \frac{1}{(1 - \beta + \beta\pi)\beta(1 - \pi)} \frac{SD}{W_t} \\ &\leq \frac{\beta(A + 1 - \delta)\pi}{1 - \beta + \beta\pi} + \frac{1}{(1 - \beta + \beta\pi)\beta(1 - \pi)} \frac{SD}{W_t}. \end{aligned}$$

Since R_t is a decreasing function in W_t for each technology, W_t must be larger than a certain critical value $\bar{W}_0 > 0$ for all $t \geq 0$. This critical value is given by

$$\bar{W}_0 := \frac{DS}{\beta(1 - \pi)((B + 1 - \delta)(1 - \beta) - \beta\pi(A - B))}, \quad (22)$$

under (21). In terms of wealth, (9) turns to

$$W_{t+1} = \tilde{H}(W_t) := \begin{cases} aW_t + \tilde{c} =: \tilde{H}_L(W_t) & \text{if } 0 < W_t \leq \frac{\xi}{\beta\pi}, \\ bW_t + \tilde{c} =: \tilde{H}_R(W_t) & \text{if } \frac{\xi}{\beta\pi} < W_t, \end{cases} \quad (23)$$

where

$$a = \frac{(A + 1 - \delta)\beta\pi}{1 - \beta(1 - \pi)}, \quad b = \frac{(B + 1 - \delta)\beta\pi}{1 - \beta(1 - \pi)}, \quad \text{and} \quad \tilde{c} = \frac{DS}{1 - \beta(1 - \pi)}.$$

By (10), recalling that $a > 1 > b > 0$ and that $\tilde{H}$ has no actual steady state as H does none by (12), we can see that $\tilde{H}$ in (23) has a trapping interval

$$I_T = \left[\tilde{H}_R(\xi/\beta\pi), \tilde{H}_L(\xi/\beta\pi) \right] = \left[\frac{b\xi}{\beta\pi} + \tilde{c}, \frac{a\xi}{\beta\pi} + \tilde{c} \right].$$

That is, if $W_s \in I_T$, then $W_t \in I_T$ for all $t \geq s$. Thus, if

$$\bar{W}_0 \leq \frac{b\xi}{\beta\pi} + \tilde{c}$$

or

$$\xi \geq \frac{(\bar{W}_0 - \tilde{c})\beta\pi}{b} =: \bar{\xi}, \quad (24)$$

and $W_0 \geq \bar{W}_0$, then $W_t \geq \bar{W}_0$ for all $t \geq 0$. $\square$

Derivation of G_M

Given that $\theta_L = \theta - \sigma$, $\theta_R = \theta + \sigma$ and

$$x'_t = h(x_t) = \frac{x_t - \sigma b}{1 - \sigma a - \sigma b},$$

θ'_L and θ'_R can be obtained as follows:

$$\theta'_L = h(\theta_L) = \frac{\theta_L - \sigma b}{1 - \sigma a - \sigma b} = \frac{\theta - \sigma - \sigma b}{1 - \sigma a - \sigma b},$$

$$\theta'_R = h(\theta_R) = \frac{\theta_R - \sigma b}{1 - \sigma a - \sigma b} = \frac{\theta + \sigma - \sigma b}{1 - \sigma a - \sigma b}.$$

Since

$$\varphi_M(x'_t) = \frac{\theta'_R - x'_t}{\theta'_R - \theta'_L},$$

$\varphi_M(h(x_t))$ can be derived as follows:

$$x'_{t+1} = \varphi_M(x'_t) = \varphi_M(h(x_t)) = \frac{\theta + \sigma - x_t}{2\sigma}.$$

Consequently,

$$x_{t+1} = h^{-1}(x'_{t+1}) = h^{-1}(\varphi_M(h(x_t))) = (1 - \sigma a - \sigma b) \cdot \frac{\theta + \sigma - x_t}{2\sigma} + \sigma b.$$

Since $x_t = \theta - \sigma y_t$, all occurrences of x are expressed in terms of y ,

$$h^{-1}(\varphi_M(h(\theta - \sigma y_t))) = (1 - \sigma a - \sigma b) \cdot \frac{1 + y_t}{2} + \sigma b.$$

Substituting this result into the following equation

$$G_M(y_t) = \frac{h^{-1}(\varphi_M(h(\theta - \sigma y_t))) + \sigma b y_t}{1 - \sigma(a - b)y_t}$$

yields

$$G_M(y_t) = \frac{(1 - \sigma a + \sigma b)(1 + y_t)}{2[1 - \sigma(a - b)y_t]}.$$

It can be verified that $\lim_{y \rightarrow -1} G_M = 0$ and $\lim_{y \rightarrow 1} G_M = 1$.

Formula for $\hat{a}$

Taking θ and b as constants, as previously assumed, we can solve the quadratic equation (20) with respect to a to obtain:

$$a = \frac{M \pm \sqrt{N}}{2b\theta},$$

where

$$M = -b^2\theta + b^2 - 2b\theta + b - 2\theta,$$

and

$$N = b^4\theta^2 - 2b^4\theta + b^4 + 4b^3\theta^2 - 6b^3\theta + 2b^3 + 8b^2\theta^2 - 8b^2\theta + b^2 + 4b\theta - 4\theta^2.$$

Discarding the solution with the negative sign, as it takes a negative value, we obtain $\hat{a} = (M + \sqrt{N})/2b\theta$. For instance, let $\theta = 0.25$ and $b = 0.5$, then $\hat{a} = (M + \sqrt{N})/2b\theta \approx 1.77$, while $(M - \sqrt{N})/2b\theta \approx -2.27 < 0$.

Derivation of $\hat{\sigma}$ for Proposition 3

This derivation is quite analogous to that of Proposition 2. Using

$$\theta'_L = h(\theta_L) = \frac{\theta_L - \sigma b}{1 - \sigma a - \sigma b} = \frac{\theta - \sigma - \sigma b}{1 - \sigma a - \sigma b},$$

$$\theta'_R = h(\theta_R) = \frac{\theta_R - \sigma b}{1 - \sigma a - \sigma b} = \frac{\theta + \sigma - \sigma b}{1 - \sigma a - \sigma b}.$$

and

$$\theta_{2n+3,L} = \frac{b^n(1-b)}{(1-b^{n+1})(1+ab^n)}, \quad \theta_{2n+3,R} = \frac{b^n(1-b)}{1-b^{n+1}},$$

we can solve the simultaneous equations:

$$\begin{cases} \theta'_L(\sigma, a) = \theta_{2n+3,L} \\ \theta'_R(\sigma, a) = \theta_{2n+3,R} \end{cases}$$

for σ and a .

First, solving $\theta'_R(\sigma, a) = \theta_{2n+3,R}$ gives σ , that is,

$$\sigma = \frac{b^n(1-b) - \theta(1-b^{n+1})}{(1-b)(1+ab^n)}.$$

Substituting this σ into $\theta'_L(\sigma, a) = \theta_{2n+3,L}$ yields

$$\frac{D_1\theta - (1+b)D_2}{D_1 - (a+b)D_2} = \frac{b^n(1-b)}{(1-b^{n+1})(1+ab^n)}, \quad (25)$$

where $D_1 = (1-b)(1+ab^n)$ and $D_2 = b^n(1-b) - \theta(1-b^{n+1})$. As (25) is a quadratic equation for a , $\hat{a}$ can be explicitly calculated, given b and θ .

Next, based on $\hat{a}$ obtained above, we solve

$$\frac{\theta - \sigma - \sigma b}{\theta_{2n+3,L}} = \frac{\theta + \sigma - \sigma b}{\theta_{2n+3,R}}$$

for $\hat{\sigma}$ to obtain

$$\hat{\sigma} = \frac{\hat{a}b^n\theta}{2 + \hat{a}b^n(1+b)}.$$

Derivation of $\omega_{2n+3}^*(x)$

This derivation is analogous to that of Proposition 5. The Frobenius-Perron operator matrix $M_{\varphi_{2n+3}}$ is given by

$$M_{\varphi_{2n+3}} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \frac{1}{a} \\ \lambda & \lambda & \cdots & \lambda & \lambda & \lambda \\ \frac{1}{b} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{1}{b} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{1}{b} & 0 & 0 \end{pmatrix}_{(2n+2) \times (2n+2)}$$

where $\lambda = \theta_{2n+3,R} - \theta_{2n+3,L} = \hat{a}b^{2n}(1-b)/[(1-b^{n+1})(1+\hat{a}b^n)]$. Then, solving $\omega M_{\varphi_{2n+3}} = \omega$, that is, the left eigenvector for eigenvalue 1, yields the following density function:

$$\omega = \omega_{2n+3}^*(x) = (\omega_1, \omega_2, \dots, \omega_{2n+1}, \omega_{2n+2}).$$

Note that $\omega_2 = (1 + \hat{a}b^n)(1 + b + \dots + b^n)$ is the maximum, while $\omega_{2n+1} = \hat{a}b^{2n}$ denotes the minimum. The remaining entries can be derived from these two entries.

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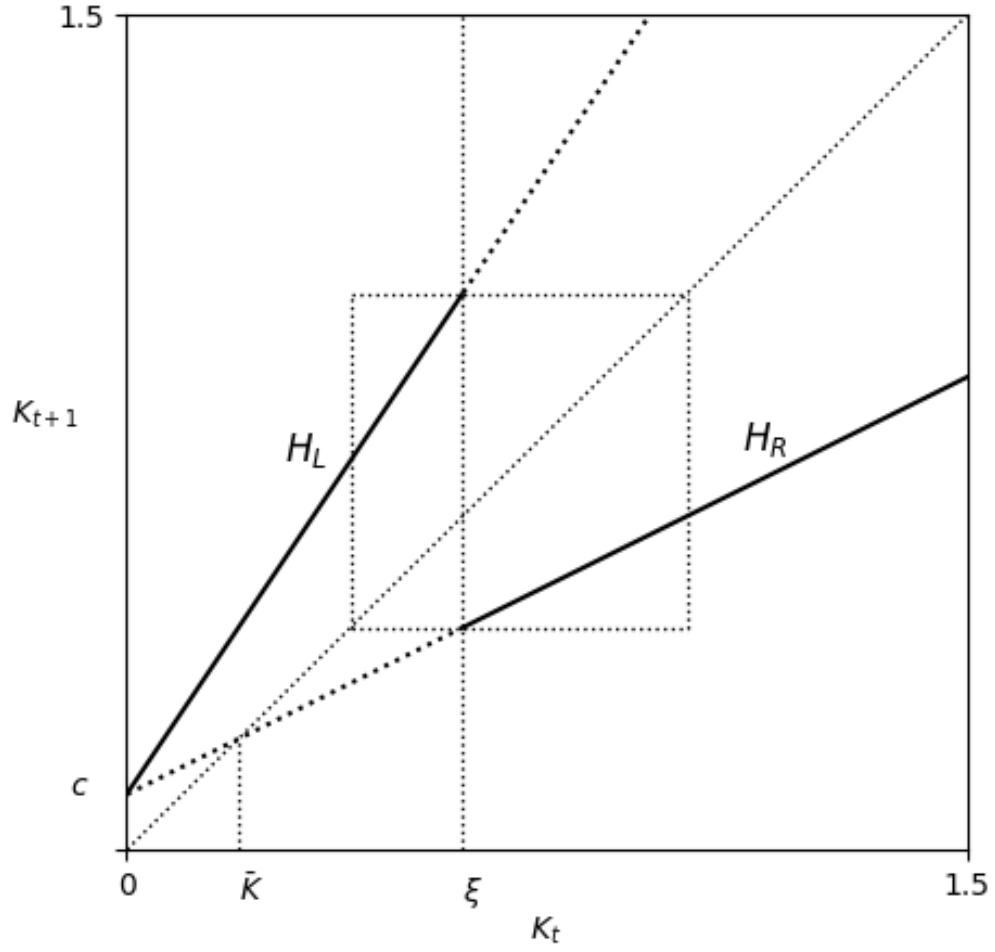


Figure 1: Plotting $K_{t+1} = H(K_t)$. Every path eventually becomes trapped in the middle box area. Parameters: $a = 1.5, b = 0.5, c = 0.1, \xi = 0.6$.

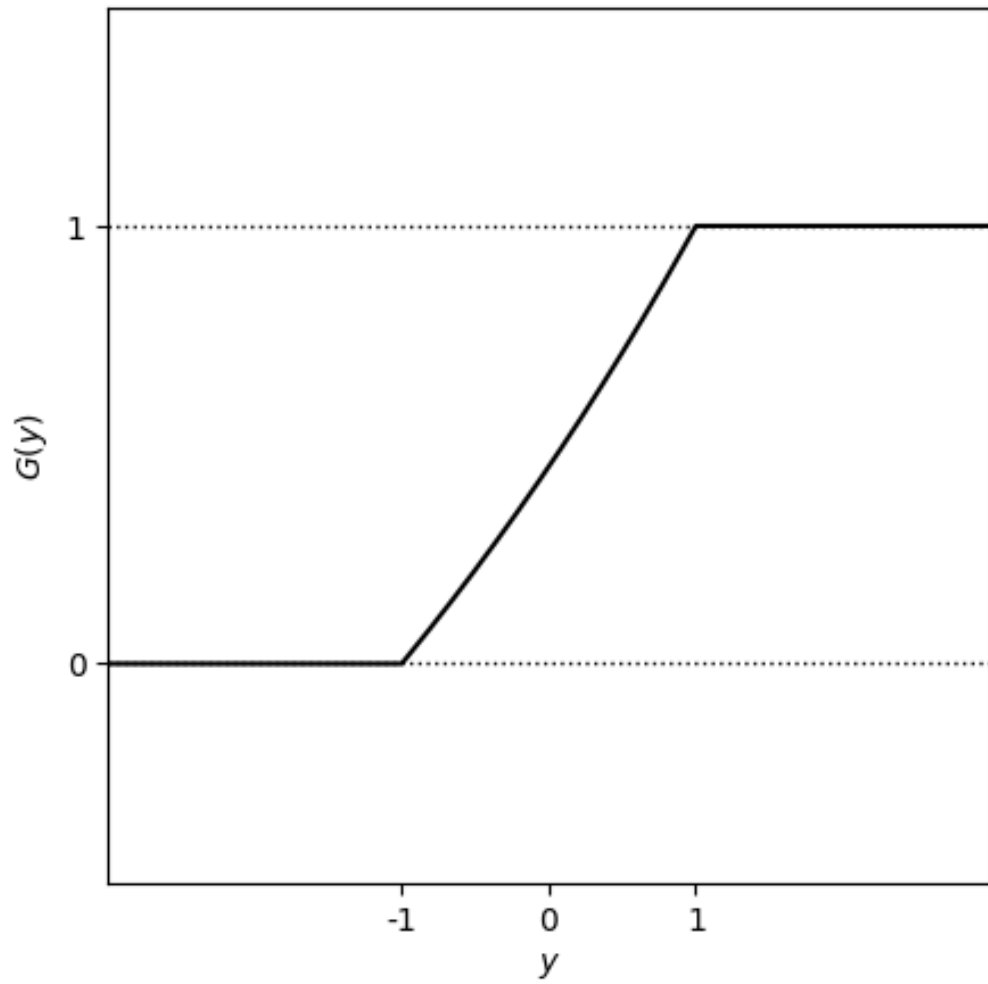


Figure 2: Graph of cumulative distribution function (CDF). $a = 1.5$, $b = 0.5$, $\sigma = 0.1$.

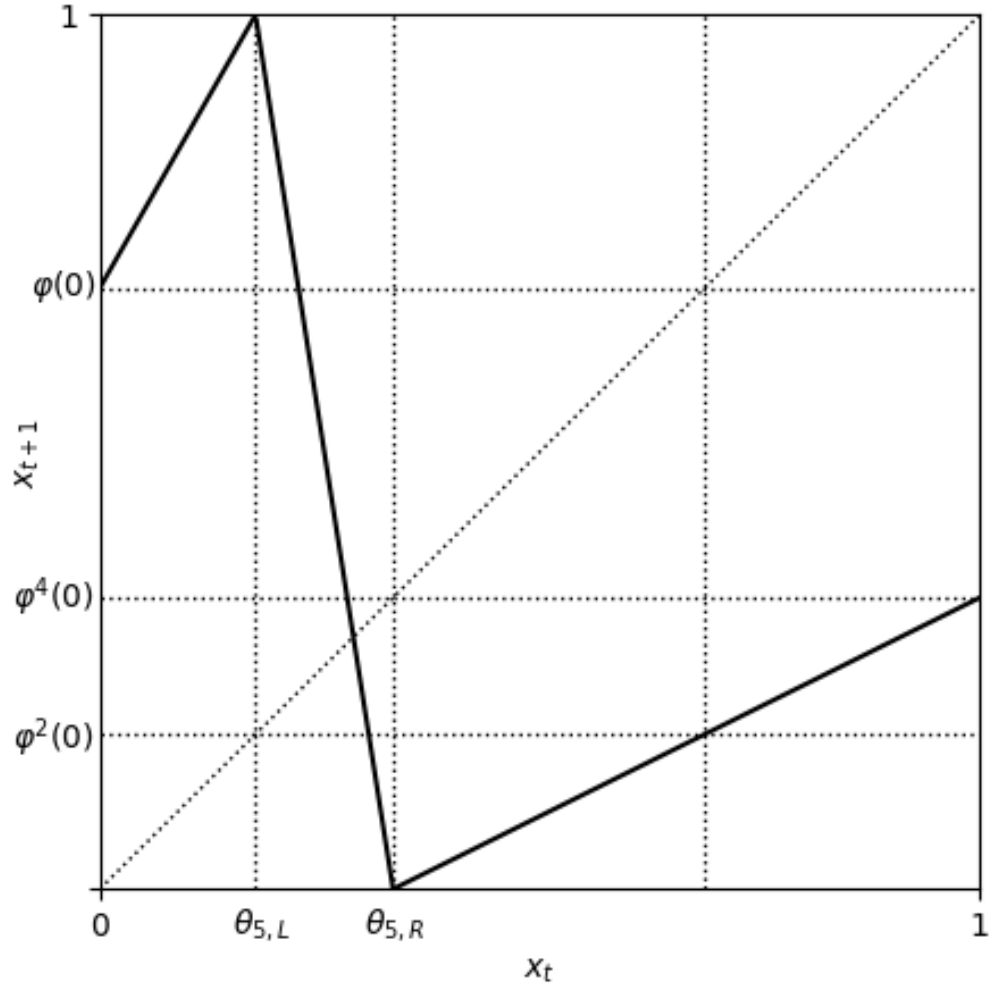


Figure 3: Period-5 Markov property. $b = 0.5$, $\theta = 0.25$, $\hat{a} \approx 1.76556$, $\hat{\sigma} \approx 0.06639$.

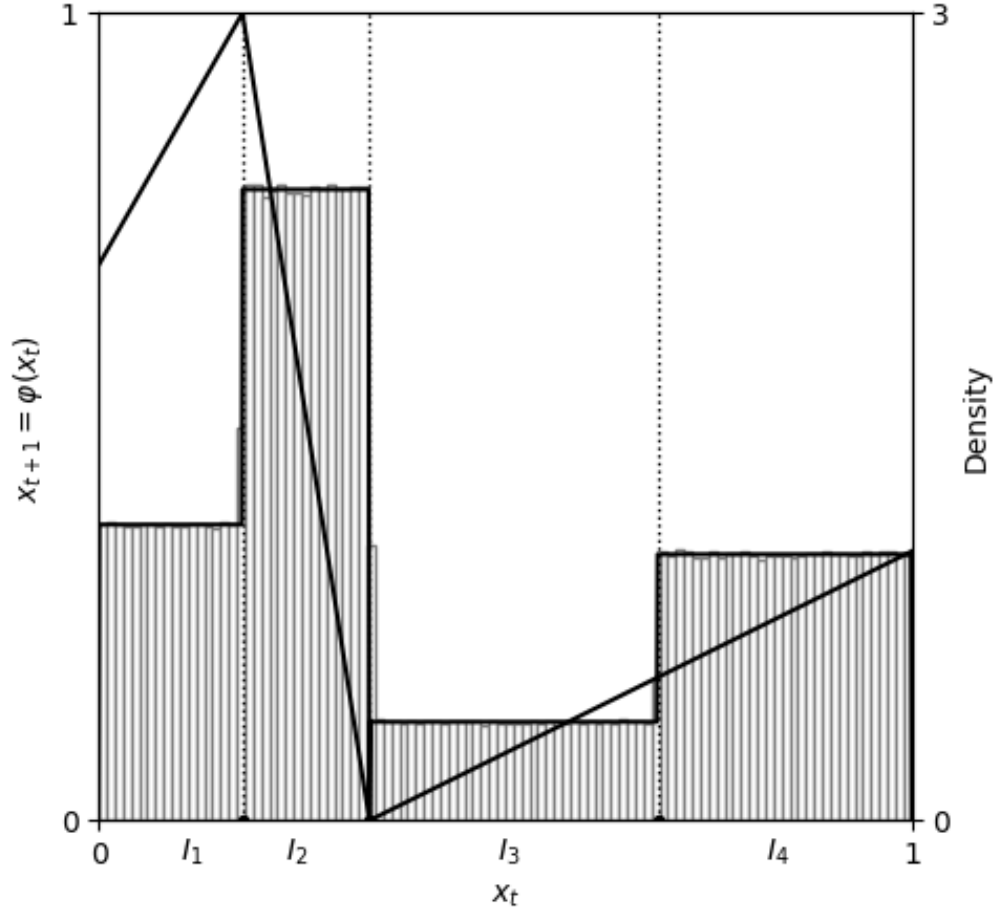


Figure 4: Theoretical and estimated invariant density for the period-5 Markov map. $b = 0.5$, $\theta = 0.25$, $\hat{a} \approx 1.76556$, $\hat{\sigma} \approx 0.06639$. Density $p(x) = \omega_5^*(x) / \sum_{i=1}^4 \omega^i |I_i|$. Simulated histogram of 10^6 iterations.

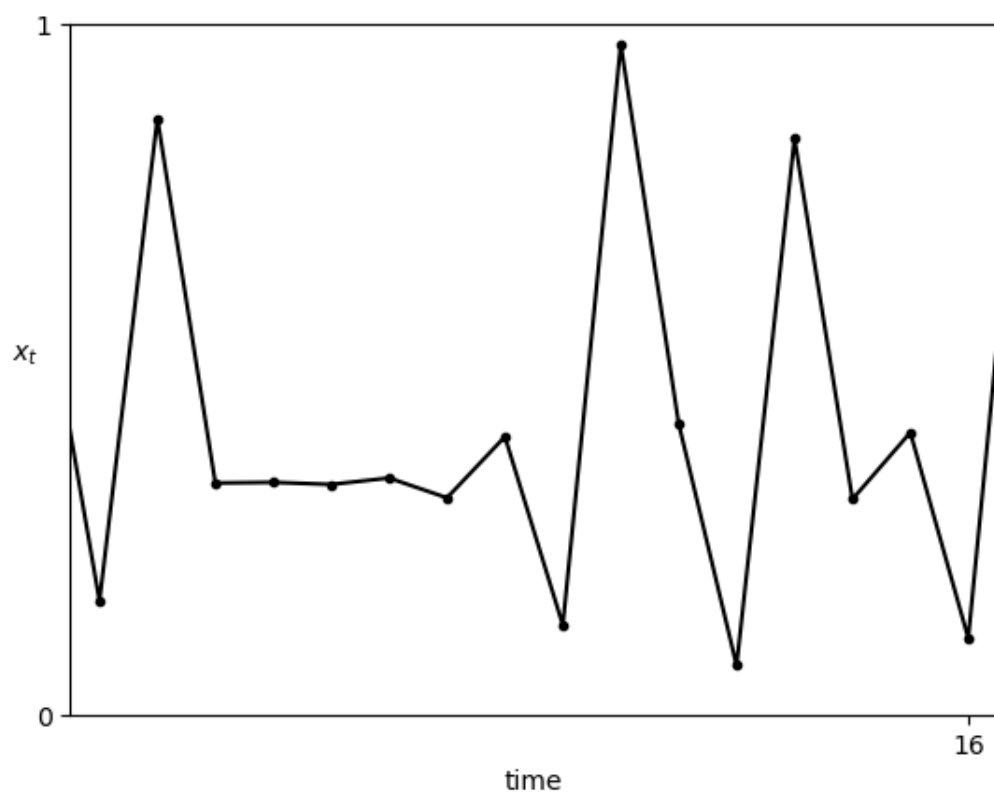


Figure 5: The chaotic fluctuations between booms and busts are reproduced endogenously.

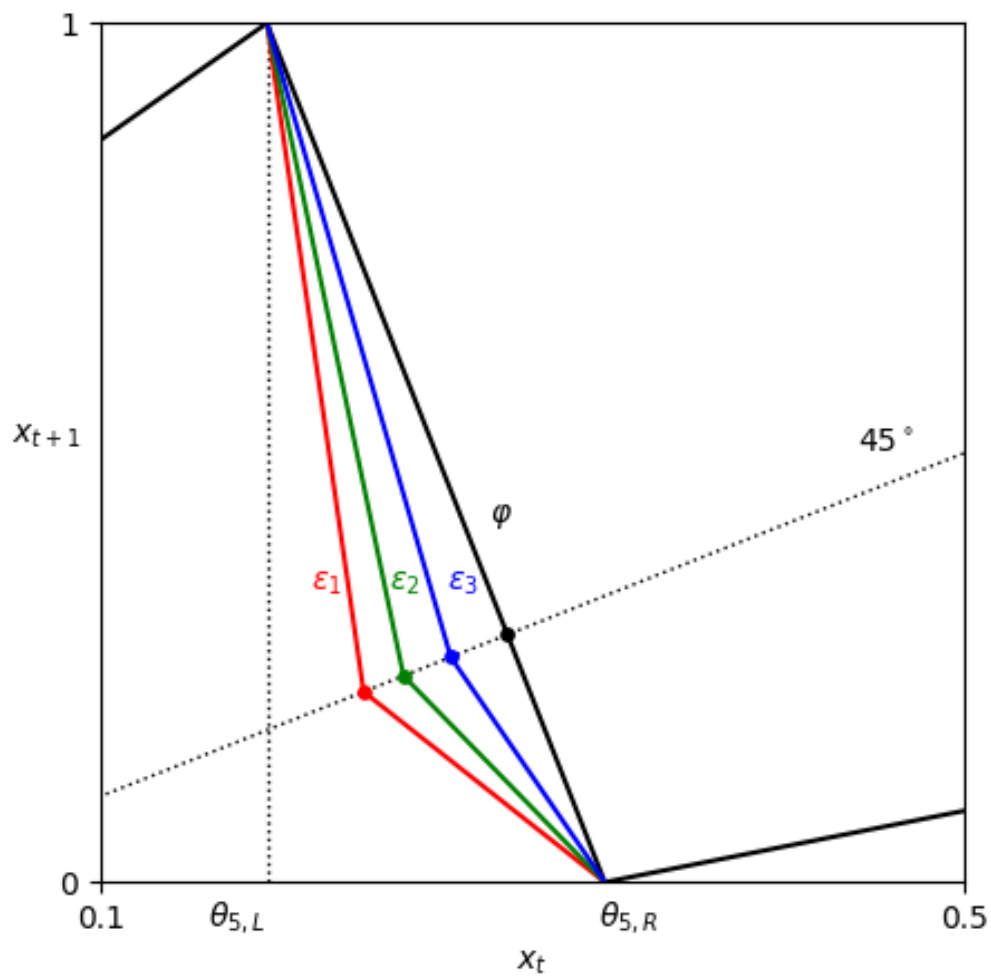


Figure 6: Perturbations for different values of ε .

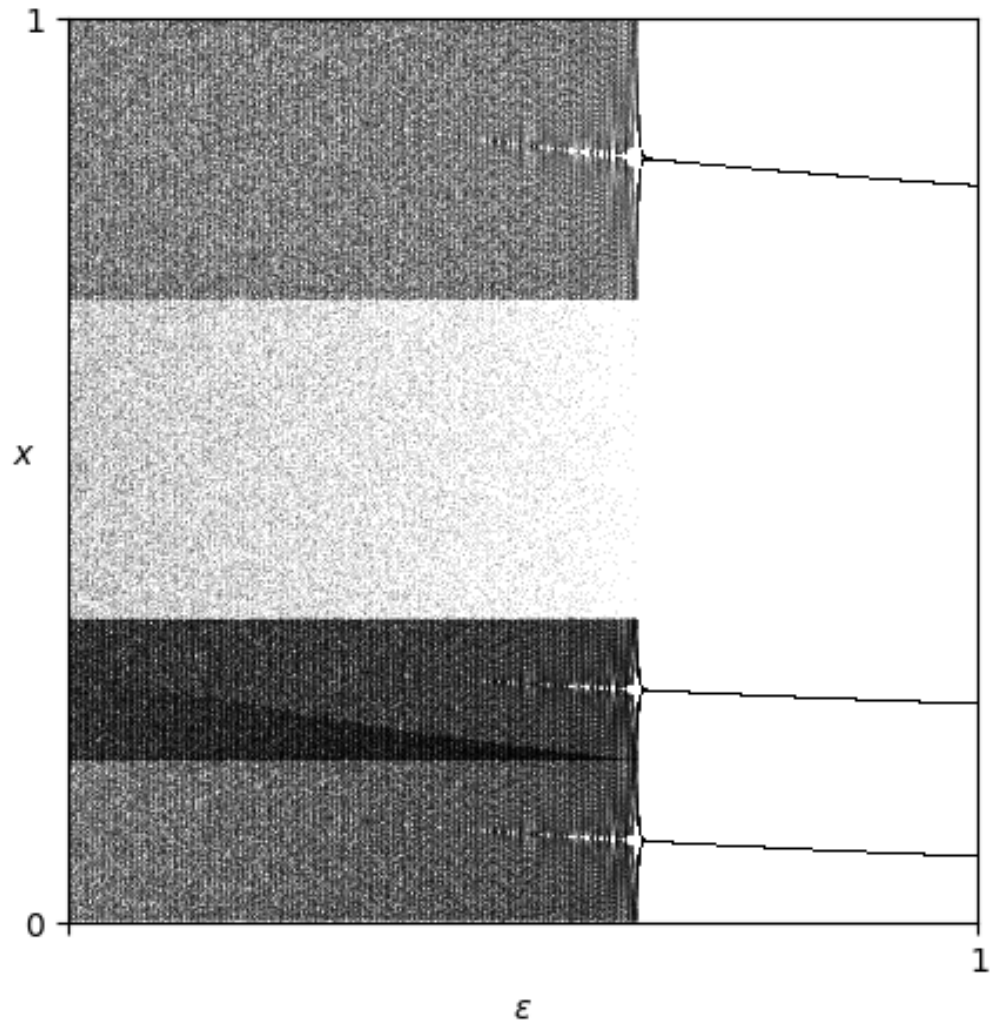


Figure 7: Bifurcation diagram with respect to ε under the period-5 Markov parameters, demonstrating the robustness with respect to small perturbations.